Lifting and Projecting Homeomorphisms

By

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1. Introduction. Let X be a pathwise connected and locally pathwise connected topological space, G the group of all self-homeomorphisms of X, and D the subgroup of maps isotopic to the identity. The homeotopy group of X is defined to be the group G/D. Let $\tilde{X}$ be a p. c., l. p. c. covering space of X, with projection p. The relationship between the homeotopy groups of $\tilde{X}$ and X is studied. It is shown (Theorem 3) that under sufficiently strong restrictions on $\tilde{X}$, X and p the homeotopy group of X is isomorphic to a factor group of the homeotopy group of $\tilde{X}$, with weaker results as one weakens the restrictions on $\tilde{X}$ and X.

The situation studied here first came to the authors' attention in an earlier investigation [1]. The homeotopy groups of 2-manifolds play an important role in the theory of Riemann surfaces, and also in the classification of 3-manifolds. It was shown in [1] that one could gain considerable insight into the structure of the homeotopy groups of surfaces by utilizing the fact that any closed compact orientable surface of genus g with $(2g+2)$ points removed can be regarded as a 2-sheeted covering of a $(2g+2)$-punctured sphere, and making use of the known properties of the homeotopy group of the punctured sphere. The development of this relationship suggested that other coverings of more general spaces might also be of interest, thus motivating the present investigation. At the conclusion of this paper (Section 4) a new application to surface topology is discussed briefly. A detailed workingout of this application will be found in [2], which should appear concurrently with the present work.

2. We begin by reviewing some well-known results about covering spaces. Let $x_0 \in X$, and suppose $\tilde{x}_0 \in p^{-1}(x_0)$. Then the covering space projection $p\colon \tilde{X} \to X$ induces a monomorphism p_* from $\pi_1(\tilde{X}, \tilde{x}_0)$ to $\pi_1(X, x_0)$. [See, for example, page 72 of [9].] For simplicity in notation, we will write $\tilde{\pi}$ for $\pi_1(\tilde{X}, \tilde{x}_0)$, π for $\pi_1(X, x_0)$, and π_c for the subgroup $p_*\tilde{\pi}$ of π.

Let $\varphi \in G$, and let $\varphi_*^{(\alpha)}$ be the automorphism of π which is induced by φ, corresponding to the choice of the path α joining x_0 to $\varphi(x_0)$. The homeomorphism φ is said to lift to $\tilde{\varphi} \in \tilde{G}$ if $\varphi p = p\tilde{\varphi}$ for every point of $\tilde{X}$. It is well-known that φ lifts iff $\varphi_*^{(\alpha)}$ leaves the subgroup π_c of π invariant, for some path α. [See, for example, page 76 of [9]; in our case the map to be lifted is the product $\varphi p\colon \tilde{X} \to X$.]

A natural homomorphism H_1 exists from the group G/D to the group $\operatorname{Aut}\pi/\operatorname{Inn}\pi$, which is defined explicitely as follows: Let $[\varphi] \in G/D$, and let φ represent $[\varphi]$. Let

$[\varphi_*^{(\alpha)}]$ be the class of $\varphi_*^{(\alpha)}$ in Aut π/Inn π, for any choice of α. Then $H_1[\varphi] = [\varphi_*^{(\alpha)}]$. It is easily verified that H_1 is well-defined and a homomorphism, and is independent of the choice of the path α. In general, H_1 is neither 1-1 nor onto, but in a wide range of cases of interest it is either or both. We will write $\tilde{H}_1$ for the corresponding homomorphism acting on $\tilde{G}/\tilde{D}$.

3. Our object is to develop a relationship between the groups $\tilde{G}/\tilde{D}$ and G/D. In the most general case we can, at best, hope for a relationship between appropriate subgroups, and with this in mind we define $G_{\tilde{x}}$ to be the subgroup of G consisting of all those elements $\varphi \in G$ which lift to $\tilde{G}$. Let $\tilde{G}_x \subseteqq \tilde{G}$ be the group of all those elements $\tilde{\varphi} \in \tilde{G}$ which preserve fibres with respect to p, i.e. which have the property that for every $\tilde{x}, \tilde{x}' \in \tilde{X}$ such that $p(\tilde{x}) = p(\tilde{x}')$: $p\tilde{\varphi}(\tilde{x}) = p\tilde{\varphi}(\tilde{x}')$. If $\tilde{\varphi} \in \tilde{G}_x$, then $\tilde{\varphi}$ projects to $\varphi \in G_{\tilde{x}}$, where φ is defined unambiguously by $\varphi = p\tilde{\varphi}p^{-1}$.

Since the groups $\tilde{G}_x$ and $G_{\tilde{x}}$ do not, in general, include the subgroups $\tilde{D}$ and D, we widen them to:

$$\tilde{S} = \{\tilde{\varphi} \in \tilde{G} \;/\; \tilde{\varphi} \text{ is isotopic to some } \tilde{\psi} \in \tilde{G}_x\},$$
$$S = \{\varphi \in G \;/\; \varphi \text{ is isotopic to some } \psi \in G_{\tilde{x}}\}.$$

We will also need:

$$\tilde{T} = \{\tilde{\varphi} \in \tilde{G} \;/\; \tilde{\varphi} \text{ is isotopic to a covering transformation}\}.$$

We can now state our main result:

Theorem 3. 3.1. *If the covering $(\tilde{X}, X, p)$ is regular, and if H_1 and $\tilde{H}_1$ are 1-1, and if the centralizer of π_c in π is trivial then*

$$(\tilde{S}/\tilde{D})/(\tilde{T}/\tilde{D}) \cong S/D.$$

3.2. *If, in addition, π_c is invariant under all those automorphisms of π which are induced by topological mappings, then*

$$(\tilde{S}/\tilde{D})/(\tilde{T}/\tilde{D}) \cong G/D.$$

3.3. *If, moreover, $\tilde{G}/\tilde{D}$ is generated by elements which can be represented by maps in $\tilde{G}_x$, then*

$$(\tilde{G}/\tilde{D})/(\tilde{T}/\tilde{D}) \cong G/D.$$

The proof of Theorem 3 is via a sequence of Lemmas. We note that Theorem 2 is a weak version of Theorem 3, under correspondingly weak restrictions on $\tilde{X}$ and X. It may, therefore, have some interest in itself.

Let i_*, $\tilde{i}_*$ denote inner automorphisms of π_c and $\tilde{\pi}$ respectively satisfying the relationship

$$p_* \tilde{i}_* = i_* p_*. \tag{1}$$

If φ is the projection of $\tilde{\varphi}$, and if α is the projection of $\tilde{\alpha}$, then since $p\tilde{\varphi} = \varphi p$, it follows that:

$$p_* \tilde{\varphi}_*^{(\tilde{\alpha})} = \varphi_*^{(\alpha)} p_* . \tag{2}$$

(Remark: $\tilde{\alpha}$ must be picked first, since if α is a curve from x_0 to $\varphi(x_0)$, the lift of α beginning at $\tilde{x}_0$ need not necessarily end at $\tilde{\varphi}(\tilde{x}_0)$.)

We now define a new homomorphism $\tilde{H}_2 \colon \tilde{G}/\tilde{D} \to \operatorname{Aut} \pi_c / \operatorname{Inn} \pi_c$ by

$$\tilde{H}_2[\tilde{\varphi}] = [p_* \tilde{\varphi}_* p^{-1}] \tag{3}$$

where $[p_* \tilde{\varphi}_* p_*^{-1}]$ denotes the class of $p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1}$ in $\operatorname{Aut} \pi_c / \operatorname{Inn} \pi_c$. The isomorphism p_*^{-1} operates on π_c, and maps it onto $\tilde{\pi}$ which is then operated on by the automorphism $\tilde{\varphi}_*^{(\tilde{\alpha})}$ and projected back to π_c. Thus $p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1}$ is an automorphism of π_c. The proof that $\tilde{H}_2[\tilde{\varphi}]$ is independent of the choice of $\tilde{\alpha}$ and depends only on the isotopy class of $\tilde{\varphi}$ is straightforward, and we omit it.

It follows from the definitions that for any $\tilde{\varphi} \in \tilde{G}$:

$$\tilde{H}_2[\tilde{\varphi}] P = P \tilde{H}_1[\tilde{\varphi}] \tag{4}$$

(where P is the isomorphism $\operatorname{Aut} \tilde{\pi} / \operatorname{Inn} \tilde{\pi} \to \operatorname{Aut} \pi_c / \operatorname{Inn} \pi_c$ induced by the isomorphism p_*).

We would like to know conditions under which a homeomorphism $\tilde{\varphi}$ of $\tilde{X}$ is in $\tilde{S}$, i.e. is isotopic to an element in $\tilde{G}_x$. The analogous problem in the base space X is to ask when a map $\varphi \in G$ is in S, and the classical solution is that $\varphi \in S$ iff there exists some path α from x_0 to $\varphi(x_0)$ such that $\varphi_*^{(\alpha)}$ leaves π_c invariant. The situation in the covering space $\tilde{X}$ is considerably more complicated, and our characterization is less satisfying:

Theorem 1. *Let $\tilde{\varphi} \in \tilde{G}$, $\tilde{\psi} \in \tilde{G}_x$. Let $\tilde{\alpha}$, $\tilde{\beta}$ be any paths from $\tilde{x}_0$ to $\tilde{\varphi}(\tilde{x}_0)$, $\tilde{\psi}(\tilde{x}_0)$ respectively. Let ψ, β be the projections of $\tilde{\psi}$, $\tilde{\beta}$ respectively. Then there is an element $\tilde{\varkappa} \in \tilde{G}$, with $[\tilde{\varkappa}] \in \ker \tilde{H}_1$, such that $\tilde{\varkappa}\tilde{\varphi}$ is isotopic to $\tilde{\psi}$, iff there exists an inner automorphism i_* of π_c such that:*

$$p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} = i_* (\psi_*^{(\beta)} | \pi_c) .$$

Proof. To establish necessity, suppose $\tilde{\varkappa}\tilde{\varphi}$ is isotopic to an element $\tilde{\psi} \in \tilde{G}_x$. Let $\tilde{\gamma}$ be a curve from $\tilde{x}_0$ to $\tilde{\varkappa}\tilde{\varphi}(\tilde{x}_0)$. Since $[\tilde{\varkappa}\tilde{\varphi}] = [\tilde{\psi}]$, therefore the images of $\tilde{\varkappa}\tilde{\varphi}$ and of $\tilde{\psi}$ under $\tilde{H}_2$ coincide, hence there exists some inner automorphism m_* of π_c such that

$$p_* (\tilde{\varkappa}\tilde{\varphi})_*^{(\tilde{\gamma})} p_*^{-1} = m_* p_* \tilde{\psi}_*^{(\tilde{\beta})} p_*^{-1} = m_* (\psi_*^{(\beta)} | \pi_c)$$

by (2). Since $[\tilde{\varkappa}] \in \ker \tilde{H}_1$, there exists $\tilde{j}_* \in \operatorname{Inn} \tilde{\pi}$ such that

$$(\tilde{\varkappa}\tilde{\varphi})_*^{(\tilde{\gamma})} = \tilde{j}_* \tilde{\varphi}_*^{(\tilde{\alpha})} .$$

Therefore:

$$\begin{aligned} j_* p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} &= m_* (\psi_*^{(\beta)} | \pi_c) \quad \text{where} \quad j_* = p_* \tilde{j}_* p_*^{-1} , \\ p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} &= j_*^{-1} m_* (\psi_*^{(\beta)} | \pi_c) = i_* (\psi_*^{(\beta)} | \pi_c) . \end{aligned}$$

Conversely, suppose that the condition in the theorem is satisfied for curves $\tilde{\alpha}$, $\tilde{\beta}$, β

and an inner automorphism i_*. Then:

$$p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} = i_* (\psi_*^{(\beta)} | \pi_c) = i_* p_* \tilde{\psi}_*^{(\tilde{\beta})} p_*^{-1}$$

using (2) so that

$$\tilde{\varphi}_*^{(\tilde{\alpha})} = \tilde{i}_* \tilde{\psi}_*^{(\tilde{\beta})}$$

where $\tilde{i}_*$ is defined by (1). Then

$$\tilde{i}_*^{-1} = \tilde{\psi}_*^{(\tilde{\beta})} [\tilde{\varphi}_*^{(\tilde{\alpha})}]^{-1} = \tilde{\psi}_*^{(\tilde{\beta})} \tilde{\varphi}_*^{-1\,((\tilde{\varphi}^{-1}(\tilde{\alpha}))^{-1})} = (\tilde{\psi} \cdot \tilde{\varphi}^{-1})_*^{((\tilde{\psi}(\tilde{\varphi}^{-1}(\tilde{\alpha}))^{-1}) \cdot \tilde{\beta})}$$

from the definitions of $\tilde{\varphi}_*$ and $\tilde{\psi}_*$. Thus $[\tilde{\psi} \cdot \tilde{\varphi}^{-1}] \in \ker \tilde{H}_1$ and we are through.

We now turn our attention to the relationship between the groups $\tilde{S}/\tilde{D}$ and S/D. In order to proceed further we assume from now on that $(\tilde{X}, X, p)$ is a regular covering (i.e. that π_c is normal in π). Let A_c be the subgroup of $\operatorname{Aut} \pi$ consisting of all automorphisms which leave the subgroup $\pi_c \subseteqq \pi$ invariant. Since the covering is regular, A_c will include the subgroup of inner automorphisms of π. We note that

$$(5) \qquad (\operatorname{Inn} \pi) | \pi_c \lhd A_c | \pi_c .$$

Inclusion follows from the fact that $\operatorname{Inn} \pi \subseteqq A_c$. To see that the inclusion is normal, choose any $\bar{a} \in (\operatorname{Inn} \pi) | \pi_c$ and any $\bar{b} \in A_c | \pi_c$, where $\bar{a}$ is the restriction of $a \in \operatorname{Inn} \pi$ and $\bar{b}$ is the restriction of $b \in A_c$. Since $\operatorname{Inn} \pi \lhd A_c \subseteqq \operatorname{Aut} \pi$, the automorphism $(bab^{-1}) \in \operatorname{Inn} \pi$. Since the covering is regular, $(bab^{-1}) | \pi_c$ is an automorphism of π_c and clearly coincides with $\bar{b} \bar{a} \bar{b}^{-1}$.

Using (5), we now define a homomorphism R (for restriction):

$$(6) \qquad R \colon A_c / \operatorname{Inn} \pi \to (A_c | \pi_c)/((\operatorname{Inn} \pi) | \pi_c) .$$

Next we note that:

$$(7) \qquad \operatorname{Inn} \pi_c \subseteqq (\operatorname{Inn} \pi) | \pi_c \subseteqq A_c | \pi_c$$

because every inner automorphism of π_c extends to an inner automorphism of π. The inclusion is normal because $\operatorname{Inn} \pi_c \lhd \operatorname{Aut} \pi_c$ and $A_c | \pi_c \subseteqq \operatorname{Aut} \pi_c$. Therefore we can define a canonical homomorphism:

$$(8) \qquad \sigma \colon (A_c | \pi_c)/\operatorname{Inn} \pi_c \to (A_c | \pi_c)/((\operatorname{Inn} \pi) | \pi_c) .$$

We assert:

Lemma 1. $R H_1 (S/D) = \sigma \tilde{H}_2 (\tilde{S}/\tilde{D})$.

Proof. Since every element of $\tilde{S}/\tilde{D}$ can be represented by the lift of an element $\varphi \in G_{\tilde{x}}$, and every element of S/D can be represented by the projection of an element $\tilde{\varphi} \in \tilde{G}_x$, we must prove that $R H_1 [\varphi] = \sigma \tilde{H}_2 [\tilde{\varphi}]$, for every $\tilde{\varphi} \in \tilde{G}_x$, where $\varphi \in G_{\tilde{x}}$ is the projection of $\tilde{\varphi}$. Using definition (3):

$$\tilde{H}_2 [\tilde{\varphi}] = p_* \tilde{\varphi}^{(\alpha)} p_*^{-1} \quad \operatorname{mod} \operatorname{Inn} \pi_c .$$

Using equation (2) we obtain:

$$\tilde{H}_2 [\tilde{\varphi}] = \varphi_*^{(\alpha)} | \pi_c \qquad \operatorname{mod} \operatorname{Inn} \pi_c .$$

Therefore from [8]:

$$\sigma \tilde{H}_2[\tilde{\varphi}] = \varphi_*^{(\alpha)} | \pi_c \qquad \bmod ((\operatorname{Inn} \pi) | \pi_c).$$

On the other hand:

$$H_1[\varphi] = \varphi_*^{(\alpha)} \qquad \bmod \operatorname{Inn} \pi$$

and since R is just restriction to π_c, the proof is complete.

To improve Lemma 1, we would like to characterize $\ker \sigma$ and $\ker R$:

Lemma 2. *Let $[\tilde{\varphi}] \in \tilde{S}/\tilde{D}$. Then $\tilde{H}_2[\tilde{\varphi}] \in \ker \sigma$ iff there exists an element*

$$\tilde{\psi} \in \tilde{G}_x \cap \ker \tilde{H}_1,$$

such that $\tilde{\varphi}\tilde{\psi}$ is isotopic to a covering transformation. (If $\tilde{H}_1$ is 1-1, this implies that $\tilde{H}_2[\tilde{\varphi}] \in \ker \sigma$ iff $\tilde{\varphi}$ is isotopic to a covering transformation.)

Proof. From the definitions, it follows that $\tilde{H}_2[\tilde{\varphi}] \in \ker \sigma$ iff

$$p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} \in (\operatorname{Inn} \pi) | \pi_c.$$

Since the covering $(\tilde{X}, X, p)$ is regular the elements of $(\operatorname{Inn} \pi) | \pi_c$ can all be induced by covering transformations. Hence $\tilde{H}_2[\tilde{\varphi}] \in \ker \sigma$ iff there exists a covering transformation $\tilde{\tau}$ and a curve $\tilde{\beta}$ such that:

$$p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} = p_* \tilde{\tau}_*^{(\tilde{\beta})} p_*^{-1} \quad (\bmod \operatorname{Inn} \pi_c).$$

But now we consult equation (4). Since p_* is 1-1, it follows from equation (4) that $\tilde{\psi} \in \ker \tilde{H}_2$ iff $\tilde{\psi} \in \ker \tilde{H}_1$, which completes the proof of Lemma 2.

Lemma 3. *Let $\varphi \in G_{\tilde{x}}$. Then $H_1[\varphi] \in \ker R$ iff φ has a lift $\tilde{\varphi}$ such that $\tilde{\varphi} \in \ker \tilde{H}_1$.*

Proof. It follows from our definitions that $H_1[\varphi] \in \ker R$ iff $\varphi_*^{(\alpha)} | \pi_c \in (\operatorname{Inn} \pi) | \pi_c$. If $\tilde{\varphi} \in \ker \tilde{H}_1$ then as in the proof of Lemma 2, $\tilde{\varphi} \in \ker \tilde{H}_2$ therefore

$$\varphi_*^{(\beta)} | \pi_c = p_* \tilde{\varphi}_*^{(\tilde{\beta})} p_*^{-1}$$

belongs to $\operatorname{Inn} \pi_c \subseteq (\operatorname{Inn} \pi) | \pi_c$.

Conversely, if $\varphi_*^{(\alpha)} | \pi_c \in (\operatorname{Inn} \pi) | \pi_c$, then there is a curve δ such that $\varphi_*^{(\alpha\delta)} | \pi_c = 1$. Let $\tilde{\varphi}$ be a lift of φ. By composing with a covering transformation if necessary we may assume $\tilde{\varphi}(\tilde{x}_0) = \widetilde{\alpha\delta}(1)$ where $\widetilde{\alpha\delta}(0) = \tilde{x}_0$. Then $p_* \tilde{\varphi}_*^{(\widetilde{\alpha\delta})} p_*^{-1} = 1$ implies $\tilde{H}_2[\tilde{\varphi}] = 1$ which implies that $\tilde{\varphi} \in \ker \tilde{H}_1$.

As a consequence of Lemma 1, 2, 3 we obtain:

Theorem 2. *Let I be the subgroup of $G_{\tilde{x}}$ defined by*

$$I = \{\varphi \in G_{\tilde{x}} \,|\, \tilde{\varphi} \in \tilde{D} \text{ for some lift } \tilde{\varphi} \text{ of } \varphi\}.$$

Let J be the mapping classes in S/D represented by elements of I. Then if $\tilde{H}_1$ is 1-1:

$$(\tilde{S}/\tilde{D})/(\tilde{T}/\tilde{D}) \cong (S/D)/J.$$

Proof. (i) From Lemma 2 we have that if $\tilde{H}_1$ is 1-1, then $T = \ker(\sigma \tilde{H}_2)$.

(ii) From Lemma 3 we have that if $\tilde{H}_1$ is 1-1, then $J = \ker RH_1$. Using (i) and (ii) in Lemma 1, we then obtain Theorem 2.

Finally, we are ready to prove Theorem 3. It suffices to show that with the additional assumption that the centralizer of π_c in π is trivial, that J is trivial. If $[\varphi] \in J$ then there is a $\tilde{\varphi}$ such that $p_* \tilde{\varphi}_*^{(\tilde{\alpha})} p_*^{-1} = \varphi_*^{(\alpha)} | \pi_c = i_* | \pi_c$ where i_* is conjugation by an element of π_c. Thus $i_*^{-1} \varphi_*^{(\alpha)} | \pi_c = 1 | \pi_c$. Let a be any automorphism of π whose restriction to π_c is the identity. Let $\gamma \in \pi_c$ and let $\delta \in \pi$. Then $\delta \gamma \delta^{-1} = \beta$ where $\beta \in \pi_c$ because π_c is normal. Now

$$a(\delta)\, a(\gamma)\, a(\delta)^{-1} = a(\beta); \qquad a(\delta)\, \gamma a(\delta)^{-1} = \beta = \delta \gamma \delta^{-1}.$$

Thus $\delta^{-1} a(\delta) \gamma = \gamma \delta^{-1} a(\delta)$, so that $\delta^{-1} a(\delta)$ belongs to the centralizer of π_c. But then $a(\delta) = \delta$, and $a = 1$. Letting $a = i_*^{-1} \varphi_*^{(\alpha)}$ we get $\varphi_*^{(\alpha)} = i_*$ on all of π. Since H_1 is 1-1 it follows that $[\varphi] = 1$. Thus part 3.1 of Theorem 3 is true. The statements of Parts 3.2 and 3.3 follow immediately.

4. Applications. Let N_{g+1} be a closed, compact non-orientable 2-manifold of genus $g + 1 \geqq 3$, and let O_g be an orientable double cover of N_{g+1}. Generators for the homeotopy groups of N_{g+1} and O_g were determined by Lickorish [4, 5, 6, 7, 8] and Chillingworth [3], but the methods they used for the non-orientable case were distinct from those used to treat the orientable surfaces. Now, it is easily verified that the triplet (O_g, N_{g+1}, p) satisfies the conditions of Theorem 3.1 [see 2]. It should, then, be possible to determine generators and also defining relations for the homeotopy groups of N_{g+1}, whenever the corresponding information is available for O_g. A companion paper to the present one, reference [2], contains a detailed working out of these ideas, which proved to shed considerable light on the structure of the homeotopy groups of N_{g+1}.

Literaturverzeichnis

[1] J. S. Birman and H. M. Hilden, Mapping class groups of closed surfaces as covering spaces. Princeton 1971.

[2] J. S. Birman and D. R. J. Chillingworth, On the homeotopy group of a non-orientable surface. To appear in Proc. Cambridge Phil. Soc.

[3] D. R. J. Chillingworth, A finite set of generators for the homeotopy group of a non-orientable surface. Proc. Cambridge Phil. Soc. **65**, 409 (1969).

[4] W. B. R. Lickorish, A representation of orientable combinatorial 3-manifolds. Ann. of Math. **76**, 531—540 (1962).

[5] W. B. R. Lickorish, A finite set of generators for the homeotopy group of a 2-manifold. Proc. Cambridge Phil. Soc. **60**, 769 (1964).

[6] W. B. R. Lickorish, On the homeotopy group of a 2-manifold (corrigendum). Proc. Cambridge Phil. Soc. **62**, 679—681 (1966).

[7] W. B. R. Lickorish, Homeomorphism of non-orientable 2-manifold. Proc. Cambridge Phil. Soc. **59**, 307—317 (1963).

[8] W. B. R. LICKORISH, On the homeomorphisms of a non-orientable surface. Proc. Cambridge Phil. Soc. **61**, 769–778 (1964).
[9] E. H. SPANIER, Algebraic Topology. New York 1966.

Eingegangen am 29. 6. 1971*)

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*) Eine Neufassung ging am 10. 1. 1972 ein.