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SPECIAL HEEGAARD SPLITTINGS FOR CLOSED, ORIENTED 3-MANIFOLDS

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§1. INTRODUCTION

LET D^4 BE an oriented 4-ball, and let $-D^4$ denote its image under an orientation-reversing homeomorphism R of period 2. The boundary of D^4 is the 3-sphere Σ, which we will assume is decomposed into homeomorphic handlebodies $\mathcal{A}$ and $\mathcal{B}$ of genus n. Let $\tilde{r} = R|\mathcal{A}$. Let $\tilde{A}(n)$ be the group of all orientation-preserving homeomorphisms of $\mathcal{A} \to \mathcal{A}$, and let $A(n)$ denote the group of homeomorphisms of $\mathrm{Bd}\mathcal{A}$ which arise by restricting maps in $\tilde{A}(n)$ to $\mathrm{Bd}\mathcal{A} = \mathrm{Bd}\mathcal{B}$. Then, for each map $\tilde{h} \in \tilde{A}(n)$ we may define a simply-connected, oriented 4-manifold $N(\tilde{h})$ by identifying D^4 with $-(D^4)$ along the $\mathcal{A}$ portion of $\mathrm{Bd}D^4$, according to the rule $\tilde{r}\tilde{h}(x) = x$, $x \in \mathcal{A}$. If $h = \tilde{h}|\mathrm{Bd}\mathcal{A}$, $r = \tilde{r}|\mathrm{Bd}\mathcal{A}$, then $\mathrm{Bd}N(\tilde{h})$ is seen to be the oriented 3-manifold $M(h)$ obtained by identifying $\mathcal{B}$, with $-\mathcal{B}$ according to the rule $\mathrm{rh}(\mathrm{x}) = \mathrm{x}$, $\mathrm{x} \in \mathrm{Bd}\mathcal{B}$. That is, $M(h)$ is defined by a *Heegaard splitting* $\mathcal{B} +_{rh} - \mathcal{B}$ of genus n, determined by the *Heegaard sewing* $h \in A(n)$.

Definition. A 3-manifold M will be said to be represented by a *special Heegaard sewing* if, for some integer n, there exists an element $h \in A(n)$ such that $M \cong M(h)$.

A system $\mathbf{x} = \{x_1, \ldots, x_n\}$ of pairwise disjoint nonseparating simple closed curves on $\mathrm{Bd}\mathcal{B}$ is a *complete system for* $\mathrm{Bd}\mathcal{B}$ if $\mathrm{Bd}\mathcal{B}$ split open along $\mathbf{x}$ is a 2-cell. The system $\mathbf{x}$ is a *complete system for* $\mathcal{B}$ if each x_i bounds a disc in $\mathcal{B}$ and if $\mathcal{B}$ split open along those discs is a 3-cell.

A special Heegaard sewing h is a special *even* sewing if there is a complete system $\mathbf{x}$ for $\mathcal{B}$ such that the algebraic intersection matrix $W(h) = \||h(x_i) \cap x_j|_0\|$ is symmetric and has even diagonal entries.

THEOREM 1. *Every closed, oriented* 3*-manifold may be represented as* $M(h)$ *for some special even sewing* h.

Note that, in view of Theorem 1, the investigation of 3-manifolds via Heegaard splittings is altered from a problem involving the study of homeomorphisms of surfaces to a problem involving the study of homeomorphisms of handlebodies.

Our special even Heegaard splittings are defined above in terms of group theoretical restrictions on the sewing map h. Alternatively, we may characterize these splittings in terms of restrictions on a Heegaard diagram for M. In order to explain the latter point of view, we require some definitions.

A Heegaard diagram for a 3-manifold is a triple $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ where $\mathrm{Bd}\mathcal{B}$ is an abstract closed orientable surface and where $\mathbf{x}$ and $\mathbf{y}$ are two complete systems of curves for $\mathrm{Bd}\mathcal{B}$. Note that each Heegaard splitting defines a multiplicity of Heegaard diagrams, obtained by choosing $\mathrm{Bd}\mathcal{B}$ to be a Heegaard surface in M and choosing $\mathbf{x}$ and $\mathbf{y}$ to be any two complete systems for the two sides of the splitting. Conversely, each Heegaard diagram determines a 3-manifold M and a Heegaard splitting of M. If $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ is a Heegaard diagram for M and if $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{z})$ is a Heegaard diagram for S^3, then $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ will be referred to as an *augmented Heegaard diagram*.

We now introduce two concepts which are analogous to the concepts of a special sewing and special even sewing. See [5] for further developments of this theme.

Definition. A Heegaard diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ is a *special Heegaard diagram* for M if there exists an augmented diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ such that the $n \times n$ matrix of geometric intersection numbers $\|\, |y_i \cap z_j| \,\|$ is the identity matrix, and is *special even* if the $n \times n$ matrix of algebraic intersection numbers $W = \|\, |y_i \cap x_j|_0 \,\|$ is symmetric and has even diagonal entries.

To get a feeling for this concept, think of the $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ curves as being colored red, blue and green respectively. The curves of any one color are, of course, pairwise disjoint, however curves of distinct colors may intersect one another many times. If M has Heegaard genus n_0, then for each $n \geq n_0$ there will be many red and blue systems which define M. The 3-sphere Σ is likewise defined by many red and green systems for every genus $n \geq 1$. We are asking that a blue system be chosen which satisfies the very restrictive property that, for some green system $\mathbf{z} = \{z_1, \ldots, z_n\}$, each blue curve y_i, $1 \leq i \leq n$, is located in the $(n-1)$-punctured torus $\mathrm{Bd}\mathcal{B} - z_1 \cup \cdots \cup z_{i-1} \cup z_{i+1} \cup \cdots \cup z_n$, and also y_i crosses z_i exactly once. We ask, further, that for such a system each blue curve y_i meet its red partner x_i an even number of times, and moreover $|y_i \cap x_j|_0 = |y_j \cap x_i|_0$ for each $1 \leq i \neq j \leq n$. These conditions sound so restrictive that one might wonder if such diagrams exist. To show that they do exist, we exhibit in Fig. 1a a special even diagram of genus 2 for the 3-sphere Σ and in Fig. 2 a special even diagram of genus 8 for the Poincaré homology sphere P, i.e. spherical dodecahedral space. In Fig. 1b we define a special even sewing s for $M(s) \cong \Sigma$. The sewing of Fig. 1b may be used to construct the diagram of Fig. 1a, by defining $\mathbf{y} = s(\mathbf{x})$. We will prove:

LEMMA 1. *A 3-manifold M admits a special (even) Heegaard diagram if and only if it admits a special (even) Heegaard sewing.*

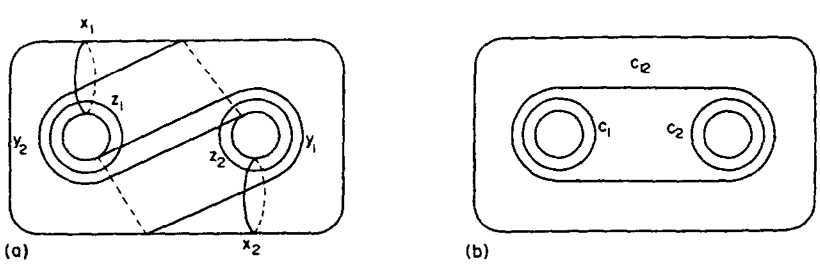
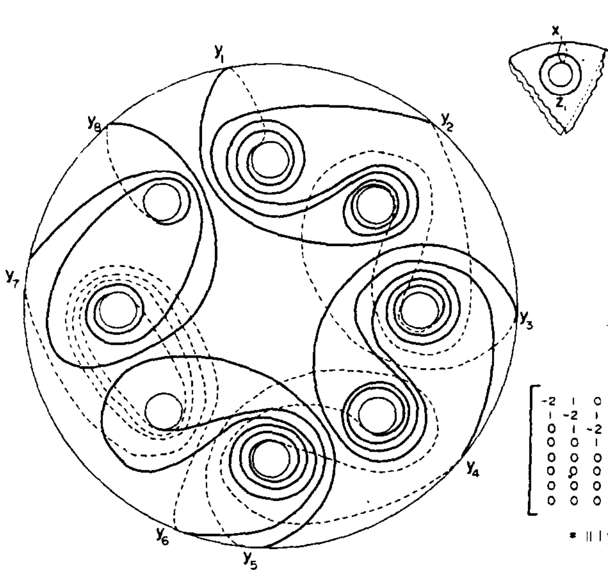


Fig. 1. (a) Special even (augmented) Heegaard diagram of genus 2 for Σ. (b) Special even sewing $s = t_{c_1}^{-1} t_{c_2}^{-1} t_{c_{12}}$ of genus 2 for $M(s) = \Sigma$ where t_c = Dehn twist about c.



$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -4 & -1 \\ 0 & 0 & 0 & 0 & 0 & & -1 & -2 \end{bmatrix}$$

$$= \|\, |y_i \cap x_j|_0 \,\|$$

Fig. 2. A special even Heegaard diagram of genus 8 for the Poincaré homology sphere P.

Lemma 1 and Theorem 1 then imply:

THEOREM 1′. *Every closed, oriented 3-manifold may be represented by a special even Heegaard diagram.*

COROLLARY 1. *If* $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ *is a special even Heegaard diagram, and if M is a $Z/2Z$-homology sphere, then the Rohlin invariant $\mu(M)$ is a* mod-16 *reduction of the signature of the symmetric matrix* $W = \||y_i \cap x_j|_0\|$.

Remark. The symmetric matrix W in a special even Heegaard diagram represents an integral symmetric quadratic form of type II (see Ch. II, [10]). A basic theorem in the theory of quadratic forms asserts that the signature of such a form is divisible by 8 (Thm 5.1, [10]). From this it follows that the signature of W is necessarily zero if the genus is less than 8. Thus, if $\mu(M) \neq 0$, the minimum genus for a special even Heegaard diagram is 8. The Rohlin invariant of the Poincaré homology sphere P is, by remarks on p. 65 of [7], non-zero, and since we have exhibited a special even diagram of genus 8 for P in Fig. 2, it follows that the manifold P has "special even genus" exactly 8. Since the Heegaard genus of P is known to be 2 (see p. 245, [13]), it follows that the *special even Heegaard genus* of a 3-manifold M is a new and meaningful topological invariant.

Outline of the paper. §2 contains the proof of Lemma 1, Theorems 1 and 1′, and Corollary 1. In §3, we apply the results of §2 to give a constructive procedure for enumerating all Z-homology 3-spheres M which have $\mu(M) \neq 0$ (or $\mu(M) = 0$). This procedure may be of interest in investigating the possible existence of index-8 Z-homology spheres.

§2. PROOFS OF LEMMA 1, THEOREMS 1 AND 1′ AND COROLLARY 1

We begin by establishing notation. Recall that $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic handlebodies, and that the 3-sphere Σ is given as $\mathcal{A} \cup \mathcal{B}$. We choose canonical complete systems $\mathbf{a}$ for $\mathcal{A}$ and $\mathbf{b}$ for $\mathcal{B}$, where $\mathbf{a}, \mathbf{b} \subset \mathrm{Bd}\mathcal{A} = \mathrm{Bd}\mathcal{B}$. These will be chosen so that the geometric intersection matrix $\||a_i \cap b_j|\|$ is the identity matrix. We will write $[a_i]$ for the homotopy class of a_i, and will assume that these curves have been chosen so that $\prod_{i=1}^{n} [a_i][b_i][a_i^{-1}][b_i^{-1}] = 1$. Let

$$H(n) = \{h\colon \mathrm{Bd}\mathcal{B} \to \mathrm{Bd}\mathcal{B} \mid h \text{ preserves orientation}\},$$

$$A(n) = \{h \in H(n) \mid h \text{ extends to a homeomorphism of } \mathcal{A}\},$$

$$B(n) = \{h \in H(n) \mid h \text{ extends to a homeomorphism of } \mathcal{B}\}.$$

Then

$$A(n) \cap B(n) = \{h \in H(n) \mid h \text{ extends to a homeomorphism of } \Sigma\}.$$

Choose $s_0 \in H(n)$ in such a way that the automorphism of $\pi_1(\mathrm{Bd}\mathcal{B})$ which is induced by s_0 acts on the homotopy classes $[a_i]$, $[b_i]$ in the following manner:

$$[a_i] \to [a_i][b_i][a_i]^{-1}, \quad [b_i] \to [a_i]^{-1}, \quad 1 \leq i \leq n. \tag{1}$$

Then

$$A(n) = s_0(B(n))s_0^{-1} = s_0^{-1}(B(n))s_0. \tag{2}$$

Note that, with our sewing convention (introduced earlier), two Heegaard sewing maps $h, h' \in H(n)$ define equivalent Heegaard splittings of the manifolds $M(h)$ and $M(h')$ if and only if

$$h' = f_2 h f_1 \quad \text{for some} \quad f_1, f_2 \in B(n). \tag{3}$$

We remark that if h is isotopic to h', then (3) is always satisfied. Note that if h is special, i.e., if $h \in A(n)$, and if $f_1, f_2 \in A(n) \cap B(n)$, then h' in formula (3) will also be special.

Proof of Lemma 1. Suppose, first that $M \cong M(h)$ for some special Heegaard sewing $h \in A(n)$. By equation (2), we have $h = s_0 f s_0^{-1}$ for some $f \in B(n)$. Let $g = hs_0 = s_0 f$. By equation (3), the Heegaard sewings g and s_0 define equivalent Heegaard splittings, hence $M(g) \cong M(s_0) \cong \Sigma$. Thus, we may define an augmented Heegaard diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{b}, h(\mathbf{b}), g(\mathbf{b}))$ for $M(h)$. Note that

$$\begin{aligned}|h(b_i) \cap g(b_j)| &= |b_i \cap h^{-1}g(b_j)| = |b_i \cap s_0(b_j)| \\ &= |b_i \cap a_j^{-1}| = \delta_{ij},\end{aligned}$$

where δ_{ij} is the Kronecker symbol. Thus $\||h(b_i) \cap g(b_j)|\|$ is the identity matrix, hence the diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{b}, h(\mathbf{b}))$ is special.

To establish the converse, suppose that $(Bd\mathcal{B}, \mathbf{x}, \mathbf{y})$ is a special Heegaard diagram for M. Then there exists a Heegaard diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{z})$ for Σ such that the intersection matrix $\||x_i \cap z_j|\|$ is the identity matrix. We may without loss of generality assume that $\mathbf{x} = \mathbf{b}$. Choose $h, g \in H(n)$ such that $h(b_i) = y_i$, $g(b_i) = z_i$, $1 \le i \le n$. Then

$$|y_i \cap z_j| = |h(b_i) \cap g(b_j)| = |b_i \cap h^{-1}g(b_j)| = \delta_{ij}.$$

Thus, the Heegaard sewing $h^{-1}g$ determines a "very good" system of meridinal pairs, in the language of Waldhausen[15], hence $M(h^{-1}g) \cong \Sigma$. By the main result in [15], and eqn (3) above, we may then find $f_1, f_2 \in B(n)$ such that $h^{-1}g = f_2 s_0 f_1$, so that $h = gf_1^{-1}s_0^{-1}f_2^{-1}$. Since $M(g) = \Sigma$, the same argument gives $g = f_4 s_0 f_3$, where $f_3, f_4 \in B(n)$. Let $h' = f_4^{-1}hf_2$. By eqn (3), we see that h and h' are equivalent Heegaard splittings of $M(h)$. By construction, $h' = s_0 f_3 f_1^{-1} s_0^{-1}$. By eqn (2), it then follows that $h' \in A(n)$. Thus h' is a special sewing. This completes the proof of Lemma 1.

Proof of Theorem 1' *and Theorem* 1'. In view of Lemma 1, it is adequate to establish Theorem 1'. Let $L = \bigcup_{i=1}^{n} L_i$ be a link in the oriented 3-sphere Σ. The link will be assumed to have associated with it a *framing* l, i.e., an integer l_i will be associated with each component L_i of L, so that the framed link $(L, l) = \bigcup_{i=1}^{n} (L_i, l_i)$ defines a 3-manifold M which is obtained by surgery on L. This surgery may be described as follows: remove from Σ pairwise disjoint tubular neighborhoods V_i of the curves L_i, and resew such V_i, identifying a meridian z_i in $\mathrm{Bd}\, V_i$ with a curve $y_i \subset \mathrm{Bd}(\Sigma - \dot{V}_i) \subset \Sigma$ which links L_i exactly l_i times. Note that the $n \times n$ intersection matrix $\||y_i \cap z_j|\|$ in such a surgery is the identity matrix. By a well known result, every M may be obtained by framed surgery in some link in Σ, hence we will assume that our manifold M is so defined. By a result due to Steve Kaplan[8] we may, further, assume that the framings l_i are all even integers.

The idea of the proof will be to modify (L, l) to a new surgery representation (L^*, l^*) such that L^* defines in a natural way the spine of a handlebody $\mathcal{A}^*$ which is half of a Heegaard splitting of Σ. The reader may find the example which is given in Fig. 3a–f helpful in following the steps in the construction.

To begin, we locate a graph G in Σ which contains the link L as a subset, and which has the property that G has a regular neighborhood $\mathcal{A}$ such that $\mathcal{B} = \Sigma - \dot{\mathcal{A}} \cong \mathcal{A}$, i.e., $\mathcal{A}$ is half of a Heegaard splitting of Σ. This is clearly possible, because L may be altered to the trivial link of n components by changing a finite number of crossings, hence we may obtain G by adding s 1-simplexes τ_k, $1 \le k \le s$, to L. In general, the genus of $\mathcal{A}$ will be $n + m$, $m \ge 0$. A typical case is illustrated in Figs. 3a,b, with $n = 3$, $m = 2$.

If $m = 0$ we proceed as follows. We may without loss of generality assume that each V_i is a subset of $\mathcal{A}$, and also that each y_i and each z_i is a simple closed curve on $\mathrm{Bd}\mathcal{A}$. Let $\mathbf{g} = \{y_1, \ldots, y_n\}$ and let $\mathbf{z} = \{z_1, \ldots, z_n\}$. Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a standard system of curves in $\mathrm{Bd}\mathcal{A}$ which are the boundaries of a complete system of meridinal discs in $\mathcal{B} = \Sigma - \mathcal{A}$. Clearly $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ is a Heegaard diagram for M, and $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{z})$ is a Heegaard diagram for Σ. The diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ is a Heegaard diagram for Σ. The diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y})$ is special because $\||y_i \cap z_j|\|$ is the identity matrix.

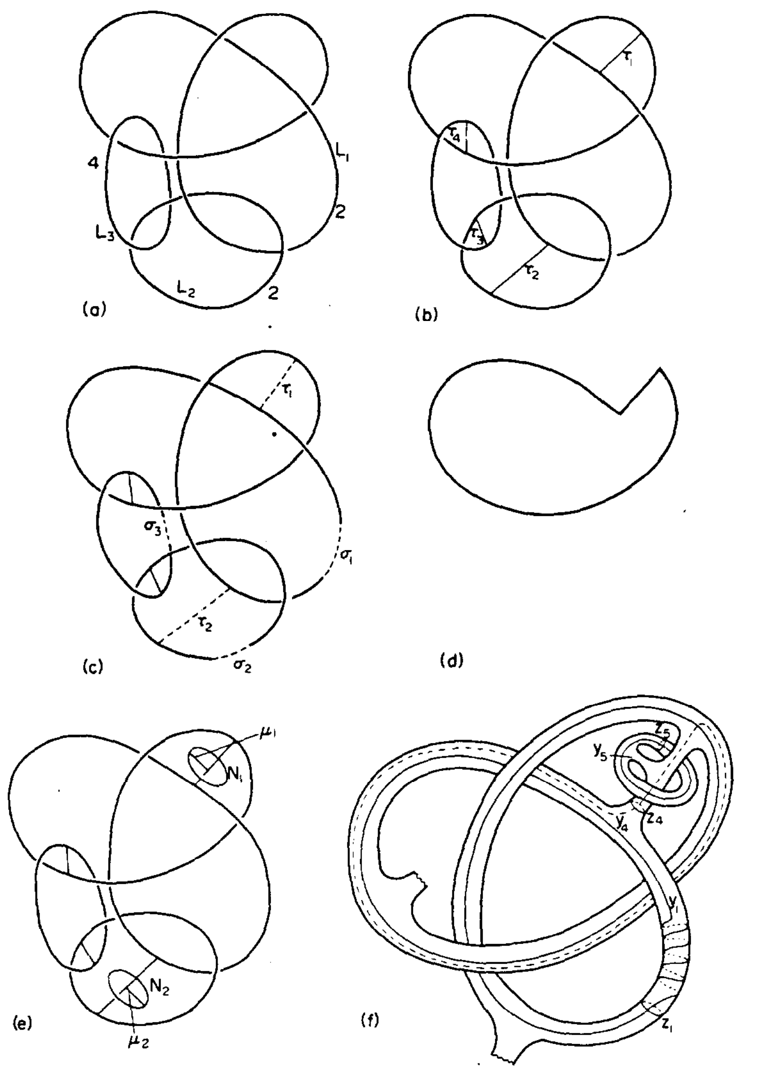


Fig. 3. (a) The framed link (L, l). (b) The graph G. (c) The maximal tree Y. (d) The unique circuit $R_1 \subset Y \cup \tau_1$. (e) The graph G^*. (f) The portion of the handlebody $\mathscr{A}^*$ which belongs to the circuits L_1, R_1 and N_1 of the graph G^*, with circuits $y_1, z_1, y_4, z_4, y_5, z_5$.

In the general case $m > 0$, it will be necessary to augment the link L (which is now regarded as a subset of the graph G) by the addition of $2m$ new circuits $R_1, N_1, R_2, N_2, \ldots, R_m, N_m$. We now describe the choice of these circuits. We begin by deleting an open 1-simplex σ_i from each component of L, choosing the σ_i's so that $\bigcup_{i=1}^{n} (L_i - \sigma_i)$ contains all of the points $\tau_k \cap L$. Delete all the τ_k. Now replace as many of the τ_k as possible, so as to obtain a maximal tree $Y \subset G$. See Fig. 3c. We may assume that the 1-simplexes τ_k were indexed in such a way that $Y = G - \bigcup_{i=1}^{n} \sigma_i - \bigcup_{k=1}^{m} \tau_k$. Now, let R_k be the unique simple closed curve in $Y \cup \tau_k$, $k = 1, \ldots, m$. See Fig. 3d.

Next, we augment the graph G by selecting m new curves $N_1, \ldots, N_m$ in Σ, one curve N_k for each curve R_k. (The curves N_k are called "neutralizing curves" in a similar construction which is used in [4]) see also [8] and [9]. See Fig. 3e. The curves N_k are to be chosen to be pairwise disjoint and unknotted, also each N_k is assumed to bound a disc D_k in Σ which is pierced once by $\tau_k \subset R_k$ but avoids all other curves $R_j (j \neq k)$ and L_i. Choose, in each D_k, a small arc μ_k which binds N_k to R_k. Let G^* be the augmented graph $G^* = G \bigcup_{k=1}^{m} (N_k \cup \mu_k)$.

Note that G^* has a regular neighborhood $\mathcal{A}^*$ (see Fig. 3f) such that $\mathcal{B}^* = \Sigma - \dot{\mathcal{A}}^*$ is a handlebody which is homeomorphic to $\mathcal{A}^*$. This is easy to see, because $\mathcal{A}^*$ may be obtained from $\mathcal{A}$ by adding m unknotted handles, one for each curve N_k, $1 \le k \le m$. The genus of $\mathcal{A}^*$ is $n^* = n + 2m$.

Our next task will be to define an augmented Heegaard diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ for a manifold M^* which, as we will see later, is homeomorphic to M. Choose a system of curves $\mathbf{x} = \{x_1, \ldots, x_{n^*}\} \subset \mathrm{Bd}\mathcal{B}^* = \mathrm{Bd}(\Sigma - \dot{\mathcal{A}}^*)$ such that $\mathbf{x}$ bounds a complete system of meridian discs in $\mathcal{B}^* = \Sigma - \dot{\mathcal{A}}^*$. (Remark: at this stage in the construction we have no control over the system $\mathbf{x}$, however later in the proof of "evenness" it will be necessary to reexamine things and select these curves with more care.) Choose curves $\mathbf{z} = \{z_1, \ldots, z_{n^*}\} \subset \mathrm{Bd}\mathcal{A}^*$ such that $\mathbf{z}$ bounds a complete system of meridian discs in $\mathcal{A}^*$. The curves $\mathbf{z}$ are to be chosen in a very particular way (cf. Fig. 3f):

(i) If $i = 1, \ldots, n$ the curve z_i is to bound a meridian disc Z_i which is pierced once by G^* at a point $q_i^* \in \dot{\sigma}_i$.

(ii) If $i = n + 2k - 1$, $k = 1, \ldots, m$, the curve z_i is to bound a meridian disc Z_i which is pierced once by G^* at a point in $\dot{\tau}_k$.

(iii) If $i = n + 2k$, $k = 1, \ldots, m$, the curve z_i is to bound a meridian disc Z_i which is pierced once by G^* at a point in N_k.

By construction $(\mathrm{Bd}(\Sigma - \dot{\mathcal{A}}^*), \mathbf{x}, \mathbf{z})$ is a Heegaard diagram for Σ.

To complete the construction, we select a curve system $\mathbf{y} = \{y_1, \ldots, y_{n^*}\} \subset \mathrm{Bd}\mathcal{A}^*$. The first n of the curves $y_1, \ldots, y_{n^*}$ will be chosen exactly as in the case $m = 0$, i.e. y_i is a curve which lies in a tubular neighborhood of L_i, also $y_i \subset \mathrm{Bd}\mathcal{A}^*$, also $lk(y_i, L_i) = l_i$. Similarly, the curves y_{n+2k}, $k = 1, \ldots, m$, will be chosen to lie in a tubular neighborhood of N_k, also $y_{n+2k} \subset \mathrm{Bd}\mathcal{A}^*$, also $lk(y_{n+2k}, N_k) = 0$. We now wish to select the remaining curves y_{n+2k-1}, $k = 1, \ldots, m$ so that each y_{n+2k-1} lies in a tubular neighborhood of R_k, also $y_{n+2k-1} \subset \mathrm{Bd}\mathcal{A}^*$, also $lk(y_{n+2k-1}, R_k) = 0$, and *also* so that each such curve y_{n+2k-1} is disjoint from each other curve y_j, $j \neq n + 2k - 1$, $j = 1, \ldots, n^* = n + 2m$. To see that these conditions are possible, let $C_i \subset \mathcal{A}^*$ be a small cylinder with one of its bases the disc Z_i, with its axis a subset of σ_i (if $i = 1, \ldots, n$) or of τ_k (if $i = n + 2k - 1$, $k = 1, \ldots, m$) or of N_k if $i = n + 2k$, $k + 1, \ldots, m$), and with its second base a meridian disc Z_i' which is parallel to Z_i and close to Z_i, also $(\mathrm{Bd}C_i - Z_i' \cap Z_i) \subset \mathrm{Bd}\mathcal{A}^*$. The closure of $\mathcal{A}^* - \bigcup_{i=1}^{n^*} C_i$ is a 3-ball with boundary a 2-sphere S^2 which contains the $2n^*$ distinguished discs $\{Z_i, Z_i';\ i = 1, \ldots, n^*\}$. The curves y_i which were already selected contain sub-arcs $\hat{y}_i \subset S^2$ which join the boundaries of these discs in pairs, with $\hat{y}_i$ joining $\mathrm{Bd}Z_i$ to $\mathrm{Bd}Z_i'$, $i = 1, \ldots, n, n + 2, n + 4, \ldots, n + 2m = n^*$. The union of these arcs does not separate $S^2 - \bigcup_{i=1}^{n^*} Z_i \cup Z_i'$, hence we may join the boundaries of the remaining disc pairs by similar arcs $\hat{y}_{n+2k-1}$ which are pairwise disjoint from one another and from the $\hat{y}_i$ which were already there. We now complete these arcs $\hat{y}_{n+2k-1}$ to simple closed curves in $\mathrm{Bd}\mathcal{A}^*$ by joining the points $\hat{y}_{n+2k-1} \cap Z_{n+2k-1}$ and $\hat{y}_{n+2k-1} \cap Z'_{n+2k-1}$ by an arc which lies in $\mathrm{Bd}C_{n+2k-1} \cap \mathrm{Bd}\mathcal{A}^*$, winding each y_{n+2k-1} around the axis of C_{n+2k-1} as many times as required so that the linking number of y_{n+2k-1} with R_k is 0 for each $k = 1, \ldots, m$. Since the cylinder C_i is disjoint from y_j if $1 \le i \neq j \le n_*$, the system of curves $\mathbf{y} = \{y_1, \ldots, y_{n^*}\}$ which have been selected will be pairwise disjoint.

Now, $(\mathrm{Bd}\mathcal{B}^*, \mathbf{x}^*, \mathbf{y}^*)$ is a Heegaard diagram for a 3-manifold M^*. It is a special diagram because we have an augmented diagram $(\mathrm{Bd}\mathcal{B}^*, \mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ such that the $n^* \times n^*$ geometric intersection matrix $\|\,|y_i \cap z_j|\,\|$ is the identity matrix. This may be seen by noting that each meridian z_j is located on the boundary of the cylinder C_j, and exactly one of the curves of $\mathbf{y}^*$, namely y_j, meets that cylinder, also y_j crosses the base curve z_j once.

To prove that $M^* \cong M$ it will be necessary to construct a new surgery presentation for M^*. Note that, by our choices, if $i = 1, \ldots, n$, the curves y_i and L_i cobound an annulus A_i in $\mathcal{A}^*$ which meets $\mathrm{Bd}\mathcal{A}^* = \mathrm{Bd}\mathcal{B}^*$ in y_i. Also, the curves y_{n+2k-1} and R_k $(k = 1, \ldots, m)$ cobound an annulus A_{n+2k-1} in $\mathcal{A}^*$ which meets $\mathrm{Bd}\mathcal{A}^*$ in y_{n+2k-1}. After a suitable isotopy, if necessary, we may assume that these annuli are chosen so that $A_i \cap A_{n+2k-1} = L_i \cap R_k$, $1 \le i \le n$, $1 \le k \le m$. We now define new curves R_k^* so that each R_k^* is arbitrarily close to but disjoint from R_k, also $R_k^* \subset A_{n+2k-1}$. Then M^* is defined by the surgery presentation $(L^*, l^*) = \bigcup_{i=1}^{n} (L_i, l_i) \bigcup_{k=1}^{m} (R_k^*, 0) \cup (N_k, 0)$. (This may be seen by going back to the earlier proof for the case $m = 0$, and repeating it, with (L^*, l^*) replacing (L, l).)

It remains to prove that M^* is homeomorphic to M. This will be true if we can show that the framed link (L^*, l^*) is equivalent to the framed link (L, l) under a finite sequence of the moves which are defined in R. Kirby's "Calculus for framed links"[9]. We refer the reader to [9] for the detailed description of those moves.

By [9], we will not alter the homeomorphism type of M^* if we replace any one component with its "band connected sum" with another component, and suitably adjust the framings. Choose any component R_k^*, $k = 1, \ldots, m$. Figure 4 shows that any

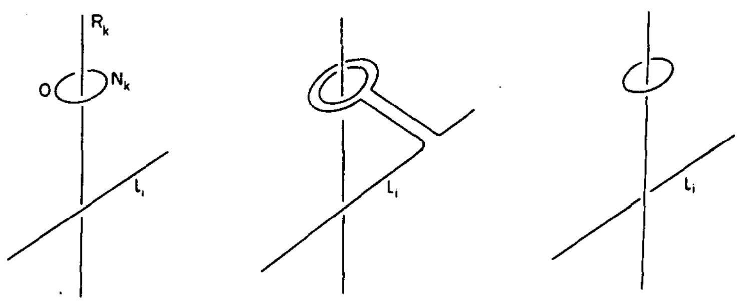


Fig. 4.

crossing of R_k^* with a component of L^* which is different from N_k, or any crossing of R_k^* with itself, can be altered by a band move. Note that N_k has framing 0 and also N_k does not link any component of L^* except R_n^*, hence by [9] the framings on the components will not be altered by these moves. Since R_k^* can be unknotted and unlinked from the rest of the curves (except N_k) by altering a finite number of crossings, it follows that an equivalent surgery presentation is obtained by removing R_k^* and N_k from the link and replacing them by a pair of unknotted, simply linked circles, each with framing 0, which are separated from the rest of the link by a 2-sphere. Since by [9] surgery on such a 2-component link defines Σ, any such pair may be deleted without altering the homeomorphism type of M^*. We do this for each curve pair (R_k, N_k). After m such operations we have altered (L^*, l^*) to (L, l), hence M^* is homeomorphic to M. Thus we have shown that every 3-manifold admits a special Heegaard diagram. Note that, in the surgery (L^*, l^*), all the framings are even integers, because by hypothesis each l_i^* with $i = 1, \ldots, n$ is even and by construction each other l_i^* is 0. However, the diagram associated to (L^*, l^*) may not be even.

To complete the proof of Theorem 1′ we must establish that M admits a special *even diagram*. It will be convenient to relabel the components of L^* as $\bigcup_{i=1}^{n+2m} L_i$, where $L_{n+2k-1} = R_k$ and $L_{n+2k} = N_k$, $k = 1, \ldots, m$. Let $W = \|w_{ij}\|$ be the $(n+2m)\times(n+2m)$ matrix with entries:

$$w_{ij} = lk(L_i, L_j) \quad \text{if} \quad i \neq j \quad \text{or} \quad l_i^* \quad \text{if} \quad i = j.$$

By [9], the matrix W is a relation matrix for $H_1(M; Z)$. It is symmetric because $lk(L_i, L_j) = lk(L_j, L_i)$ and its diagonal entries are even because the framings are all even. We also have a second relation matrix for $H_1(M; Z)$, namely the $(n+2m)\times(n+2m)$ matrix of intersection numbers $\tilde{W} = \||y_i \cap x_j|_0\|$ which is determined by the Heegaard diagram $(\mathrm{Bd}(\Sigma - \mathscr{A}^*), \mathbf{x}^*, \mathbf{y}^*)$. From this it follows that there exist unimodular matrices R, S such that $\tilde{W} = RWS$. Let $T = S^{-1}R'$. Then $\tilde{W}T = RWR'$. Since the properties of having even diagonal entries and of being symmetric are preserved under congruence, we may conclude that $\tilde{W}T$ is symmetric and has even diagonal entries for some unimodular matrix $T = \|t_{ij}\|$.

Now, the curves $\mathbf{x}^*$ bound a complete system of meridian discs for the handlebody $\mathcal{B}^* = \Sigma - \mathcal{A}^*$, hence they may be regarded as defining a basis $[\mathbf{x}^*]$ for the free abelian group $H_1(\mathcal{B}^*; Z)$. Let $[\mathbf{x}'^*] = [\mathbf{x}^*]T$. Then $[\mathbf{x}'^*]$ is another such basis, and the transformation $[\mathbf{x}^*] \to [\mathbf{x}'^*]$ is an automorphism of $H_1[\mathcal{B}^*; Z]$. By Theorem 3.1 of [6] and Theorem N4 of [11] each such automorphism is geometrically induced. Hence, there exists a system $\mathbf{x}'^*$ of curves in $\mathrm{Bd}(\mathcal{B}^*)$ which bound a complete system of meridian discs in $\mathcal{B}^*$ such that the intersection matrix $\||y_i \cap x_j'^*|_0\|$ is symmetric and has even diagonal entries. The Heegaard diagram $(\mathrm{Bd}\mathcal{B}^*, \mathbf{x}'^*, \mathbf{y}^*)$ will then be a special even diagram with augmented diagram $(\mathrm{Bd}\mathcal{B}, \mathbf{x}'^*, \mathbf{y}^*, \mathbf{z}^*)$. This completes the proof of Theorem 1′ and hence also of Theorem 1.

Proof of Corollary 1. It was shown at the beginning of this paper that each special sewing $h \in A(n)$ defines a 4-manifold $N(\tilde{h})$. Let $\varphi_N: H_2(N(\tilde{h})) \times H_2(N(\tilde{h}) \to Z$ be the bilinear form of homology intersection numbers. In Lemma 5 of [2], it is proved that the symmetric matrix $W(h)$ represents φ_N, and is also a relation matrix for $H_1(M(h); Z)$. If the sewing is even, it then follows that the manifold $N(\tilde{h})$ is parallelizable. Thus, each 3-manifold $M(h)$ which is defined by a special even sewing h has associated with it a parallelizable 4-manifold $N(\tilde{h})$, with $\mathrm{Bd}N(\tilde{h}) \cong M(h)$. If $\det W(h) \equiv 1 \pmod 2$, so that $M(h)$ is a $Z/2Z$-homology sphere, then the mod 16 reduction of the signature of $W(h)$ will be the Rohlin invariant (see [7, Ch. 7]) $\mu(M)$ of $M(h)$. This proves Corollary 1.

§3. THE CONSTRUCTION OF INDEX 8 Z-HOMOLOGY SPHERES

In this section, we will use Theorem 1 to give a simple constructive procedure for enumerating index 8 Z-homology spheres. An example of the construction will be given.

Let the 3-sphere $\Sigma = \mathcal{A} \cup \mathcal{B}$ be regarded as $E^3 \cup \{\infty\}$ where $\mathcal{B}$ is regarded as $D^2 \times I$, with D^2 a disk with n holes which is a subset of E^2 (see Fig. 5). Choose standard

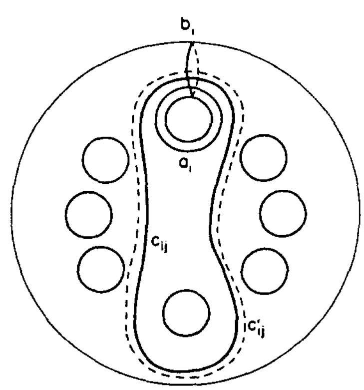


Fig. 5.

systems $\mathbf{a}$ and $\mathbf{b}$ for $\mathcal{A}$ and $\mathcal{B}$. If $h \in H(n)$ then h induces an automorphism h_* of $H_1(\mathrm{Bd}\mathcal{B}; Z)$ which may be identified with the symplectic matrix $h_* = \|\Delta_{ij}\|$, where

$$h_*(a_i) \sim \sum_{j=1}^{n} \Delta_{ij} a_j + \Delta_{i,j+n} b_j$$

$$h_*(b_i) \sim \sum_{j=1}^{n} \Delta_{i+n,j} a_j + \Delta_{i+n,j+n} b_j.$$

THEOREM 2. *Let $W = W(h)$ be any $n \times n$ symmetric unimodular matrix over Z which represents a symmetric bilinear form of even type and signature* 8(mod 16). *Then* *there is an element $h \in A(n)$ such that $h_* = \begin{bmatrix} I & 0 \\ W(h) & I \end{bmatrix}$. For any such element the manifold $M(h)$ is an index 8 Z-homology sphere.*

Proof. We first show that it is possible to choose an element h which satisfies the hypotheses of Theorem 2. Let c_{ij}, $1 \le i \ne j \le n$ be a simple closed curve on Bd$\mathcal{B}$ which encircles the ith and the jth handles in the manner indicated in Fig. 5. Let $c_{ii} = a_i$, $1 \le i \le n$. Let c'_{ij} be a curve which is disjoint from and homologous to c_{ij}. Let t_{ij} (respectively t'_{ij}) denote a Dehn twist about c_{ij} (respectively c'_{ij}). Since c_{ij} and c'_{ij} bound discs in $\mathcal{A}$, it is immediate that t_{ij} and t'_{ij} are in $A(n)$. With our choice of conventions, we have

$$(t_{ij})_* = (t'_{ij})_* = \begin{bmatrix} I & 0 \\ T_{ij} & I \end{bmatrix}$$

where T_{ij} is a matrix which has zeros everywhere except at the intersections of the ith and jth rows and columns, where the entry is 1 or -1, depending on the choice of conventions for the positive sense of the twist t_{ij}, $1 \le i,j \le n$. Then the matrix $\begin{bmatrix} I & 0 \\ W & I \end{bmatrix}$ may be expressed as a product of powers of the matrices $(t_{ij})_*$ (or $(t'_{ij})_*$), hence one may lift h_* to the corresponding product of powers of the t_{ij} (or t'_{ij}). Clearly this construction may be modified by composing h with any map in $A(n)$ which induces the identity automorphism of $H_1(\mathrm{Bd}\mathcal{B}; Z)$, thus there are many possibilities for h.

By our earlier observations, the matrix $W(h)$ is an homology relation matrix for $M(h)$, which is then seen to be a Z-homology sphere, because $\det W = \pm 1$. Since $h \in A(n)$ it may be extended to a map $\tilde{h} \in \tilde{A}(n)$, as defined in the introduction, and so we may construct a 4-manifold $N(\tilde{h})$ with $\mathrm{Bd}N(\tilde{h}) \cong M(h)$, and $W(h)$ represents φ_N. Since $W(h)$ has even type and signature 8 (mod 16), it then follows that $M(h)$ has index 8.

Example. We will use the methods of Theorem 2 to produce an infinite sequence of index 8 Z-homology spheres. The information which is given in §6, Chap. II of [10] allows us to produce for each even integer m, a $4m \times 4m$ symmetric unimodular integer matrix V_{4m} having even diagonal entries, with the property that the signature of V_{4m} is $4m$. The lower right 3×3 corner of V_{4m} is the matrix

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -4 & -1 \\ 0 & -1 & -m \end{bmatrix}$$

The remaining entries are -2's along the main diagonal, $+1$'s along the two diagonals adjacent to the main diagonal, and zeros elsewhere. We will be interested in choosing m so that the signature of V_{4m} is 8(mod 16), hence $m \equiv 2 (\mathrm{mod}\ 4)$.

Let m be congruent to 2(mod 4). Define†

$$h_{4m} = t_{4m,4m}^{-(m-1)} t_{4m-1,4m-1}^{-1} t_{4m-2,4m-2}^{-1} t_{4m-3,4m-3}^{-4} \cdots t_{3,3}^{-4}$$
$$t_{2,2}^{-4} t_{1,1}^{-3} t'^{-2}_{4m-2,4m-1} t'_{4m-4,4m-3} \cdots t'_{4,5} t'_{2,3}\ t_{4m-1,4m}^{-1} t_{4m-3,4m-2} \cdots t_{3,4} t_{1,2}.$$

Then $h_{4m} \in A(4m)$ and

$$h_* = \begin{bmatrix} I & 0 \\ V_{4m} & I \end{bmatrix}.$$

By Theorem 2 it then follows that $M(h_{4m})$ is an index 8 Z-homology sphere which is defined by a Heegaard sewing of genus $4m$, for each even integer $m \equiv 2(\mathrm{mod}\ 4)$.

Using the Van-Kampen theorem, one may now show by calculation that $\pi_1 M(h_{4m})$ is a group with two generators x, y and defining relations $x^{4m-3}(xy)^2 = y^{2m-1}(xy)^{-2m} = 1$. On adding the relation $(xy)^2 = 1$ this presentation goes over to a presentation with

†We have used both t_{ij} and t'_{ij} to simplify the computation of $\pi_1(M(h))$.

generators x, y and defining relations $x^{4m-3} = y^{2m-1} = (xy)^2 = 1$. By [3], the group $\pi_1(M(h_{4m}))$ is finite if and only if $(1/4m-3)+(1/2m-1)+(1/2) > 1$, i.e. $m = 2$. In the exceptional case $m = 2$ the substitution $a = xyx$, $b = x^{-1}$ exhibits $\pi_1(M(h_8))$ as the group generated by a, b with defining relations $a^3 = b^5 = (ab)^2$. The manifold $M(h_8)$ is the Poincaré homology sphere P. The Heegaard diagram which is defined by the special sewing h_8 is the diagram which was given earlier, in Fig. 2.

Remark. New results of the author and Jerome Powell show that each M^3 admits a special even Heegaard diagram which satisfies several additional restrictions beyond those considered here. These include

(i) The associated diagram (Bd$\mathcal{B}$,x,z) for Σ may be assumed to be standard, i.e., $\|\|x_i \cap z_j\|\| = Id$.

(ii) The algebraic intersection matrix $\|\|y_i \cap x_j|_0\|$ coincides with the geometric intersection matrix $\|\|y_i \cap x_j\|\|$, hence, the latter is also symmetric.

From (i) it follows that the construction in Theorem 2 gives *all* index 8 Z-homology sphere. These and other results be reported upon in a forthcoming manuscript.

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