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BY

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ABSTRACT. This paper concerns itself with the relationship between two seemingly different methods for representing a closed, orientable 3-manifold: on the one hand as a Heegaard splitting, and on the other hand as a branched covering of the 3-sphere. The ability to pass back and forth between these two representations will be applied in several different ways:

1. It will be established that there is an effective algorithm to decide whether a 3-manifold of Heegaard genus 2 is a 3-sphere.

2. We will show that the natural map from 6-plat representations of knots and links to genus 2 closed oriented 3-manifolds is injective and surjective. This relates the question of whether or not Heegaard splittings of closed, oriented 3-manifolds are "unique" to the question of whether plat representations of knots and links are "unique".

3. We will give a counterexample to a conjecture (unpublished) of W. Haken, which would have implied that S^3 could be identified (in the class of all simply-connected 3-manifolds) by the property that certain canonical presentations for $\pi_1 S^3$ are always "nice".

The final section of the paper studies a special class of genus 2 Heegaard splittings: the 2-fold covers of S^3 which are branched over closed 3-braids. It is established that no counterexamples to the "genus 2 Poincaré conjecture" occur in this class of 3-manifolds.

1. **Introduction.** We begin our study in §2 with a brief discussion of several different methods for presenting links, and the relationship between them. In particular, we discuss two ways to form a link from a braid, as a "closed n-braid", or as a "plat". These results will be used later. §3 contains a brief review of known facts about twist maps on surfaces.

In §4 we develop the central results of this paper. We study the relationship between representations of closed, orientable 3-manifolds by Heegaard splittings, and as branched coverings of S^3. Theorem 1 is a generalization of a theo-

rem due to Alexander [1], which states that every closed, orientable 3-manifold
may be represented as a branched covering of S^3, branched over a link. Our
method of proof is very different from Alexander's, and moreover it contains cer-
tain extra information about the relationship between Heegaard genus and
Heegaard "sewing maps", on the one hand, and the branch set and number of
sheets, on the other hand. In the general case, the covering will not be regular,
however in certain special cases (notably all genus 2 Heegaard splittings) the cover-
ing will not only be regular, but even cyclic.

   The idea used in the proof of Theorem 1 may be specialized to the case of
Heegaard splittings which enjoy a certain symmetry. We give these 3-manifolds
the name "p-symmetric Heegaard splittings", and establish (in Theorems 2–5) the
relationship between p-symmetric Heegaard splittings and p-fold cyclic coverings
of S^3. The results in Theorems 2–5 were announced by the authors in [8]. We
note that Theorem 5 was also discovered independently by O. Ja. Viro [32].
Also, that Theorem 5 generalizes a result due to Schubert [30].

   §§5–7 are concerned with applications of the results of §4. In §5 we give
an algorithm to decide whether a 2-symmetric 3-manifold is S^3. Our algorithm
applies to all 3-manifolds of Heegaard genus $g \leqslant 2$, and to a proper subset of 3-
manifolds of Heegaard genus $g \geqslant 3$. The algorithm was announced by the authors
in [8]. At the conclusion of §5 the algorithm is applied to give a new proof that
every genus 1 homology sphere is S^3.

   In §6 we define a function F from equivalence classes of plat presentations
of knots and links to equivalence classes of Heegaard splittings of closed, oriented
3-manifolds and we show that F is injective and surjective.

   §7 contains applications to problems concerning presentations for $\pi_1 S^3$. In
Lemma 13 we give an algorithm for obtaining a "canonical" presentation for the
fundamental group of a 3-manifold which is defined by a Heegaard splitting,
where it is assumed that the action of the "sewing maps" on the fundamental
group of the Heegaard surface is known. In Lemma 14 we characterize the class
of sewing maps which have the property that the corresponding Heegaard splittings
define S^3. These two lemmas are used in Theorem 8 to obtain a counterexample
to a conjecture of W. Haken about canonical presentations of $\pi_1 S^3$—an unfortu-
nate result, because it points up the difficulty in solving the "trivial group problem"
for 3-manifolds. Finally, at the end of §7, we discuss symmetries in our canoni-
cal presentations for fundamental groups of "2-symmetric" 3-manifolds (Theorem
9).

   The final section of the paper contains a brief proof that one will not find
counterexamples to the Poincaré conjecture among 2-fold coverings of S^3 which
are branched over closed 3-braids. This may be regarded as a small step in an
attempt to prove the "genus 2 Poincaré conjecture".

2. **Links, closed braids and plats.** A *link* will be understood to mean the union of $\mu \geqslant 1$ tame polygonal disjoint simple closed curves in S^3. The term *knot* will be used when we wish to stress the fact that $\mu = 1$. Two links are equivalent if there is an orientation-preserving homeomorphism of $S^3 \longrightarrow S^3$ which maps one onto the other. A *link type* is the equivalence class of a tame link.

Let E^2 be the Euclidean plane, let $\Pi_n E^2$ be the n-fold product space, and let $F_n E^2$ denote the subspace of $\Pi_n E^2$ consisting of all points $(p_1, \ldots, p_n) \in \Pi_n E^2$ which have the property $p_i \neq p_j$ if $i \neq j$. Let $B_n E^2$ denote the quotient space of $F_n E^2$ obtained by identifying points which differ only by a permutation. Let $(p_{10}, \ldots, p_{n0}) \in F_n E^2$, and let $\vec{p}_0$ be its image in $B_n E^2$. The classical braid group B_n is defined to be in the fundamental group $\pi_1(B_n E^2, \vec{p}_0)$. Each element $\beta \in B_n$ may be represented by a loop $(I, \partial I) \longrightarrow (B_n E^2, \vec{p}_0)$ which lifts to a path $(I, 0) \longrightarrow (F_n E^2, (p_{10}, \ldots, p_{n0}))$. This path may be described by the image set $(p_1(t), \ldots, p_n(t))$, $0 \leqslant t \leqslant 1$, where $(p_1(0), \ldots, p_n(0)) = (p_{10}, \ldots, p_{n0})$ and $(p_1(1), \ldots, p_n(1)) = (p_{\mu_1 0}, \ldots, p_{\mu_n 0})$ is a permutation of $(p_{10}, \ldots, p_{n0})$. We will call this image set, or any other obtained in a similar manner from another representative of the same element $\beta \in B_n$, a *geometric n-braid*. We may think of a geometric n-braid as a configuration of n "strings" suspended between parallel planes $t = 0$ and $t = 1$ in E^3.

The braid group B_n has been studied extensively in the literature. For a review of the literature on braids and related topics, see [6]. Original references include [3], [4], [9], [14], [19]. We will make particular use of the following well-known facts:

1. The braid group may be generated by a standard set of "elementary" braids $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i denotes a braid in which the ith string crosses over the $(i + 1)$st, as illustrated in Figure 1.

2. The group B_n has a faithful representation as the full group of auto-homeomorphisms of an n-punctured disc, where admissible maps are required to keep the boundary fixed pointwise. The elementary braid σ_i may be visualized in this representation as a twist which interchanges the ith puncture and the $(i + 1)$st puncture, and is the identity map outside a disc which encloses these two punctures, but avoids all other punctures. Generators $x_1, \ldots, x_n$ can be chosen for the fundamental group of the punctured disc in such a way that the elementary braid σ_i induces the automorphism $\sigma_i(x_i) = x_{i+1}$, $\sigma_i(x_{i+1}) = x_{i+1}^{-1} x_i x_{i+1}$, $\sigma_i(x_j) = x_j$ is $j \neq i, i + 1$.
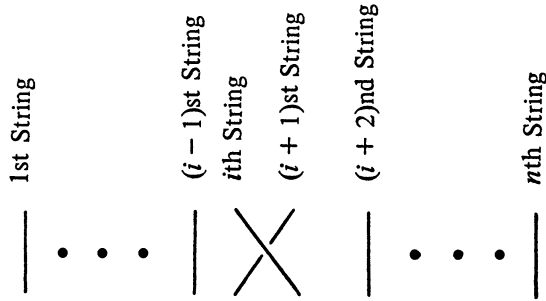
FIGURE 1. Geometric n-braid representing the element σ_i

A link type is said to be represented as a *closed n-braid* if it has a representative which is obtained from a geometric n-braid by identifying the points $p_i(0)$ and $p_i(1)$ for each $i = 1, \ldots, n$. It was proved by Alexander [2], [6] that every link type may be represented by a closed n-braid for some (nonunique) integer n. The *braid number* of a link is the smallest such integer.

A link type is said to be displayed as a *2m-plat* if it is obtained from a geometric braid on $2m$ strings by identifying the points $p_{2i-1}(0)$ and $p_{2i}(0)$, and also $p_{2i-1}(1)$ and $p_{2i}(1)$, for each $i = 1, \ldots, m$. It was established by Reidemeister in [29] that every link type may be represented as a $2m$-plat for some (nonunique) integer m. For another proof, see [6].

A link type is said to have an *m-bridge presentation* if it is represented as the union of two subsets A, B of E^3, parametrized by x, y, z coordinates, as follows:

A is the disjoint union of m arcs lying in the plane $z = 0$, with endpoints in the set $\{(i, 0, 0); i = 1, \ldots, 2m\}$.

B is the disjoint union of m arcs lying in the plane $y = 0$, the jth arc having its endpoints at $(2j - 1, 0, 0)$ and $(2j, 0, 0)$, and its interior in the subset of the plane defined by the condition $z > 0$.

The relationship between plat and bridge representations of a link, and between bridge number and braid number, is expressed by:

LEMMA 1. *If a link type is represented in an m-bridge presentation, then it may also be represented as a 2m-plat; conversely, if it is represented as a 2m-plat, then it may also be represented in an m-bridge presentation. Thus the bridge number is also the smallest integer b such that the link can be represented by a 2b-plat.*

PROOF. See [6].

LEMMA 2. *The bridge number of a link is less than or equal to its braid number.*

PROOF. See [6].

3. Twist maps on surfaces.

Let Y_g be a solid handlebody of genus g, and let ∂Y_g denote its surface (a closed, orientable surface of genus g). In the sections which follow we will need to know certain information about self-homeomorphisms of ∂Y_g.

Let c be a simple closed curve on the surface ∂Y_g. Let N_c be a neighborhood of c which is homeomorphic to a cylinder. Assume that N_c has cylindrical coordinates (y, θ), $-1 \leqslant y \leqslant +1$, $0 \leqslant \theta \leqslant 2\pi$. A twist t_c about c is defined by the identity map outside N_c, with $t_c(y, \theta) = (y, \theta + \pi(y + 1))$.

It was proved by M. Dehn [11] that every homeomorphism of ∂Y_g is isotopic to a product of twists. Dehn's result was later reproved and simplified by W.B.R. Lickorish [17], [18], who showed:

LEMMA 3 (LICKORISH [18]). *Let* $\Phi: \partial Y_g \longrightarrow \partial Y_g$ *be a homeomorphism. Then* Φ *is isotopic to a product of the twists* $t_{c_1}, \ldots, t_{c_{3g-1}}$ *about the curves* $c_1, \ldots, c_{3g-1}$ *illustrated in Figure 2.*
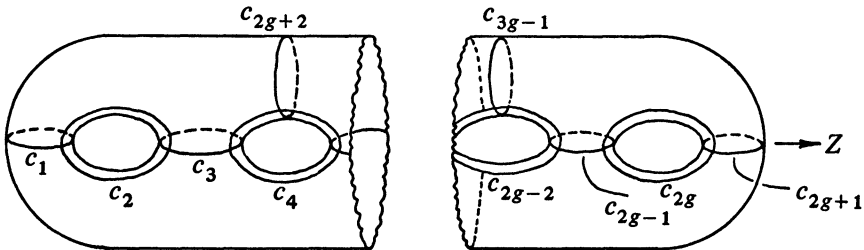
FIGURE 2. Generators for the mapping class group of a closed, orientable surface of genus g

4. Heegaard splittings of branched coverings of S^3.

Let M, N be triangulated n-dimensional manifolds, and let $\pi: M \longrightarrow N$ be a simplicial map. The map π is said to be a *branched covering space projection* if the restriction of π to the complement of the $(n-2)$-dimensional skeleton of the triangulation is a covering space projection. The *branch set* $A \subset N$ is the set of points $z \in N$ which have the property that z has no neighborhood U such that the restriction of π to an arc-component of $\pi^{-1}(U)$ is a covering. The set $L = \pi^{-1}(A)$ will be referred to as the *branch cover*, and we will say that M *is a covering space of N, branched over A*. We will also refer to the pair (M, L) as a *branched covering* of (N, A), writing $\pi: (M, L) \longrightarrow (N, A)$.

Let (M, L) be a branched covering of (N, A), with projection π. Let $\hat\pi$ be the restriction of π to $(M - L)$. Then $\hat\pi: (M - L) \longrightarrow (N - A)$ is an ordinary covering space projection, which will be referred to as the *associated unbranched covering space*.

We will say that M is a *p-fold cyclic covering of N branched over A* if the group $\hat{\pi}_* \pi_1(M - L)$ is the kernel of a homeomorphism from $\pi_1(N - A)$ onto a cyclic group of order p.

It was proved by Alexander in [1] that every closed, orientable 3-manifold may be represented as a branched covering of S^3. Alexander also stated (without proof) that the branch set may be chosen to be a 1-manifold. This fact has been widely accepted in the literature, although no published proof exists. (The gap will, however, be filled in Theorem 1, below.)[3]

We will also be interested in a second method of representing closed, orientable 3-manifolds, as Heegaard splittings. Let Y_g be a solid handlebody of genus g. Let Y'_g be a second copy of Y_g, and let $\tau: Y_g \longrightarrow Y'_g$ be the map that identifies a point $z \in Y_g$ with its corresponding point $z' \in Y'_g$. Let Φ be an orientation-preserving self-homeomorphism of ∂Y_g. Then we may use Φ to define a map which "sews" ∂Y_g to $\partial Y'_g$ by the rule

$$(1) \qquad\qquad \tau \Phi(z) = z, \qquad z \in \partial Y_g.$$

The identification space $Y_g \cup_{\tau \Phi} Y'_g$ will be a closed, orientable 3-manifold which is represented by a *Heegaard splitting* of genus g. It is a classical result that every closed, orientable 3-manifold may be obtained in this way, for some (nonunique) integer g and surface homeomorphism Φ. A 3-manifold which is so represented will be said to have *Heegaard genus g* if it admits a Heegaard splitting of genus g, but no Heegaard splitting of genus $< g$.

The following result will be established below:

THEOREM 1.* *Let $M = Y_g \cup_{\tau \Phi} Y'_g$ be a closed orientable 3-manifold of Heegaard genus g.*

(1.1) *If $g \leqslant 2$, then M may be represented as a 2-fold cyclic covering of S^3, branched over a link of bridge number $g + 1$.*

(1.2) *If $g \geqslant 3$, then M may be represented as a $(4g - 4)$-sheeted branched covering of S^3, with the branching set a 1-manifold of at most $4g - 4$ components.*

Theorem 1 generalizes a result of Alexander [1]. The method used here to prove Theorem 1 is very different from that employed by Alexander, and may be of some interest in its own right. Statement (1.1) of Theorem 1 was established by the authors in [8], and will also be reproved below in a somewhat more general context. We remark that our proof of Theorem 1 is constructive in that it

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[3] See W. B. R. Lickorish, *3-manifolds as branched covers*, Proc. Cambridge Philos. Soc. **74** (1973), 449–451, for another new proof of Alexander's theorem, different from that given here.

Note added in proof. Hilden and Montesinos have shown that any closed oriented 3-manifold is a 3-fold branched covering of S^3. See Bull. Amer. Math. Soc. **80** (1974), pp. 845–846 and 1243–1244.

allows us to relate the branching set in an explicit manner to the Heegaard "sewing map".

We will also be interested in certain special types of Heegaard splittings, which we will refer to as *p-symmetric Heegaard splittings*. Let Y_g, Y'_g, τ, Φ be as defined above. It will be assumed further that there is given a piecewise-linear homeomorphism $P. E^3 \longrightarrow E^3$ of period p, and that Y_g is left invariant under the action of P. Note that the homeomorphisms P and τ define in a natural way a transformation $P' = \tau P \tau^{-1}$ which acts on Y'_g, that P' also has period p, and that Y'_g is left invariant under the action of P'. The Heegaard splitting $Y_g \cup_{\tau\Phi} Y'_g$ will be said to be *p-symmetric* if

(i) There is an integer p_0, with $1 \leqslant p_0 \leqslant p$, such that

(2) $$(\Phi)(P|\partial Y_g)(\Phi^{-1}) = (P|\partial Y_g)^{p_0}.$$

(ii) The orbit space of Y_g under the action of P is a 3-ball.

(iii) The fixed point set of P = fixed point set of P^k for each k, $1 \leqslant k < p$.

(iv) The image of the fixed point set of P is an unknotted set of arcs in the ball Y_g/P.

It will be proved below that the class of 3-manifolds which admit *p*-symmetric Heegaard splittings coincides with the class of 3-manifolds which may be represented as *p*-fold cyclic coverings of S^3 branched over a link.

To state the latter result precisely, we define a new concept: the *p-symmetric Heegaard genus* of a 3-manifold M is the smallest integer g such that M admits a *p*-symmetric Heegaard splitting of genus g. We will establish

THEOREM 2. *Let $g \geqslant 0, p \geqslant 1, b \geqslant 1$ be integers which are related by the equation*

(3) $$g = (b - 1)(p - 1).$$

Then every closed, orientable 3-manifold of p-symmetric Heegaard genus g admits a representation as a p-fold cyclic covering of S^3 branched over a link[4] of bridge number $\leqslant b$.

THEOREM 3. *The p-fold cyclic covering of S^3 branched over a knot of braid number b is a closed, orientable 3-manifold of p-symmetric Heegaard genus $g \leqslant (b - 1)(p - 1)$.*

THEOREM 4. *Every closed, orientable 3-manifold of 2-symmetric Heegaard genus g is a 2-fold covering of S^3 branched over a $(g + 1)$-bridge link.*

THEOREM 5. *Every closed, orientable 3-manifold of Heegaard genus $g \leqslant 2$*

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[4] The reader is reminded that a "link" may have $\mu \geqslant 1$ components, while a "knot" is a link of 1 component.

*is a 2-fold covering of S^3 branched over a link of bridge number $g + 1$. Converse-
ly, the 2-fold covering of S^3 branched over a link of bridge number $b \leqslant 3$ is
a closed, orientable 3-manifold of Heegaard genus $b - 1$.*

Theorems 2–5 were announced in [8]. Theorem 5 was also discovered
independently by O. Ja. Viro in [32]. We note that Theorem 5 reduces to a
result of Schubert [30] in the case $g = 1$ and $b = 2$; our method of proof is,
however, different from his method.

We will begin our proof by establishing Theorems 2 and 3. Theorems 4 and
5 will then follow, easily. Statement (1.1) of Theorem 1 is implied by Theorem
5. The proof of statement (1.2) of Theorem 1 is somewhat more complicated,
and will be treated last.

Note that all of the results above are trivially true if $g = 0$, hence we will
always assume $g \geqslant 1$.

PROOF OF THEOREM 2. The definition of a P-symmetric Heegaard splitting
implies that the map $q: Y_g \longrightarrow Y_g/P$ is a branched covering space. The restriction of
q to ∂Y_g defines a branched covering space $\partial Y_g \longrightarrow \partial(Y_g/P)$, and since Y_g/P is a
3-ball, it follows that its boundary is a 2-sphere. Since the covering is cyclic of
order p, standard results on classification of surfaces imply that g, p and b are
related by (3).

To elucidate the structure of the branch set, it will be convenient to use an
explicit representation for a handlebody Y_g of genus g as a subset of Euclidean
3-space E^3. This handlebody will be constructed in such a way that it is invariant
under a rotation P of period p about the z-axis of E^3, and also in such a way
that Y_g/P is a 3-ball. The handlebody Y_g will then be used to construct a p-
symmetric 3-manifold $M = Y_g \cup_{\tau \Phi} Y_g'$.

Let (r, θ, z) denote cylindrical coordinates in E^3. Let $g \geqslant 1, p \geqslant 2, b \geqslant 2$
be integers which satisfy the equality (3). Let $Y_g = Y_{b,p}$ denote a handlebody
of genus $g = (b - 1)(p - 1)$, which will now be described in terms of its cylin-
drical coordinates.

Let $K(b, p)$ be the union of the set of line segments $\{[(1, 2\pi j/p, i),$
$(0, 0, i)]; j = 1, \ldots, p; i = 1, \ldots, b\}$ and $\{[(1, 2\pi j/p, 1), (1, 2\pi j/p, b)],$
$j = 1, \ldots, p\}$. (See Figure 3.) The handlebody $Y_{b,p}$ will be regarded as the
set of points whose distance from $K(b, p)$ is less than or equal to $(1/10)^p$. Let
P denote rotation about the z axis by $2\pi/p$. It is easily verified that $K(b, p)$
and $Y(b, p)$ are left invariant as sets by P, that $Y(b, p)$ has genus $(b - 1)(p - 1)$
and that $F_{b,p} = \{(r, \theta, z) \in Y_{b,p} | \pi/p \leqslant \theta \leqslant 3\pi/p\}$ is a fundamental set for P.
(See Figure 4.)

The natural projection $Y_{b,p} \longrightarrow Y_{b,p}/P$ is a branched covering space pro-
jection. The branch cover (preimage of the branch set), denoted by $L_{b,p}$, is the
point set $Y_{b,p} \cap z$ axis. It consists of the b disjoint arcs $[(0, 0, i - 1/10^p),$
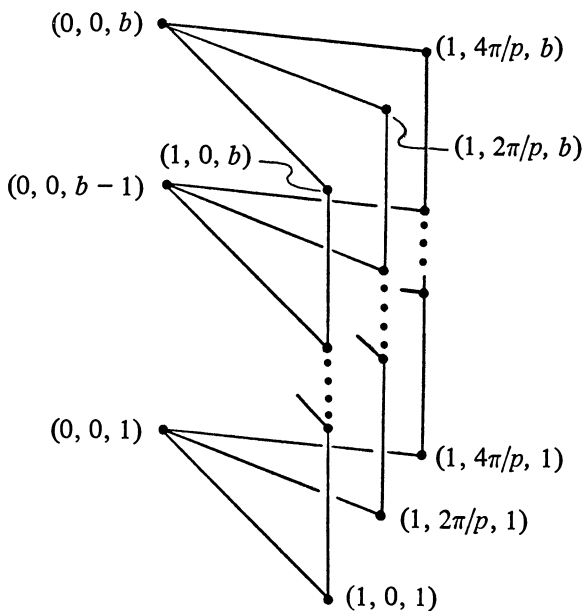
FIGURE 3. Part of the set $K(b, p)$
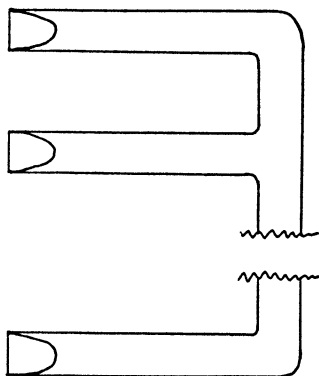
FIGURE 4. Fundamental set $F_{b,p}$ for action of P on $K(b, p)$

$(0, 0, i + 1/10^p)]$ where $i = 1, \ldots, b$. The underlying space $(Y_{b,p}/P, L_{b,p}/P)$ is obtained by identifying the points $(r, \pi/p, z)$ and $(r, 3\pi/p, z)$ in $F_{b,p}$. It is a 3-ball with b distinguished unknotted, unlinked arcs. For simplicity we will hereafter denote $(Y_{b,p}/P, L_{b,p}/P)$ by (D, A_b).

Let $Y'_{b,p}$ be another copy of $Y_{b,p}$, and let $\tau: Y_{b,p} \longrightarrow Y'_{b,p}$ be the map that identifies a point $z \in Y_{b,p}$ with its corresponding point $z' \in Y'_{b,p}$. For simplicity, we will use the same symbol τ to denote the map that identifies a

point z with its corresponding point z' whether the domain be $Y_{b,p}$, $\partial Y_{b,p}$, $L_{b,p}$, D or ∂D. It will be clear from the context what we intend the domain to be. Recall that $Y_{b,p}$ is invariant under the rotation P of $2\pi/p$ about the z-axis. It then follows that $Y'_{b,p}$ will be invariant under the map $P' = \tau P \tau^{-1}$, which also has period p. In a similar fashion, we may use τ to define the branch cover $L'_{b,p} = \tau L'_{b,p} \tau^{-1}$, the set $(Y'_{b,p}/P', L'_{b,p}/P') = (D', A'_b)$ and the branched covering space projection $\pi': (Y'_{b,p}, L'_{b,p}) \rightarrow (D', A'_b)$, all in a natural way.

Let Φ be an orientation-preserving homeomorphism which satisfies the condition (2). Then $M = Y_{b,p} \cup_{\tau\Phi} Y'_{b,p}$ is a closed orientable 3-manifold which is exhibited as a p-symmetric Heegaard splitting. Let $\varphi = \pi\Phi\pi^{-1}$. Then the map $\pi \cup \pi': Y_{b,p} \cup_{\tau\Phi} Y'_{b,p} \rightarrow D \cup_{\tau\varphi} D'$ is a p-fold cyclic covering space projection branched over $A_b \cup_{\tau\varphi} A'_b$. Since $D \cup_{\tau\varphi} D'$ is a Heegaard splitting of genus 0, it can only be S^3. Hence $M = Y_{b,p} \cup_{\tau\Phi} Y'_{b,p}$ is a p-fold cyclic covering space of S^3, branched over $A_b \cup_{\tau\varphi} A'_b$.

It remains to show that $A_b \cup_{\tau\varphi} A'_b$ is a link which is represented by a $2b$-plat. To see this, recall that the point set A_b is a collection of b unknotted, unlinked arcs $A_{b1}, \ldots, A_{bb}$ which are contained in the 3-ball D, with ∂A_b a set of $2b$ points on ∂D (cf. Figure 5). The point set A'_b is the image of A_b under τ.
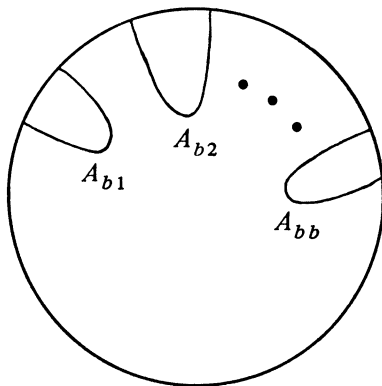
FIGURE 5. The set (D, A_b)

The sewing map $\tau\varphi = \tau\pi\Phi\pi^{-1}$ has the property $\tau\varphi(\partial A_b) = \partial A'_b$. It is now clear that $A_b \cup_{\tau\varphi} A'_b$ can only be a 1-manifold which is represented as a $2b$-plat imbedded in the 3-sphere $D \cup_{\tau\varphi} D'$. This completes the proof of Theorem 2. $\square$

We turn our attention next to Theorem 3, which is a partial converse to Theorem 2. As a preparatory step, we investigate further the branched covering space projection $\pi: Y_{b,p} \rightarrow D$ constructed in the proof of Theorem 2, considering in particular the restriction of this branched covering space projection to the surface $\partial Y_{b,p}$. The associated unbranched covering space will be the regular cyclic

covering space $(\hat{\pi}, \partial Y_{b,p} - \partial L_{b,p}, \partial D - \partial A_b)$ of the 2-sphere ∂D with the $2b$ points ∂A_b removed. We next describe a set of generators for the fundamental group $\pi_1(\partial D - \partial A_k)$ of the base space.

The point set

$$(\partial F_{b,p} \cap \text{ the half-plane } \theta = 2\pi/p) \cup \{(r, \theta, z) \in \partial F_{b,p} | \theta = \pi/p \text{ or } 3\pi/p\}$$

describes a simple closed curve c on ∂D that contains all the branch points. Pick a point x_0 not on c. Let $S_1, Q_1', \ldots, S_b, Q_b$ be the ordered array of points on ∂D whose coordinates on $F_{b,p}$ are $(0, 0, 1 - 1/10^p)$, $(0, 0, 1 + 1/10^p)$, $\ldots$, $(0, 0, b + 1/10^p)$. These are now the branch points. Let s_i be a simple closed curve which starts at x_0, first intersects the curve c between the points S_i and S_{i+1}, and then intersects c exactly one more time, enclosing the point S_i (but no other point S_j or Q_k). Let q_i be a simple closed curve which starts at x_0, first intersects c between Q_{i-1} and Q_i (where $Q_0 = S_b$), and then intersects c exactly one more time, enclosing the point Q_i (but no other point Q_j or S_k). Notice that s_i and q_i have opposite orientation. The group $\pi_1(\partial D - \partial A_k)$ is a free group, which admits the presentation

(4) $$\langle s_1, \ldots, s_b, q_1, \ldots, q_b; s_1 q_1^{-1} s_2 \cdots q_b^{-1} \rangle.$$

The subgroup belonging to the covering space $\pi: (\partial Y_{b,p} - \partial L_{b,p}) \rightarrow (\partial D - \partial A_b)$ is the set of words which have total exponent sum a multiple of p.

LEMMA 4. *If $p = 2$, every orientation-preserving homeomorphism of ∂D which leaves the set $\partial D \cap A_b$ invariant lifts to a homeomorphism of $\partial Y_{b,p}$ which leaves the set $\partial Y_{b,p} \cap L_{b,p}$ invariant. If $p \geq 3$, an orientation-preserving homeomorphism of ∂D which leaves the set $\partial D \cap A_b$ invariant lifts to a homeomorphism of $\partial Y_{b,p}$ which leaves the set $\partial Y_{b,p} \cap L_{b,p}$ invariant if and only if the homeomorphism leaves the point set $S = \{S_1, \ldots, S_b\}$ invariant.*

PROOF OF LEMMA 4. It is well known that a homeomorphism η of ∂D which fixes the branch points as a set lifts if and only if η_* leaves the subgroup of the covering invariant [22]. If η preserves orientation, then η_* necessarily takes each s-type generator (respectively q-type generator) into a conjugate of another s-generator (respectively q-generator) or into a conjugate of some q^{-1} (respectively s^{-1}). (To see this, think of s_i as a curve which traverses an arc α from the base point to a point near S_i, then winds about S_i in a small circular path, and then traverses α in the reverse direction, and recall also that s_i and q_i wind in opposite directions.) If $p = 2$, the elements s_i and $w q_j^{-1} w^{-1}$ have the same exponent sum (mod 2), so that every η_* leaves the subgroup of the covering invariant. But if $p > 2$, then s_i and $w q_j^{-1} w^{-1}$ do not have the same exponent sum (mod b) and the subgroup of the covering is left invariant if and only if η_*

maps s-generators into conjugates of other s-generators. This happens if and only if η_* leaves the set $S = \{S_1, \ldots, S_b\}$ invariant. This proves Lemma 4. $\square$

One more preparatory lemma is needed for the proof of Theorem 3. Observe that the statement of Theorem 3 refers only to knots, not to links. This is not surprising, in view of the following:

LEMMA 5. *There is a unique p-fold cyclic covering of S^3 branched over a link of $\mu \geqslant 1$ components if and only if either $p = 2$ or $\mu = 1$.*

PROOF OF LEMMA 5. The p-fold coverings of S^3 branched over a link L are in 1-1 correspondence with nontrivial homomorphisms w of $\pi_1(S^3 - L)$ into the symmetric group on p letters, by the following rule: one considers p copies of S^3, each with the distinguished subset L, and identifies them along L according to pasting instructions given by w. If the covering is cyclic, then the correspondence will be with equivalence classes of nontrivial homomorphisms of $\pi_1(S^3 - L)$ onto the cyclic group Z_p, where two such homomorphisms α, β are equivalent if there is an automorphism δ of Z_p such that $\alpha = \beta\delta$. Since Z_p is abelian, any such homomorphism factors through the commutator quotient group of $\pi_1(S^3 - L)$, which is a μ-fold direct sum $Z \oplus \cdots \oplus Z$. The pre-images of the generators of the summands are necessarily meridians of L, and there is one summand Z for each component of the link. The meridians of the link are mapped onto the elements $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the meridian of the kth component has an entry 1 in the kth copy of Z. Since the elements $(0, \ldots, 0, 1, 0, \ldots, 0)$ must each be mapped onto nontrivial elements of Z_p, it follows that the only cases where there is precisely one admissible mapping is if either $p = 2$ or $\mu = 1$. This completes the proof of Lemma 5. $\square$

PROOF OF THEOREM 3. Let K be a closed b-braid knot,[5] and let p be any integer $\geqslant 2$. By the argument used in the proof of Theorem 2, we may always represent (S^3, K) as

$$(S^3, K) = (D, A_b) \cup_\varphi (D', A_b')$$

for some autohomeomorphism $\varphi: (\partial D - \partial A_b) \longrightarrow (\partial D - \partial A_b)$. We may then reverse the entire argument used in the proof of Theorem 2 to construct the p-fold cyclic covering of D (respectively D') branched over A_b (respectively A_b'). This covering space will be a handlebody $Y_{b,p}$ (respectively $Y_{b,p}'$) of genus $g = (b - 1)(p - 1)$.

Since K is a closed b-braid, the condition of Lemma 4 will be satisfied, hence the homeomorphism φ lifts to a homeomorphism Φ of $\partial Y_{b,p}$. The

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[5] The reader is referred to §2 for the definition of a closed b-braid. Note that (by Lemmas 1 and 2) every closed b-braid is also a $2b$-plat, and hence has bridge number $\leqslant b$; however the converse need not be true.

homeomorphism Φ may be used to construct a 3-manifold $M = Y_{b,p} \cup_{\tau\Phi} Y'_{b,p}$, which is now seen to be a p-symmetric Heegaard splitting of genus $g = (b - 1)(p - 1)$. Also, M is a p-fold cyclic covering space of S^3, branched over K. By Lemma 5, this covering space is unique. This completes the proof of Theorem 3. $\square$

Our next result, Theorem 4, is a sharpened version of Theorems 2 and 3, which holds in the special case $p = 2$.

PROOF OF THEOREM 4. To establish Theorem 4, we must prove

(4.1) Every closed orientable 3-manifold of 2-symmetric Heegaard genus g is a 2-fold covering of S^3 branched over a link of at most $g + 1$ bridges.

(4.2) Every 2-fold covering of S^3 branched over a link of $g + 1$ bridges has 2-symmetric Heegaard genus at most g.

Statement (4.1) is precisely Theorem 2, for the special case $p = 2$. Statement (4.2) is a strong version of Theorem 3, which holds when $p = 2$. The proof of statement (4.2) may be accomplished by a sequence of steps just like those used to prove Theorem 3, except that

(a) If $p = 2$, Lemma 4 tells us that *every* autohomeomorphism φ of $(\partial D - \partial A_b)$ lifts to $\partial Y_{b,p}$. Hence we may consider all $(g + 1)$-bridge links, instead of restricting our attention to the sublcass of closed $(g + 1)$-braids.

(b) If $p = 2$, the statement of Lemma 5 allows us to consider links of $\mu \geqslant 1$ components, instead of restricting our attention to the case $\mu = 1$, i.e. to proper knots.

This completes the proof of Theorem 4. $\square$

PROOF OF THEOREM 5. Theorem 5 will follow immediately from Theorem 4, if we can show that every closed, orientable 3-manifold of Heegaard genus $g = 1$ or 2 also has 2-symmetric Heegaard genus g. This is true because every homeomorphism Φ of the boundary ∂Y_g of a handlebody of genus $g \leqslant 2$ is isotopic to a homeomorphism which commutes with a particular involution $T: \partial Y_g \rightarrow \partial Y_g$, which is such that $\partial Y_g/T$ is a 2-sphere. To verify this latter statement, note that, by Lemma 3 of §3, every homeomorphism Φ of ∂Y_g is isotopic to a product of the Lickorish twists $t_{c_1}, \ldots, t_{c_{3g-1}}$, and that if $g \leqslant 2$ each such twist may be chosen to commute with a rotation T of period 2 about the x axis in Figure 4.([6]) Since the quotient space $\partial Y_g/T$ is a 2-sphere in this case, the proof of Theorem 5 is complete.

Finally we approach the proof of Theorem 1. Statement (1.1) of Theorem 1, which relates to the case of $g \leqslant 2$, is contained in Theorem 5, hence we may restrict our attention to the case $g \geqslant 3$. Our proof will be, essentially, an escalation of the technique used in the proof of Theorems 2 and 3, however in the

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([6]) The special symbol T will be used instead of P to refer to this *particular* rotation.

general case to be considered now the Heegaard splitting $Y_2 \cup_{\tau \Phi} Y'_g$ is not assumed to be p-symmetric, and the covering space projection we will define when we develop a representation of $M = Y_g \cup_{\tau \Phi} Y'_g$ as a branched covering of S^3 will not be a regular covering space.
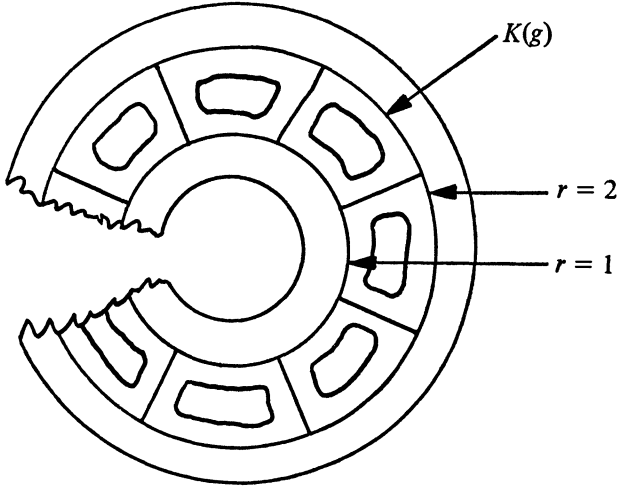

FIGURE 6.   Projection of Y_g onto the plane $z = 0$

PROOF OF THEOREM 1. As before, let (r, θ, z) denote cylindrical coordinates in E^3. The reader is referred to Figure 6. Let $K(g), g \geqslant 3$, be the union of the line segments $\{[(1, (2k + 1)\pi/(g - 1), 0), (2, (2k + 1)\pi/(g - 1), 0)] ; k = 0, \ldots, g - 2\}$ and the circles $z = 0, r = 1$ and $z = 0, r = 2$. Let Y_g be a regular neighborhood of $K(g)$ which is invariant under a rotation L of $2\pi/(g - 1)$ radians about the z axis, and also under a rotation B_k of π radians about the lines $z = 0$, $\theta = (2k + 1)\pi/(g - 1)$, for each $k = 0, \ldots, g - 2$. We also assume that the intersection of Y_g with each half-plane $\theta = \theta_0$ has central symmetry, and we denote the homeomorphism of Y_g obtained by simultaneously performing the central symmetry on each such section by M.

Note that M commutes with L and also with $B_0, \ldots, B_{g-2}$. Thus we have defined an action on Y_g by the finite group G generated by L, M and B_0, $\ldots, B_{g-2}$. The group G is an extension of order 2 of a dihedral group of order $2(g - 1)$. Nontrivial elements of G with fixed points are $\{M, B_k$ and $MB_k, k = 0, \ldots, g - 2\}$. The fixed point sets of M, B_k and MB_k are the intersection of Y_g with the circle $z = 0, r = 1\frac{1}{2}$; the line $z = 0, \theta = 2k\pi/(g - 1)$, and the line $r = 1\frac{1}{2}, \theta = 2k\pi/(g - 1)$ respectively. (See Figure 7.)

Let $T_g = \partial Y_g$. The group G acts on T_g by restriction. Let a_i, b_i, c_i, d_i and e be the curves on T_g depicted in Figure 7. Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ and ϵ be the Dehn twists about these curves (see §3 for the definition of a Dehn twist). We
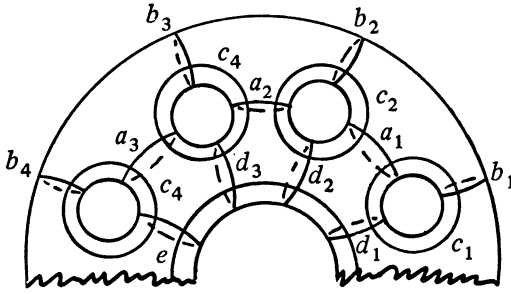
FIGURE 7. Curves on $T_g = \partial Y_g$

may choose these twists and curves in such a way that $\alpha_k M = M\alpha_k$, $\beta_k B_k = B_k \beta_k$, $\gamma_k B_k = B_k \gamma_k$, $\delta_k B_k = B_k \delta_k$ and $\epsilon B_k = B_k \epsilon$ for each $k = 0, \ldots, g - 2$. By Lemma 3, this collection of Dehn twists generates the mapping class group of T_g.

LEMMA 6. *Given any Dehn twist ω in the above collection, there is an element $\chi(\omega) \in G$ such that:*

(i) *ω commutes with $\chi(\omega)$.*

(ii) *$\chi(\omega) = B_k$ or $\chi(\omega) = M$; $k = 0, \ldots, g - 2$.*

(iii) *If $\underline{\omega}$ is the projection of ω to the manifold $T_g/\chi(\omega)$, then $\underline{\omega}$ is isotopic to the identity map.*

(iv) *Let $P(\omega)$ be the image of the fixed point set of $\chi(\omega)$ under the mapping $\pi: T_g \longrightarrow T_g/\chi(\omega)$. Let $Q(\omega)$ be the image under π of the remaining fixed points. Then the isotopy $\underline{\omega}_t$ of $\underline{\omega}$ may be chosen so that $P \cap \underline{\omega}_t(Q) = \emptyset$ for all $0 \leqslant t \leqslant 1$.*

PROOF OF LEMMA 6. If $\omega = \beta_k$, γ_k or δ_k choose $\chi(\omega) = B_k$. If $\omega = \alpha_k$, choose $\chi(\omega) = M$. If $\omega = \epsilon$ choose $\chi(\omega) = B_k$ for any k. Conditions (i) and (ii) above are satisfied. In each case the support of $\underline{\omega}$ on $T_g/\chi(\omega)$ is a disc, hence (iii) is also satisfied.

If $\omega = \beta_i$ or δ_i, the disc supporting $\underline{\omega}$ contains no points of $Q(\omega)$ at all. If $\omega = \gamma_i$ or α_i, then there is only one point of $Q(\omega)$ in the disc supporting $\underline{\omega}$. In fact $\underline{\omega}$ is a twist centered at this point. In either of these cases $\underline{\omega}$ can be untwisted without moving the points of $Q(\omega)$ at all, and (iv) is easily satisfied. Finally, if $\omega = \epsilon$, then $\underline{\omega}$ is a twist exchanging a pair of points of $P(\omega)$. The line segment connecting these points contains all the points of $Q(\omega)$ in the support of $\underline{\omega}$ and is left invariant as a set by $\underline{\omega}$ (the pre-image of this line segment is the support of ϵ intersected with the $x - y$ plane). As $\underline{\omega}$ is untwisted, $\underline{\omega}_t(Q(\omega))$ traces out the path indicated in Figure 8, easily avoiding the set $P(\omega)$. This completes the proof of Lemma 6. $\square$
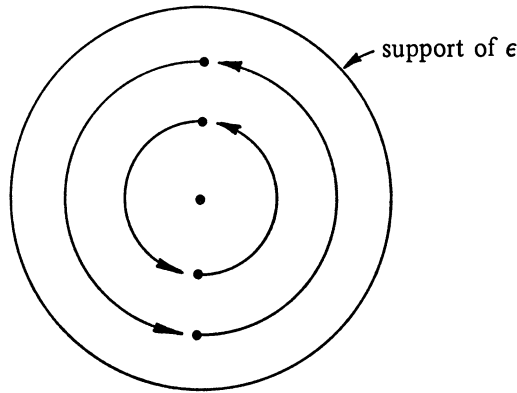
FIGURE 8. Effect of ϵ_t on points in $Q(\epsilon)$

Continuing the proof of Theorem 1, let Y_g' be another copy of Y_g. Let τ be the map that identifies Y_g with Y_g'. Let $T_g' = \tau(T_g) = \tau(\partial Y_g) = \partial Y_g'$. In what follows we shall allow the group G to act on the sets T_g' and Y_g', and allow the Dehn twists to act on $T_g \times \{n + 1\}$, considered as a subset of the set $T_g \times [n, n + 1]$. This is always done in the obvious way.

Let τ_j: $T_g \times \{2j - 1\} \longrightarrow T_g \times \{2j\}$ be defined by $\tau_j(z, 2j - 1) = (z, 2j)$. Let ω_j be a Dehn twist acting on $T_g \times \{2j - 1\}$. We will use the symbol $T_g \times [2j - 2, 2j - 1] \cup_{\tau_j \omega_j} T_g \times [2j, 2j + 1]$ to indicate that $T_g \times \{2j - 1\}$ is to be identified with $T_g \times \{2j\}$ by the rule $(\omega_j(z), 2j - 1) = (z, 2j)$.

LEMMA 7. *Every closed, orientable 3-manifold M of Heegaard genus g admits a representation of the form*:

(5)
$$M = Y_g \cup_{\tau_0} T_g \times [0, 1] \cup_{\tau_1 \omega_1} T_g \times [2, 3] \cup_{\tau_2 \omega_2} \cdots$$
$$\cup_{\tau_m \omega_m} T_g \times [2m, 2m + 1] \cup_{\tau_{m+1}} Y_g'$$

where τ_0 and τ_{m+1} are the obvious identifications of $\partial Y_g = T_g$ with $T_g \times \{0\}$ and of $T_g \times \{2m + 1\}$ with $T_g' = \partial Y_g'$.

PROOF OF LEMMA 7. Since the Dehn twists generate the mapping class group, M has Heegaard genus g, and T_g has a collar neighborhood, the Heegaard splitting of M yields the representation

$$M = Y_g \cup_{\tau_0} T_g \times [0, 1] \cup_{\tau_1} T_g \times [2, 3] \cup \cdots$$
$$\cup_{\tau_m} T_g \times [2m, 2m + 1] \cup_{\tau_{m+1} \omega_m \cdots \omega_2 \omega_1} Y_g'.$$

A homeomorphism between M and a manifold M' with the required form is defined below:

$$Y_g \cup_{\tau_0} T_g \times [0, 1] \cup_{\tau_1} T_g \times [2, 3] \cup \cdots \cup_{\tau_m} T_g \times [2m, 2m + 1] \cup_{\tau_{m+1}\omega_m \cdots \omega_2 \omega_1} Y_g'$$

$$\Big\downarrow h_0 \qquad \Big\downarrow h_1 \qquad \Big\downarrow h_2 \qquad \cdots \qquad \Big\downarrow h_{m+1} \qquad \Big\downarrow h_{m+2}$$

$$Y_g \cup_{\tau_0} T_g \times [0, 1] \cup_{\tau_1\omega_1} T_g \times [2, 3] \cup \cdots \cup_{\tau_m\omega_m} T_g \times [2m, 2m + 1] \cup_{\tau_{m+1}} Y_g'$$

Define $h_0(z) = z$, $h_{m+2}(z) = z$, $h_1(z, t) = (z, t)$ and $h_j(z, t) = (\omega_{j-1} \cdots \omega_2 \omega_1(z), t)$ for each $j = 2, \ldots, m + 1$. This proves Lemma 7. $\square$

Before proving the next lemma we introduce a definition which we will make use of only in the remainder of this section. Given a branched covering space projection $\pi: M \rightarrow N$, where M and N are compact manifolds, a point $x \in M$ is called *ordinary* if p is a local homeomorphism in some neighborhood of x. A point that is not ordinary is called *irregular*, and the set of irregular points is called the *irregular set*. Notice that the branch cover is precisely the inverse image of the image under π of the irregular set.

The next lemma is a weak version of Theorem 1.

LEMMA 8. *Let M be defined by equation (5). Then there is a branched covering $\pi: M \rightarrow S^3$, where S^3 is represented by*

$$S^3 = (Y_g/G) \cup_{\tau_0} (T_g/G) \times [0, 1] \cup_{\tau_1} (T_g/G) \times [2, 3] \cdots$$

$$\cup_{\tau_m} (T_g/G) \times [2m, 2m + 1] \cup_{\tau_{m+1}} (Y_g'/G)$$

such that

(i) π *restricted to* $Y_g \cup Y_g' \cup T_g \times [2m, 2m + 1] \cup T_g \times \{0, 2, 4, \ldots, 2m\}$ *is the natural projection.*

(ii) $\pi: (T_g) \times [2j, 2j + 1] \rightarrow (T_g/G) \times [2j, 2j + 1]$ *preserves the second coordinate* $j = 0, \ldots, m$.

(iii) *The irregular set is a 1-complex. The singular points of this 1-complex are the points $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ and $\tau(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ in Y_g and Y_g' respectively.*

PROOF OF LEMMA 8. First note that (T_g/G) and (Y_g/G) are S^2 and D^3 respectively. To see this observe that a fundamental set for the action of G on Y_g is $Y_g \cap \{(r, \theta, z)| 0 \leq r \leq 1\frac{1}{2}, 0 \leq \theta \leq \pi/(g - 1)\}$. (Y_g/G) is obtained from the fundamental set by making the identifications $(r, \theta, z) = (r, \theta, -z)$ for points (r, θ, z) lying in the half-planes $\theta = 0$ or $\theta = \pi/(g - 1)$ or the cylinder $r = 1\frac{1}{2}$ (see Figure 9).

Next, we use condition (i) to define π on the set denoted in (i). We wish to extend the definition of π to the rest of M and then check that conditions (ii) and (iii) are satisfied. Pick some j: $0 \leq j \leq m - 1$. We wish to extend the
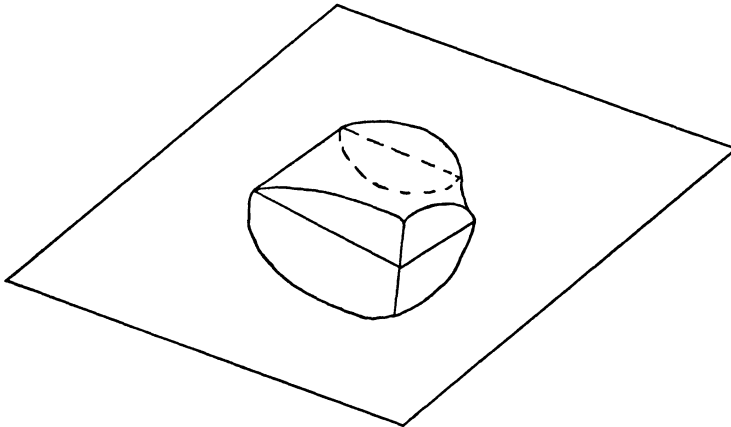
FIGURE 9. Fundamental set for action of G on Y_g

definition of π from $T_g \times \{2j\} \cup T_g \times \{2j+2\}$ to $T_g \times [2j, 2j+1] \cup_{\tau_{j+1}\omega_{j+1}}$ $T_g \times \{2j+2\}$. Let $\chi(\omega_{j+1}) \in G$ be the element whose existence is guaranteed by Lemma 6. Let $\underline{\omega}_{j+1,t}$ be the isotopy whose existence is guaranteed by Lemma 6, parts (iii) and (iv) $(\underline{\omega}_{j+1,0} = \text{id}; \underline{\omega}_{j+1,1} = \underline{\omega}_{j+1})$. Let

$$p_j\colon T_g \times [2j, 2j+1] \cup_{\tau_{j+1}\omega_{j+1}} T_g \times \{2j+2\}$$

$$\longrightarrow (T_g/(\chi(\omega_{j+1}))) \times [2j, 2j+1] \cup_{\tau_{j+1}\underline{\omega}_{j+1}} (T_g/(\chi(\omega_{j+1}))) \times \{2j+2\}$$

be the natural projection. Let

$$q_j\colon (T_g/\chi(\omega_{j+1})) \times [2j, 2j+1] \cup_{\tau_{j+1}\underline{\omega}_{j+1}} (T_g/\chi(\omega_j)) \times \{2j+2\}$$

$$\longrightarrow (T_g/\dot\chi(\omega_{j+1})) \times [2j, 2j+1] \cup_{\tau_{j+1}} (T_g/\chi(\omega_{j+1})) \times \{2j+2\}$$

be the homeomorphism $(x, t) \longrightarrow (\underline{\omega}_{j+1,t-2j}(x), t)$, $2j \leqslant t \leqslant 2j+1$, and $(x, 2j+2) \longrightarrow (x, 2j+2)$. Note that q_j is well defined and that q_j restricted to $T_g \times \{2j, 2j+1\}$ is the identity. Let

$$\gamma_j\colon (T_g/\chi(\omega_{j+1})) \times [2j, 2j+1] \cup_{\tau_{j+1}} (T_g/\chi(\omega_{j+1})) \times \{2j+2\}$$

$$\longrightarrow (T_g/G) \times [2j, 2j+1] \cup_{\tau_{j+1}} (T_g/G) \times \{2j+2\}$$

be the natural projection. Now define π on $T_g \times [2j, 2j+1] \cup T_g \times \{2j,+2\}$ to be $\gamma_j q_j p_j$ for each j, and note that π is well defined because $\gamma_j q_j p_j$ is just the natural projection on the sets $T_g \times \{2j, 2j+2\}$.

Now we investigate the irregular set of π restricted to $T_g \times [2j, 2j+1]$. Since it is easily checked that p_j, q_j, and γ_j are all branched coverings, this set is equal to $P_j \cup p_j^{-1}q_j^{-1}Q_j$ where P_j is the irregular set of the natural projection $p_j\colon T_g \times [2j, 2j+1] \longrightarrow (T_g/\chi(\omega_j)) \times [2j, 2j+1]$ and Q_j is the irregular set of

the natural projection

$$\gamma_j: (T_g/\chi(\omega_j)) \times [2j, 2j + 1] \longrightarrow (T_g/G) \times [2j, 2j + 1].$$

By the definition of q_j (utilizing part (iv) of Lemma 6) P_j and $p_j^{-1}q_j^{-1}Q_j$ are disjoint sets. Thus the irregular set consists of $8(g-1)$ arcs, whose intersection with the set $T_g \times \{t\}$ consists of $8(g-1)$ points. Putting together all the irregular sets of π restricted to $T_g \times [2j, 2j + 1]$ we see that the irregular set of π restricted to $T_g \times [0, 1] \cup_{\tau_1\omega_1} T_g \times [2, 3] \cup \cdots \cup_{\tau_m\omega_m} T_g \times [2m, 2m + 1]$ consists of $8(g-1)$ disjoint arcs. Thus all the singular points of the irregular set belong to the set $Y_g \cup Y_g'$. Checking the definition of π on these sets $Y_g \cup Y_g'$ we see that the singular points of the irregular set are the points $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ and $\tau(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$, $k = 0, \ldots, g - 2$. This completes the proof of Lemma 8. $\square$

We may now complete the proof of Theorem 1, part (1.2). First we eliminate the singular point from the irregular set. In view of Lemma 8, and the definition of π, all we need to do is to show that we can modify the map π of Lemma 8 in a neighborhood of a point $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ so as to eliminate the singular point.

Let H be the subgroup of G consisting of elements that fix the point $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$. Let $\eta_1: Y_g \longrightarrow (Y_g/M)$, $\eta_2: (Y_g/M) \longrightarrow (Y_g/H)$ and $\eta_3: (Y_g/H) \longrightarrow (Y_g/G)$ be the natural projections. Then π restricted to Y_g equals $\eta_3\eta_2\eta_1$. The irregular set of η_1, denoted B_1, is the circle $(z = 0, r = 1\frac{1}{2}) \cap Y_g$. The irregular set of η_2, denoted B_2, is η_1 (the ray $(z = 0, \theta = (2k+1)\pi/(g-1) \cap Y_g)$). The map η_3 is a local homeomorphism in a neighborhood of $\eta_2\eta_1(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$. Let ψ be a homeomorphism of Y_g/M supported in a small ball D centered at $\eta_1(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ such that $\psi(\eta_1(B_1)) \cap B_2 \cap D = \varnothing$. The map $\eta_3\eta_2\psi\eta_1$ has no singular points in its irregular set and agrees with π except in a small neighborhood of $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$. We can repeat the process at each of the points $(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ and $\tau(1\frac{1}{2}, (2k + 1)\pi/(g - 1), 0)$ until all singular points have been eliminated from the irregular set of π.

The irregular set, which we shall call S, is now a 1-manifold, but its image has singular points. To eliminate these we shall further modify the map π so that π restricted to S is an embedding.

Let M be the set of multiple points of S, i.e. $M = \{x \in S| \pi(y) = \pi(x)$ for some $y \in S$ with $y \neq x\}$. Let $U_i, i = 1, \ldots, n$, be a covering of $\pi(S)$ by open balls such that the restriction of π to any component of $\pi^{-1}(U_i)$ embeds the intersection of S with that component. Suppose V is a component of $\pi^{-1}(U_i)$, for some i, such that $V \cap M \neq \varnothing$. Let ψ be a homeomorphism of S^3 supported on U_i such that $\psi(\pi(V \cap S)) \cap \pi(S) = \varnothing$. (Such a homeomorphism exists by a

general position argument.) Now define $\pi'(x) = \pi(x)$ if $x \notin V$ and $\pi'(x) = \psi(\pi(x))$ if $x \in V$.

The branched covering π' has the following properties: (1) For any i, $i = 1, \ldots, n$, $\pi'^{-1}(U_i) = \pi^{-1}(U_i)$. (2) The set of multiple points for π', M' is contained in M. (3) $V \cap M' = \emptyset$, but $V \cap M \neq \emptyset$. We can iterate this process until no component of the inverse image of any U_i contains a multiple point. The final branched covering we obtain, $\tilde{\pi}$, embeds the irregular set. Thus the branch set $\tilde{\pi}(S)$ is a one-manifold. Since $\tilde{\pi}$ is a local homeomorphism on the complement of S, the branch cover $\tilde{\pi}^{-1}\tilde{\pi}(S)$ is also a 1-manifold.

5. **An algorithm to decide whether a 3-manifold which admits a 2-symmetric Heegaard splitting is S^3.** In this section we apply the results of §4 to establish that there is an effective algorithm to decide whether a 3-manifold which admits a 2-symmetric Heegaard splitting is S^3. The reader is reminded that this class includes all 3-manifolds of Heegaard genus $g \leqslant 2$. This result was established earlier by the authors in [8]. As an application, we will use our algorithm to give a new and very simple proof that every genus 1 homology sphere is S^3.

Let Y_g be a handlebody of genus g which is imbedded in 3-space in the manner indicated in Figure 4, so that it is invariant under a rotation T of 180° about the z axis. Note that the orbit space Y_g/T is a 3-ball D, and that T has period $p = 2$. Let Y_g' be a second copy of Y_g, and let $\tau\colon Y_g \longrightarrow Y_g'$ be the natural map. Let $\Phi\colon \partial Y_g \longrightarrow \partial Y_g$ be a homeomorphism which commutes with the restriction of T to ∂Y_g, and let $M = Y_g \cup_{\tau\Phi} Y_g'$ be the 3-manifold defined by using the rule in equation (1) to identify points on ∂Y_g and $\partial Y_g'$. Thus M is defined by the 2-symmetric Heegaard splitting $Y_g \cup_{\tau\Phi} Y_g'$. Recall that every 3-manifold of Heegaard genus $g \leqslant 2$ admits such a representation, because by Lemma 3 every homeomorphism Φ of ∂Y_g is isotopic to a product of Lickorish twists $t_{c_1}, \ldots, t_{c_{3g-1}}$ about the curves $c_1, \ldots, c_{3g-1}$ in Figure 4, and if $g \leqslant 2$ each such twist may be defined in such a way that it commutes with T. If $g \geqslant 3$, we restrict ourselves to twists about $c_1, \ldots, c_{2g+1}$.

Let $[\Phi]$ denote the isotopy class of Φ. We will assume that $[\Phi]$ is defined as a product of twists, i.e., that we are given

$$(6) \qquad [\Phi] = [t_{c_{\mu_r}}^{\epsilon_r}] \cdots [t_{c_{\mu_2}}^{\epsilon_2}][t_{c_{\mu_1}}^{\epsilon_1}], \qquad \epsilon_i = \pm 1, \mu_i = 1, \ldots, 2g+1.$$

The algorithm. Let $[\Phi]$ be given by equation (6).

1. Construct the $(2g + 2)$-string braid

$$(7) \qquad \beta = \sigma_{\mu_r}^{\epsilon_r} \cdots \sigma_{\mu_1}^{\epsilon_1}$$

where σ_i $(i = 1, \ldots, 2g + 1)$ is a standard generator of the $(2g + 2)$-string Artin braid group. [See Figure 1.]

2. Construct the $(2g + 2)$-plat L determined by the geometric braid β (see §2).

3. Apply the algorithm given by Haken in [16], or by Schubert in [31] to decide whether L is the trivial knot type.

We assert:

THEOREM 6. *The 2-symmetric 3-manifold $M = Y_g \cup_{\tau\Phi} Y'_g$ is S^3 if and only if L defines the trivial knot type.*

PROOF OF THEOREM 6. Suppose that M is homeomorphic to S^3. Then T is an involution of the 3-sphere, hence by a theorem of Waldhausen [33] the fixed point set F of T is an unknotted circle. Hence its image $\pi(F)$ under the collapsing map $\pi: M \longrightarrow M/T$ is also the trivial knot type. We claim that $\pi(F)$ is precisely the plat defined by the $(2g + 2)$-braid β in equation (7). If we can establish this, it will follow that a necessary condition for $M \sim S^3$ is that this plat be trivial. To see that this condition is also sufficient, note that if $\pi(F)$ is the trivial knot type, then by the construction described in the proof of Theorem 2, §4, M is the 2-fold cyclic covering of

$$(8) \qquad M/T = (Y_g \cup_{\tau\Phi} Y'_g)/T = D \cup_{\tau\varphi} D' \sim S^3$$

branched over the trivial knot. But then M can only be S^3, hence the condition is also sufficient.

It remains to establish that $\pi(F)$ is the plat defined by the $(2g + 2)$-string braid β. To see this, recall that it was established in the proof of Theorem 2 that $\pi(F)$ is the point set $A_b \cup_{\tau\varphi} A'_b \subset D \cup_{\tau\varphi} D'$, where A_b is a collection of $b = 2g + 2$ unknotted arcs which are contained in the 3-ball D, and $A'_b = \tau(A_b)$ is the corresponding collection in $D' = \tau(D)$ (cf. Figure 5); also that $D \cup_{\tau\varphi} D'$ is S^3 (the 3-balls D and D' are sewn together by the surface mapping $\tau\varphi: \partial D \longrightarrow \partial D'$, where $\varphi = \pi\Phi\pi^{-1}$). The mapping $\Phi: \partial Y_g \longrightarrow \partial Y_g$ is the twist product given in equation (6), hence $[\varphi]$ will be the corresponding product of twists $[\pi_{c_{\mu_i}}^{\epsilon_i} \pi^{-1}]$, and $[\pi t_{c_{\mu_i}} \pi^{-1}]$ is precisely the elementary braid autohomeomorphism σ_i of the surface $(\partial D - \partial A_b)$, as defined in §2.[7] Thus Φ projects to

$$[\varphi] = \beta = \sigma_{\mu_r}^{\epsilon_r} \cdots \sigma_{\mu_2}^{\epsilon_2} \sigma_{\mu_1}^{\epsilon_1}$$

hence $\pi(F)$ is precisely the plat determined by β.[8] This proves Theorem 6. □

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[7] The fact that $[\pi t_{c_i} \pi^{-1}] = \sigma_i$ is clear from the definitions of the twists t_{c_i} and σ_i and the projection π. For an explicit proof, see [6, Chapter 4].

[8] Note that we have followed the convention that mappings are applied from right to left. Thus $\sigma_{\mu_r}^{\epsilon_r} \cdots \sigma_{\mu_1}^{\epsilon_1}$ means "apply $\sigma_{\mu_1}^{\epsilon_1}$ first, then $\sigma_{\mu_2}^{\epsilon_2}, \cdots$". This convention is opposite to the convention normally followed by group theorists, e.g., as in [20, pp. 173–179]. However, since we are not concerned with *orientations*, it will not matter whether we construct our plat from the braid $\sigma_{\mu_1}^{\epsilon_1} \cdots \sigma_{\mu_r}^{\epsilon_r}$ or from the braid $\sigma_{\mu_r}^{\epsilon_r} \cdots \sigma_{\mu_1}^{\epsilon_1}$, as one may be obtained from the other by turning the braid diagram upside down and over.

REMARK. We were unable to generalize our algorithm to the case of 3-manifolds which are represented by p-symmetric Heegaard splittings, where $p > 2$, because the "Smith conjecture" [34] for periods $p > 2$ has not been established. If this gap were filled, our algorithm could be adapted to an arbitrary p-symmetric Heegaard splitting. By a recent result of Montesinos [24] this would still be a proper subset of all closed orientable 3-manifolds.

As an application of our algorithm, we now give a new and very brief proof of a classical result: every genus 1 homology 3-sphere is S^3. The algorithm will also be applied later (in §7) to the more difficult cases of Heegaard splittings of genus 2.

COROLLARY 1. *Let* $M = Y_1 \cup_{\tau\Phi} Y_1'$ *be a 3-manifold which is defined by a genus 1 Heegaard splitting. Then* $M \sim S^3$ *if and only if* $H_1(M) = 0$.

PROOF OF COROLLARY 1. Let a, b (respectively a', b') be generators of $\pi_1 \partial Y_1$ (respectively $\pi_1 \partial Y_1'$), with $\tau(a) = a'$, $\tau(b) = b'$. Suppose that $\Phi: \partial Y_1 \rightarrow \partial Y_1$ has the action:

$$[\Phi]: a \rightarrow a^{n_{11}} b^{n_{12}},$$
(9)
$$b \rightarrow a^{n_{21}} b^{n_{22}}, \qquad n_{11}n_{22} - n_{12}n_{21} = 1.$$

Suppose also that the natural inclusion maps from $\pi_1 \partial Y_1 \rightarrow \pi_1 Y_1$ (respectively $\pi \partial Y_1' \rightarrow \pi_1 Y_1'$) have kernels generated by b (respectively b'). With these conventions, a straightforward application of van Kampen's theorem shows that $\pi_1 M$ admits the presentation:

(10) $$\langle a; a^{n_{21}} = 1 \rangle.$$

Thus $H_1(M) = 0$ if and only if $n_{21} = \pm 1$.

Our proof will be complete if we can show that $n_{21} = \pm 1$ implies $M \sim S^3$.

Let t_{c_1}, t_{c_2} be Dehn twists about representatives of $b \simeq c_1$, $a \simeq c_2$ respectively (cf. Figure 2). These twists induce the automorphisms

$$[t_{c_1*}]: a \rightarrow ab^{-1}, \qquad [t_{c_2*}]: a \rightarrow a,$$
(11)
$$b \rightarrow b, \qquad\qquad b \rightarrow ab.$$

An easy calculation shows that:

$$[\Phi] = [t_{c_1}]^{n_{22}-1} [t_{c_2}]^{n_{21}} [t_{c_1}]^{1-n_{11}}, \qquad n_{21} = \pm 1.$$

We may now apply our algorithm to decide whether $M \sim S^3$. To accomplish this, replace the map Φ by the 4-braid

$$\beta = \sigma_1^{n_{22}-1} \sigma_2^{n_{21}} \sigma_1^{1-n_{11}}, \qquad n_{21} = \pm 1,$$

and construct the 4-plat determined by β (see Figure 10). It is immediate that this plat defines the trivial knot type for every choice of the integers n_{11}, n_{22}, hence $M \sim S^3$. $\square$
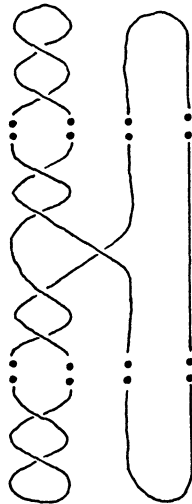

FIGURE 10. 4-plat corresponding to the genus 1 Heegaard splittings
$$\Phi = t_{c_1}^{n_{22}-1} t_{c_2}^{n_{21}} t_{c_1}^{1-n_{11}} \quad (n_{21} = +1)$$

6. Equivalence classes of genus two 3-manifolds and equivalence classes of 6-plat presentations of knots and links.

In this section we prove the theorem:

THEOREM 7. *Let $Y = Y_g$ be a solid handlebody of genus g, which is imbedded in E^3 in the manner indicated in Figure 4. Let $T: Y \longrightarrow Y$ be the involution which is induced by rotation about the z axis in Figure 4. Let $\alpha: Y \longrightarrow Y$ be a homeomorphism which has the property that $\alpha|\partial Y$ commutes with $T|\partial Y$. Then there is a homeomorphism $\beta: Y \longrightarrow Y$ such that $\beta|\partial Y = \alpha|\partial Y$, and β commutes with the involution T.*

We then use this theorem to show that the natural map from equivalence classes of 6-plat presentations of knots and links to equivalence classes of Heegaard splittings of genus two 3-manifolds is injective and surjective.

In what follows, let b denote the set $Y \cap z$ axis, which is the branch cover of the natural projection $p: Y \longrightarrow Y/\tau$. The symbol D denotes the 3-ball Y/τ, and $a = p(b)$ is the branch set.

LEMMA 9. *Let f be a proper map of D^2 into Y, such that $f(\partial D^2)$ is a simple closed curve in ∂Y which is disjoint from $Tf(\partial D^2)$. There is a proper map g of D into Y such that $g|\partial D^2 = f|\partial D^2$, g is an embedding, and $g(D^2) \cap Tg(D^2) = \varnothing$.*

PROOF OF LEMMA 9. The idea of the proof is to modify f so that $f(D^2)$ no longer intersects b, then project f to obtain a Dehn disc in $D^3 - a$ and use the Dehn lemma.

Assume $f(D^2)$ intersects b n times. We begin by finding a homotopy, $f \simeq f_4$, fixed on ∂D^2, such that f_4 intersects b n-times, and $f_4(D^2) \cap Tf_4(D^2)$ is a 1-complex in which each vertex is the boundary of exactly two or four 1-simplices. We do this via a sequence of homotopies, each fixed on ∂D^2. Each subsequent function in the following list will be assumed to have all the properties listed for the functions preceding it.

0. The function f_0 intersects b exactly n times and is simplicial with respect to some triangulation of D^2.

1. The function $f_1 \simeq f_0$ is nondegenerate, also $f_1(\text{1-skeleton}) \cap b = \varnothing$.

2. The function $f_2 \simeq f_1$. Also, if σ_1^2 and σ_2^2 are 2-simplices such that $f_2(\sigma_1^2) \cap b \neq \varnothing$ and $f_2(\sigma_2^2) \cap \neq \varnothing$ then $f_2(\sigma_1^2) \cap f_2(\sigma_2^2) = \varnothing$ (we may have to pass to a finer triangulation). Also, if σ_2 is a 2-simplex such that $f_2(\sigma_2) \cap b = \varnothing$, then $f_2(\sigma_2) \cap \tau f_2(\sigma_2) = \varnothing$.

3. The function $f_3 \simeq f_2$. If $f_3(\sigma^2) \cap b \neq \varnothing$, then $f_3(\sigma^2)$ is not perpendicular to b.

4. The function $f_4 \simeq f_3$, also $pf_4(D^2)$ is a normal Dehn disc in D^3. This means that singularities of $pf_4(D^2)$ do not occur on the boundary, and are either double points, along which two sheets cross, or triple points at which three sheets cut, or branch points. ($Q \in pf_4(D^2)$ is a branch point if a small sphere centered at Q cuts $pf_4(D^2)$ in a single nonsimple curve.) This may be accomplished by a small homotopy of pf_3 fixed on $\partial D^2 \cup$ neighborhood of $f_3^{-1}(b)$, which may be lifted to a homotopy of f_3. We need not homotopy f_3 at the points of $f_3^{-1}(b)$ because by 1, 2 and 3, all the singularities of $pf_3(D^2)$ in a neighborhood of a are of the right type.

By 2 and 3 each point P of $f_4(D^2) \cap Tf_4(D^2) \cap b$ has a neighborhood N such that $f_4(D^2) \cap Tf_4(D^2) \cap N$ is a line segment with P in its interior.

If $Q \in f_4(D^2) \cap Tf_4(D^2) \cap (Y - b)$, let N be a spherical neighborhood of Q such that $N \cap TN = \varnothing$. The map $p: (f_4(D^2) \cup Tf_4(D^2)) \cap N \to pf_4(D^2) \cap pN$ is a homeomorphism. Since p of any point of $f_4(D^2) \cap Tf_4(D^2) \cap N$ is a singular point of $pf_4(D^2) \cap N$ we see that $f_4(D^2) \cap Tf_4(D^2)$ is a 1-complex. If $p(Q)$ is a double point then it is the boundary of exactly two one-simplexes in $f_4(D^2) \cap Tf_4(D^2)$. If $p(Q)$ is a triple point then it is the boundary of exactly four one-simplexes in $f_4(D^2) \cap Tf_4(D^2)$. Now $p(Q)$ cannot be a branch point since if that were so the inverse image of the intersection of $pf_4(D^2)$ with a small sphere centered at $p(Q)$ would be a simple closed curve in either $f_4(N)$ or $Tf_4(N)$. Since $f_4(N) \cap Tf_4(N) = \varnothing$, Q could not belong to $f_4(D^2) \cap Tf_4(D^2)$. Thus $f_4(D)$ intersects b m times, $f_4(D^2) \cap Tf_4(D^2)$ is a 1-complex in which each

vertex is the boundary of either two or four 1-simplices.

Now we show that each component of $f_4(D^2) \cap Tf_4(D^2)$ contains an even number of points of b. Let K be a component of $f_4(D^2) \cap Tf_4(D^2)$ that contains a point in b. Since $f_4(D^2) \cap Tf_4(D^2)$ is invariant under T, and the point in b is left fixed by T, an easy connectedness argument shows K is invariant under T. Triangulate K so the map $p: K \longrightarrow K/T$ is simplicial. There are an even number of 1-simplices in K for this triangulation. We remove the midpoint of each 1-simplex and all the vertices. Note that the number of components is divisible by four, and the closure of each component contains exactly one vertex. Now count the number of components. This number equals $2 \times$ (# vertices in $K \cap b$) + $4 \times$ (# vertices in $p(K) - a$ that are the boundaries of two 1-simplices) + $8 \times$ (# vertices in $p(K)$ that are the boundaries of four 1-simplices). Thus the number of vertices in $K \cap b$ is even. Now we show how to modify f_4 so as to eliminate two intersections of $f_4(D)$ with b. Let K be a component of $f_4(D^2) \cap Tf_4(D^2)$ containing points of b and let δ be a simple arc in $p(K)$ containing two distinct points of a for endpoints and no points of a in its interior. $p^{-1}(\delta)$ is a simple closed curve in Y and $f_4^{-1}p^{-1}(\delta)$ contains a simple closed curve in D^2 bounding a disc we shall call E. We may choose a parametrization, $\gamma(s)$, $0 \leqslant s \leqslant 1$, of $f_4^{-1}p^{-1}(\delta)$ such that $f_4(\gamma(s)) = Tf_4(\gamma(1-s))$, $0 \leqslant s \leqslant 1$. Now let Q be a homeomorphism of the disc E such that $Q(\gamma(s)) = \gamma(1-s)$. Define the function $f_5: f_5(x) = f_4(x)$ if $x \notin E$, and $f_5(x) = Tf_4(Q(x))$ if $x \in E$. Note that f_5 is well defined since $f_5(\gamma(s)) = Tf_4(Q(\gamma(s))) = Tf_4(\gamma(1-s)) = f_4(\gamma(s))$. We have not altered the fact that $f_5(D) \cap b$ is n points. Now examine f_4 and f_5 in a neighborhood of $\gamma(0)$. Note that $\gamma(0)$ belongs to the interior of a simplex σ^2 which f_4 linearly embeds in Y, in such a way that $f_4(\sigma^2)$ is not perpendicular to b. Thus there is a unique line segment in $f_4(\sigma^2)$ that is perpendicular to b. This line segment and b define a plane P, which divides E^3 into two closed half-spaces H_1 and H_2. Examining the definition of f_5 we see that there is a neighborhood N of $\gamma(0)$ such that $f_5(N)$ lies entirely in one of these half-spaces. Now it is easy to find a homotopy $f_6 \simeq f_5$ fixed on the complement of the interior of N such that $f_6(N) \cap b = \varnothing$. In a similar manner we may find a neighborhood N' of $\gamma(\frac{1}{2})$ and a homotopy $f_7 \simeq f_6$ fixed on the complement of the interior of N' such that $f_7(N') \cap b = \varnothing$. In summary we have found a function f_7 such that $f_7|\partial D^2 = f|\partial D^2$, $f_7(D^2) \cap b$ contains two fewer points then $f(D^2) \cap b$, and if $f_7(\sigma^2) \cap b \neq \varnothing$ then $f_7(\sigma^2) \cap b$ belongs to the interior of $f_7(\sigma^2)$. Repeating the same process $n/2$ times we finally arrive at a function g_1 such that $g_1(D^2) \cap b = \varnothing$ and $g_1|\partial D^2 = f|\partial D^2$.

Since $g_1(\partial D^2)$ is a simple closed curve in ∂Y disjoint from $Tg_1(\partial D^2)$, it follows that $pg_1(D^2)$ is a singular Dehn disc in $D^3 - a$. By the Dehn lemma [26] there is a function $\tilde{g}$ such that $\tilde{g}(D^2)$ is a nonsingular disc in $D^3 - a$ and

$\tilde{g}|\partial D^2 = g_1|\partial D^2$. We may lift $\tilde{g}$ to a function $g: D^2 \longrightarrow Y$ such that g satisfies the conclusion of Lemma 9. $\square$

PROOF OF THEOREM 7. Now note that (D^3, a) is a 3-ball with $g + 1$ distinguished disjoint unknotted, unlinked, and proper arcs, where g is the genus of Y. Since $\alpha|\partial Y$ commutes with T it projects to a homeomorphism $\underline{\alpha}$ of the 2-sphere ∂D^3 that takes ∂a into itself. Let $\underline{D}_1^2, \ldots, \underline{D}_g^2$ be a set of disjoint proper discs in $D^3 - a$ such that $D^3 - \bigcup_{i=1}^g \underline{D}_i^2$ consists of $g + 1$ components each of which contains one distinguished arc. Let D_i^2 be one of the two lifts of $\underline{D}_i^2$ to Y. This function α restricted to D_i^2 satisfies the hypothesis of the preliminary lemma. Hence there is a disc E_i^2 such that $\alpha(\partial D_i^2) = \partial E_i^2$ and $E_i^2 \cap TE_i^2 = \varnothing$. Let $\underline{E}_i^2 = p(E_i^2)$. In general the set $\{\underline{E}_i^2\}$ will not be pairwise disjoint. However, we may systematically eliminate intersections and replace the $\{\underline{E}_i^2\}$ with a set of pairwise disjoint proper discs $\{\underline{F}_i^2\}$ such that $\underline{F}_i^2 = \underline{\alpha}(\partial \underline{D}_i^2)$. To see how to do that suppose $\underline{E}_i^2$ and $\underline{E}_j^2$ intersect. We may suppose they are in general position and so they intersect in a set of circles. Let c be a circle of intersection such that c bounds a disc G^2 in $\underline{E}_i^2$ not containing any other intersection points in $\underline{E}_i^2$ and c bounds a disc H^2 in E_j^2. Since H^2 and G^2 do not intersect they bound a ball J^3 in $D^3 - a$. There is an isotopy φ supported on a regular neighborhood of the ball J^3 such that φ "pushes" G^2 through H^2. If we replace the disc $\underline{E}_i^2$ with $\varphi(\underline{E}_i^2)$ we reduce the number of circles of intersection by at least one. We can continue to do this, being careful at each stage not to introduce new intersections, until all the intersections are eliminated. Now extend $\underline{\alpha}$ to $\partial D^3 \cup \bigcup_{i=1}^g \underline{D}_i^2$ by letting $\underline{\alpha}$ map $\underline{D}_i^2$ homeomorphically onto $\underline{F}_i^2$. Since $\partial \underline{D}_i^2$ divided ∂D^3 into $g + 1$ regions $R_0^2, \ldots, R_g^2$ each containing two points of ∂a, the regions $\underline{\alpha}(R_0^2), \ldots, \underline{\alpha}(R_g^2)$ also each contain two points of ∂a. Thus we have $g + 1$ regions $S_0^3, \ldots, S_g^3$ each of which is a ball containing one arc of a and another $g + 1$ regions $T_0^3, \ldots, T_g^3$, each of which is a ball containing one arc of a. We also have a homeomorphism $\underline{\alpha}: \partial S_i^3 \longrightarrow \partial T_i^3$ that preserves the set ∂a. Now we extend $\underline{\alpha}$ in any way at all to a homeomorphism $\underline{\alpha}: S_i \longrightarrow T_i$ and, making use of the fact that any two unknotted arcs in a ball are isotopic, find an isotopy $\underline{\psi}$ such that $\underline{\psi\alpha}$ is a homeomorphism sending a into a and $\underline{\psi\alpha}|\partial D^3 = \underline{\alpha}$. Now we may lift $\underline{\psi\alpha}$ to get the desired homeomorphism β of Y. This completes the proof of Theorem 7. $\square$

We recall from §4 that a $2b$ plat is a representation of a knot or link of form $(D, A_b) \cup_{\tau\delta} (D', A_b')$ where D is a 3-ball containing A_b, a set of b unlinked unknotted arcs, (D', A_b') is another copy of (D, A_b), τ is the map identifying D with D', restricted to ∂D, and δ is a homeomorphism of ∂D leaving the points in ∂A_b invariant as a set.

We say the plat $(D, A_b) \cup_{\tau\delta} (D', A_b')$ is equivalent to the plat $(D, A_b) \cup_{\tau\psi}$ (D', A_b') if there are orientation-preserving homeomorphisms α, α' of (D, A_b) and (D', A_b') respectively, such that $(\alpha'|\partial D')\tau\delta = \tau\psi(\alpha|\partial D)$.

Recall from §4 that $Y_g \cup_{\tau_\mu} Y_g'$ defines a Heegaard splitting of a 3-manifold; $\pi: (Y_g, L_b) \to (D, A_b)$ and $\pi': (Y_g', L_g') \to (D', A_b')$ are b-fold branched cyclic coverings, with branch covers the sets L_b and L_b' respectively. We say the Heegaard splittings $Y_g \cup_{\tau_\mu} Y_g'$ and $Y_g \cup_{\tau'\nu} Y_g'$ are equivalent if there are orientation preserving homeomorphisms $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ of Y_g and Y_g' respectively such that $(\widetilde{\alpha}'|\partial Y_g')\tau\mu = \tau\nu(\widetilde{\alpha}|\partial Y_g)$.

Let $F_{b,g}$ be the map that assigns to an equivalence class of plats $\{(D, A_b) \cup_{\tau\delta} (D', A_b')\}$ the equivalence class of Heegaard splittings $\{Y_g \cup_{\tau_\mu} Y_g'\}$ where μ is the lift to ∂Y_g of the homeomorphism δ of ∂D. $F_{b,g}$ is well defined because homeomorphisms α, α' of (D, A_b) and (D', A_b') lift to homeomorphisms $\widetilde{\alpha}$, $\widetilde{\alpha}'$ of Y_g, Y_g' and the commutative diagram defining equivalence downstairs lifts to a commutative diagram defining equivalence upstairs. At this point, we specialize to the case $b = 3$, $g = 2$ and denote $F_{3,2}$ simply by F. We wish to show that F is injective and surjective. (For $g \geqslant 3$, $F_{b,g}$ is probably not surjective as there are isotopy classes of homeomorphisms of ∂Y_g that have no representative commuting with the covering transformation.)

THEOREM 8. *The natural map F from equivalence classes of 6-plats to equivalence classes of genus 2 Heegaard splittings is injective and surjective.*

PROOF OF THEOREM 8. Surjectivity follows easily from the fact that any homeomorphism of ∂Y_2 is isotopic to one commuting with the covering transformation in the covering $\pi: \partial Y_2 \to \partial D$.

Suppose $Y_2 \cup_{\tau_\mu} Y_2' \equiv Y_2 \cup_{\tau\nu} Y_2'$. Then there are orientation-preserving homeomorphisms $\widetilde{\alpha}$, $\widetilde{\alpha}'$ of Y_2 and Y_2' such that $(\widetilde{\alpha}'|\partial Y_2')\tau\mu = \tau\nu(\widetilde{\alpha}|\partial Y_2)$. Since the genus is 2, $\widetilde{\alpha}|\partial Y_2$ is isotopic to a homeomorphism $\widetilde{\alpha}_1$ of ∂Y_2 where $\widetilde{\alpha}_1$ commutes with the covering transformation. We can extend the isotopy to all of Y_2. Call the isotopy $\widetilde{\alpha}_t$. Then we can use $\widetilde{\alpha}_t$ to define an isotopy of $\widetilde{\alpha}'|\partial Y_2'$, defining $\widetilde{\alpha}_t'|\partial Y_2'$ by the equation $(\widetilde{\alpha}_t'|\partial Y_2')\tau\mu = \tau\nu(\widetilde{\alpha}_t|\partial Y_2)$. This is possible since μ, ν, τ and $\widetilde{\alpha}_t$ are all homeomorphisms. Thus $(\widetilde{\alpha}_1'|\partial Y_2')\tau\mu = \tau\nu(\widetilde{\alpha}_1|\partial Y_2)$. Now $\widetilde{\alpha}_1'$ must be fibre preserving since $\widetilde{\alpha}_1, \mu, \nu$ and τ are all fibre preserving. Thus $\widetilde{\alpha}_1'$ commutes with the covering transformation. Since the isotopy $\widetilde{\alpha}_t'$ may be extended to an isotopy of Y_2' we could use the homeomorphisms $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_1'$ in place of the homeomorphisms $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ to define the equivalence of Heegaard splittings. Thus we may assume that $\widetilde{\alpha}|\partial Y_2$ and $\widetilde{\alpha}'|\partial Y_2'$ commute with the covering transformation. By Theorem 7 we may choose extensions of $\widetilde{\alpha}|\partial Y_2$ and $\widetilde{\alpha}'|\partial Y_2'$ that commute with the covering transformation in the coverings $\pi: Y_2 \to D$ and $\pi': Y_2' \to D'$. Thus we may assume $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ had this property to begin with. Now $\widetilde{\alpha}$, and $\widetilde{\alpha}'$ project to homeomorphisms α and α' with $(\alpha'|\partial D')\tau\delta = \tau\psi(\alpha|\partial D)$ so that $(D, A_3) \cup_{\tau\delta} (D', A_3') \equiv (D, A_3) \cup_{\tau\psi} (D', A_3')$. $\square$

Theorem 8 sheds light on a question raised by Papakyriakopoulos [28,

p. 330]. He asks whether any two Heegaard splittings of a 3-manifold are equiv-
alent.[9] (His definition of equivalence is somewhat different from ours in that
he allows homeomorphisms of $M = Y_g \cup_{\tau\mu} Y'_g$ onto $M = Y_g \cup_{\tau\nu} Y'_g$ that map
Y_g onto Y'_g. We could have used a similar definition of equivalence (suitably
modifying our definition of equivalence of plats) and obtained a theorem similar
to Theorem 8, but the exposition would have been more complicated.) Theorem
8 shows that, for genus 2, the existence of two nonequivalent 6-plat presentations
of a prime knot or link would imply a negative answer to Papakyriakopoulos'
question.

Theorem 8 and the results of §4 indicate that the following question is
interesting, and may be easier to answer than the analogous question about 3-
manifolds: Are any two "Heegaard splittings" of a prime knot equivalent?[*] i.e.
let $D \cup_{\tau\psi} D'$ be a Heegaard splitting of S^3. Let A_b be a collection of b unknotted
and unlinked arcs in D with endpoints on ∂D, and let $A'_b = \tau(A_b)$. Then $L_\psi =
A_b \cup_{\tau\psi} A'_b$ is a link in $S^3 = D \cup_{\tau\psi} D'$, represented as a $2b$-plat.[10] Let B_{2b}
be the group of all autohomeomorphisms of $(\partial D, A_b)$, and let G be the subgroup
of those autohomeomorphisms which extend to (D, A_b).

Question. Suppose that L is prime. Are all $2b$-plat representatives of the
link type of L_ψ obtainable as $A_b \cup_{\tau\varphi} A'_b$, where $\varphi = \psi_1 \cdot \psi \cdot \psi_2$, with $\psi_1, \psi_2
\in G$?

7. **Presentations for** $\pi_1 S^3$. If a 3-manifold M is defined by a Heegaard
splitting $Y_g \cup_{\tau\Phi} Y'_g$, and if the action of the defining homeomorphism on $\pi_1(\partial Y_g)$
is known, then there is a very easy algorithm which allows us to determine a pre-
sentation for $\pi_1(M)$. This algorithm will be presented in Lemma 10. We will then
study the question: If $M \sim S^3$, what types of "canonical" presentations can we
expect for $\pi_1(M) = 1$? To attack this question, we first characterize algebraically
the class of elements in the mapping class group of the surface ∂Y_g which have
the property that they define Heegaard splittings of S^3 (Lemma 11). It will then
be established in Theorem 9 that $\pi_1 S^3$ can, in fact, have very "bad" presentations
which arise in a natural manner in connection with Heegaard splittings.

We will end this section with an observation which relates in particular to
2-symmetric Heegaard splittings: If $M = Y_g \cup_{\tau\Phi} Y'_g$, is 2-symmetric, then $\pi_1 M$

---

[9] For composite manifolds the question was answered in the negative by Engmann,
*Nicht-homöomorphe Heegaard-Zerlegungen vom Geschlect 2 der zusammenhängenden
Summe zweier Linsenräume*, Abh. Math. Sem. Univ. Hamburg **35** (1971), 33–38.

Note added in proof. This question has been answered in the negative. Theorem 8
was used by Birman, Gonzalez-Acuña, and Montesinos to exhibit two inequivalent Heegaard
splittings of a prime genus two Z homology sphere, *Heegaard splittings of prime 3-manifolds
are not unique* (to appear).

[10] The autohomeomorphism ψ defines the $2b$-braid which determines our plat, as
in equations (6) and (7) of §5.

has a "symmetric" presentation (Theorem 10).

To determine a presentation for the fundamental group of a 3-manifold which is defined by a Heegaard splitting $Y_g \cup_{\tau\Phi} Y'_g$, we first choose canonical generators for $\pi_1(\partial Y_g)$. These generators will be denoted by $w_1, \ldots, w_{2g}$ and will be assumed to satisfy the single defining relation

(12) $$\prod_{k=1}^{g} w_k w_{g+k} w_k^{-1} w_{g+k}^{-1} = 1.$$

Let $\vec{u} = \{u_1, \ldots, u_g\}$ be a free basis for $\pi_1 Y_g$, where if $i: \partial Y_g \rightarrow Y_g$ denotes the inclusion map, and $i_*: \pi_1 \partial Y_g \rightarrow \pi_1 Y_g$ denotes the homomorphism induced by i, then

(13)
$$i_*: w_j \rightarrow u_j, \qquad 1 \leqslant j \leqslant g,$$
$$i_*: w_j \rightarrow 1, \qquad g+1 \leqslant j \leqslant 2g.$$

Let $w'_j, u'_j, i', i'_*, \vec{u}'$ be defined in an analogous manner for the handlebody Y'_g. Suppose also that the homeomorphism $\tau: Y_g \rightarrow Y'_g$ has the effect:

(14) $$\tau_*: w_j \rightarrow w'_j, \qquad 1 \leqslant j \leqslant 2g.$$

The defining homomorphism $\Phi: \partial Y_g \rightarrow \partial Y_g$ will be assumed to induce an automorphism $\Phi_*: \pi_1 \partial Y_g \rightarrow \pi_1 \partial Y_g$ which acts as follows:

(15)
$$\Phi_*: w_j \rightarrow U_j(w_1, \ldots, w_{2g}),$$
$$\Phi_*^{-1}: w_j \rightarrow U'_j(w_1, \ldots, w_{2g}), \qquad 1 \leqslant j \leqslant 2g.$$

Then we claim

LEMMA 10. $\pi_1(Y_g \cup_{\tau\Phi} Y'_g)$ admits the following two presentations:

(i) $\langle u'_1, \ldots, u'_g; U_k(u'_1, \ldots, u'_g, 1, \ldots, 1), g+1 \leqslant k \leqslant 2g\rangle$.

(ii) $\langle u_1, \ldots, u_g; U'_k(u_1, \ldots, u_g, 1, \ldots, 1), g+1 \leqslant k \leqslant 2g\rangle$.

PROOF OF LEMMA 10. The presentations of Lemma 10 are obtained by a straightforward application of the van Kampen theorem. □

The presentation (i) (or (ii)) of Lemma 10 defines a homomorphism from the free group F_g of rank g with free basis $\vec{u}'$ (or $\vec{u}$) onto $\pi_1(Y_g \cup_{\tau\Phi} Y'_g)$. A natural question to ask, if one is attempting to understand the relationship between the algebraic and geometric properties of 3-manifolds, is whether the trivial group always appears in these particular "canonical" presentations in a "nice" way. For example, an obvious first guess would be that perhaps $\pi_1(Y_g \cup_{\tau\Phi} Y'_g) = 1$ only if the set $\vec{U} = \{U_k(u'_1, \ldots, u'_g, 1, \ldots, 1)\}$ are a set of primitive elements in the free group F_g. This is false, as will be seen below. (This appears to be a known result, although we were unable to locate an explicit reference in the published literature.) More subtly, it has been conjectured by W. Haken that if $Y_g \cup_{\tau\Phi} Y'_g$

is S^3, then at least one of the presentations (i) or (ii) of Lemma 10 has the property that the relators are a set of conjugate primitive elements. This conjecture is, unfortunately, laid to rest below:

THEOREM 9. *There exists a genus 2 Heegaard splitting of S^3 which has the property that neither of the sets $\{U_3(u_1', u_2', 1, 1), U_4(u_1', u_2', 1, 1)\}$ or $\{U_3'(u_1, u_2, 1, 1), U_4'(u_1, u_2, \cdot 1, 1)\}$ is a conjugate primitive set.*

PROOF. Our proof begins with the establishment of several lemmas, the first of which may have some interest in its own right.

LEMMA 11. *Let $Y \cup_{\tau \Phi_0} Y'$ be any "standard" Heegaard splitting of S^3, and let $Y_g \cup_{\tau \Phi} Y_g'$ be an arbitrary Heegaard splitting. Then $Y_g \cup_{\tau \Phi} Y_g'$ is S^3 if and only if*

$$(16) \qquad\qquad \Phi = \Phi_1 \Phi_0 \Phi_2,$$

when Φ_1 and Φ_2 are surface homeomorphisms which extend to homeomorphisms of Y_g.

PROOF OF LEMMA 11. Lemma 11 will be shown to follow from results of Waldhausen in [33]. It was established by Waldhausen that every "minimal" Heegaard splitting of S^3 has genus 0. (The reader is referred to [33] for the definition of a minimal Heegaard splitting.) An induction based on genus then implies that if $Y_g \cup_{\tau \Phi} Y_g' \sim S^3$, then there exists a homeomorphism

$$(17) \qquad\qquad h: Y_g \cup_{\tau \Phi} Y_g' \longrightarrow Y_g \cup_{\tau \Phi_0} Y_g'$$

which maps $Y_g \rightarrow Y_g$, $Y_g' \rightarrow Y_g'$, and is consistent on the boundary, that is, the diagram

$$
\begin{array}{ccc}
\partial Y_g & \xrightarrow{\ \tau_* \Phi\ } & \partial Y_g' \\[4pt]
{\scriptstyle h|\partial Y_g} \Big\downarrow & & \Big\downarrow {\scriptstyle h|\partial Y_g'} \\[4pt]
\partial Y_g & \xrightarrow{\ \tau_* \Phi_0\ } & \partial Y_g'
\end{array}
$$

is commutative. Thus

$$(18) \qquad\qquad \Phi = \tau_*^{-1}(h|\partial Y_g')\tau_* \Phi_0(h|\partial Y_g).$$

Setting $\Phi_1 = \tau_*^{-1}(h|\partial Y_g')\tau_*$ and $\Phi_2 = h|\partial Y_g$, we then obtain the desired result.

To establish sufficiency, suppose that $M = Y_g \cup_{\tau \Phi} Y_g'$, where Φ satisfies condition (18) of Lemma 11. Let h_1, h_2 be extensions of Φ_1, Φ_2 respectively to Y_g. We may then define a homeomorphism $h: Y_g \cup_{\tau \Phi} Y_g' \rightarrow Y_g \cup_{\tau \Phi_0} Y_g'$ by the rule $h|Y_g = h_2$ and $h|Y_g' = \tau^{-1}h_1\tau$. Condition (18) ensures that h is well

defined on ∂Y_g. Since $Y_g \cup_{\tau \Phi_0} Y'_g \sim S^3$, it then follows that $Y_g \cup_{\tau \Phi} Y'_g$ is also S^3. This proves Lemma 11. $\square$

Our next two lemmas are very easy:

LEMMA 12. *Let $\vec{U}$ be a set of g elements of F_g. Let $\beta \in \mathrm{Aut}\ F_g$. Then $\beta(\vec{U})$ is a conjugate primitive set only if $\vec{U}$ is a conjugate primitive set.*

PROOF OF LEMMA 12. This follows immediately from the definition of a primitive set and a conjugate primitive set. $\square$

LEMMA 13. *Let $\Phi_1 : \partial Y_g \rightarrow \partial Y_g$ be a homeomorphism which extends to Y_g. Suppose that the action of Φ_{1_*} is given by*

$$\Phi_{1_*} : w_j \rightarrow R_j(w_1, \ldots, w_{2g}), \quad 1 \leqslant j \leqslant 2g.$$

Then

(19) $\qquad R_j(u_1, \ldots, u_g, 1, \ldots, 1) = 1 \quad \text{for each } g + 1 \leqslant j \leqslant 2g$

and

(20) $\qquad \{R_j(u_1, \ldots, u_g, 1, \ldots, 1), 1 \leqslant j \leqslant g\}$

are a basis for the free group with free basis $u_1, \ldots, u_g$.

PROOF OF LEMMA 13. Let $i_* : \pi_1(\partial Y_g) \rightarrow \pi_1 Y_g$ be the homomorphism induced by the inclusion map. A necessary and sufficient condition for Φ_1 to extend to X_g is that Φ_{1_*} leave $\ker i_*$ invariant [22], and since $\ker i_*$ is the normal closure of $\{w_{g+1}, \ldots, w_g\}$ in $\pi_1(\partial Y_g)$, it follows that (19) must be true, and also Φ_{1_*} induces an automorphism $\Phi_{1_{**}} : \pi_1 Y_g \rightarrow \pi_1 Y_g$, defined by

(21) $\qquad \Phi_{1_{**}} : u_j \rightarrow R_j(u_1, \ldots, u_g, 1, \ldots, 1), \quad 1 \leqslant j \leqslant g.$

This proves Lemma 13. $\square$

We are now ready to prove Theorem 9. We begin by producing a genus 2 Heegaard splitting of S^3 which has the property that, in the presentation (i), neither of the relators is conjugate to a primitive element. Explicitly,[11] let

(22) $\qquad \Phi = t_4^{-1} t_5^2 t_4^{-2} t_2^{-2} t_1 t_2^{-1} t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} t_5 t_4 t_3 t_2 t_1,$

where t_1, t_2, t_3, t_4, t_5 are the Lickorish twists about the curves c_1, c_2, c_3, c_4, c_5 defined in Figure 4 (cf. §3). To see that $Y_2 \cup_{\tau \Phi} Y'_2$ is S^3, we note that the algorithm given in §4 is applicable, because every genus 2 Heegaard splitting is 2-symmetric. We thus construct the 6-plat corresponding to Φ (Figure 11). Since the plat defines the trivial knot type, it follows from Theorem 6 that $Y_2 \cup_{\tau \Phi} Y'_2 \sim S^3$.

---

[11] This example is a modification of an example which was shown to the authors by C. Miller, who attributes it to Reidemeister. We were unable to find a reference in Reidemeister's works.
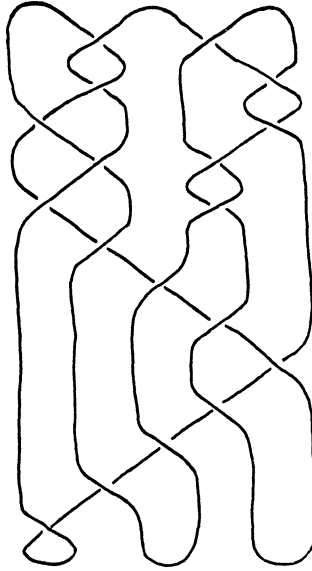
FIGURE 11. Plat corresponding to the genus 2 Heegaard splitting
$$\Phi = t_4^{-1} t_5^2 t_4^{-2} t_2^{-2} t_1 t_2^{-1} t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} t_5 t_4 t_3 t_2 t_1$$

To compute the action of Φ_* on $\pi_1(\partial Y_2)$ we first observe that the twists $t_1, \ldots, t_5$ induce the following automorphisms of $\pi_1(\partial Y_2)$:

(23)

$$
\begin{aligned}
&t_{1_*}: w_1 \longrightarrow w_1 w_3^{-1}, &\qquad& t_{2_*}: w_3 \longrightarrow w_3 w_1, \\
&t_{3_*}: w_1 \longrightarrow w_1 w_3^{-1} w_2 w_4 w_2^{-1}, &\qquad& t_{4_*}: w_4 \longrightarrow w_4 w_2, \\
&\phantom{t_{3_*}:}\; w_2 \longrightarrow w_2 w_4^{-1} w_2^{-1} w_3 w_2, &\qquad& t_{5_*}: w_2 \longrightarrow w_2 w_4^{-1}, \\
&\phantom{t_{3_*}:}\; w_3 \longrightarrow w_2 w_4^{-1} w_2^{-1} w_3 w_2 w_4 w_2^{-1}, & &
\end{aligned}
$$

where every generator which is not listed explicitly is kept fixed. The action of Φ_* on the generators w_3, w_4 of $\pi_1 \partial Y_2$ may now be computed, using equations (22) and (23):

(24)

$$
\begin{aligned}
\Phi_*: w_3 &\longrightarrow w_2^2 w_4^{-1} w_2 w_4^{-1} w_1 w_3 w_1^{-3}, \\
w_4 &\longrightarrow w_1^3 w_3^{-1} w_2 w_4 w_2^{-2} w_4 w_2^{-1} w_4 w_2^{-2}.
\end{aligned}
$$

Thus, from Lemma 10, $\pi_1(Y_2 \cup_{\tau\Phi} Y_2')$ admits the presentation

(25)   (i) $\langle u_1', u_2': u_2'^3 u_1'^{-2}, u_1'^3 u_2'^{-4} \rangle$.

One verifies immediately by a standard test (see [20, Corollary N4]) that neither $u_2'^3 u_1'^{-2}$ nor $u_1'^3 u_2'^{-4}$ is conjugate to a primitive element in the free group with free basis u_1', u_2'. If the presentation (ii) had a similar property, we would be done: however if one does a similar calculation, replacing the automorphism Φ_* by Φ_*^{-1}, one discovers that the words $U_3'(u_1, u_2, 1, 1)$ and $U_4'(u_1, u_2, 1, 1)$ are

in fact primitive. Hence we need to do some more work.

Let $\Phi_0 \colon \partial Y_2 \longrightarrow \partial Y_2$ be the twist product

(26) $$\Phi_0 = t_4 t_5 t_4 t_1 t_2 t_1.$$

Applying the algorithms of §5, we verify that the 6-plat determined by the 6-braid $\sigma_4 \sigma_5 \sigma_4 \sigma_1 \sigma_2 \sigma_1$ defines the trivial knot type, hence $Y_2 \cup_{\tau \Phi_0} Y_2' \sim S^3$. Therefore, by Lemma 11, it follows that $\Phi = \Phi_1 \Phi_0 \Phi_2$, where Φ_1 and Φ_2 extend to Y_2. We will now show that if

(27) $$\Psi = \Phi_2^{-1} \Phi_0 \Phi_2,$$

then $Y_2 \cup_{\tau \Psi} Y_2'$ is S^3, and also $\pi_1(Y_2 \cup_{\tau \Psi} Y_2')$ admits two canonical presentations (i) and (ii), as described in Lemma 10, which have the property that the relators in each of these presentations fail to be a conjugate primitive set. This will provide the example needed to complete the proof of our theorem.

To see that $Y_2 \cup_{\tau \Psi} Y_2'$ is S^3, note that Φ_2 extends to Y_2, hence also Φ_2^{-1} extends to Y_2, hence by Lemma 11 it must be true that $Y_2 \cup_{\tau \Psi} Y_2' \sim S^3$.

To see that our canonical presentation has the required properties, suppose that the action of Φ_{1_*} and Φ_{2_*} on $\pi(\partial Y_2)$ is given by

(28)
$$\Phi_{1_*} \colon w_j \longrightarrow R_j(w_1, w_2, w_3, w_4)$$
$$\Phi_{2_*} \colon w_j \longrightarrow S_j(w_1, w_2, w_3, w_4)$$
$$\Phi_{2_*}^{-1} \colon w_j \longrightarrow T_j(w_1, w_2, w_3, w_4), \quad 1 \leqslant j \leqslant 4.$$

Note that the action of Φ_{0_*} and $\Phi_{0_*}^{-1}$ may be computed from equations (23) and (26) to be

(29)
$$\Phi_{0_*} \colon w_1 \longrightarrow w_3^{-1}, \qquad \Phi_{0_*}^{-1} \colon w_1 \longrightarrow w_1 w_2 w_1^{-1},$$
$$w_2 \longrightarrow w_4^{-1}, \qquad w_2 \longrightarrow w_2 w_4 w_2^{-1},$$
$$w_3 \longrightarrow w_3 w_1 w_3^{-1}, \qquad w_3 \longrightarrow w_1^{-1},$$
$$w_4 \longrightarrow w_4 w_2 w_4^{-1}, \qquad w_4 \longrightarrow w_2^{-1}.$$

Observe that, since Φ_1 and Φ_2 (and hence also Φ_2^{-1}) extend to Y_2, it follows from Lemma 11 that

(30) $R_j(w_1, w_2, 1, 1) = S_j(w_1, w_2, 1, 1) = T_j(w_1, w_2, 1, 1) = 1$ if $j = 3, 4$.

Using this fact, and composing automorphisms to compute the action of Ψ_*, we then find that the two canonical presentations for $\pi_1(Y_2 \cup_{\tau \Psi} Y_2')$ are

(i)
$$\langle u_1', u_2' \colon S_3(1, 1, T_1(u_1', u_2', 1, 1), T_2(u_1', u_2', 1, 1)),$$
$$S_4(1, 1, T_1(u_1', u_2', 1, 1), T_2(u_1', u_2', 1, 1)) \rangle \rangle.$$

(ii)
$$\langle u_1, u_2 : S_3(1, 1, T_1^{-1}(u_1, u_2, 1, 1), T_2^{-1}(u_1, u_2, 1, 1)),$$
$$S_4(1, 1, T_1^{-1}(u_1, u_2, 1, 1), T_2^{-1}(u_1, u_2, 1, 1))\rangle.$$

Note also that the relators in the canonical presentation (i) for $\pi_1(Y_2 \cup_{\tau\Phi} Y_2')$ are $u_2'^3 u_1'^{-2}$ and $u_1'^3 u_2'^{-4}$. Hence, composing automorphisms to obtain the action of Φ_*, we must have

(31) $$\langle u_2'^3 u_1'^{-2} = S_3(1, 1, R_1(u_1', u_2', 1, 1), R_2(u_1', u_2', 1, 1))\rangle,$$

(32) $$\langle u_1'^3 u_2'^{-4} = S_4(1, 1, R_1(u_1', u_2', 1, 1), R_2(u_1', u_2', 1, 1))\rangle.$$

As observed before, these are *not* a conjugate primitive set.

Now, since Φ_1 and Φ_2 (and hence also Φ_2^{-1}) extend to Y_2, it follows from Lemma 13 that the set $\{R_1(u_1', u_2', 1, 1), R_2(u_1', u_2', 1, 1)\}$ and the set $\{T_1(u_1', u_2', 1, 1), T_2(u_1', u_2', 1, 1)\}$ are each bases for the free group F_2 with free basis u_1', u_2'.

Hence the relators in the presentation (i) must be the images of $u_2'^3 u_1'^{-2}$ and $u_1'^3 u_2'^{-4}$ under a change in basis. Also $\{T_1^{-1}(u_1, u_2, 1, 1), T_2^{-1}(u_1, u_2, 1, 1)\}$ are a basis for F_2, hence the relators in the presentation (ii) are the image of $u_2^3 u_1^{-2}$ and $u_1^3 u_2^{-4}$ under a change in basis. But then Lemma 12 implies that the relators in the presentations (i) and (ii) are not conjugate primitive sets. This completes the proof of Theorem 9.   □

We conclude this section with a result which relates to the special case of 2-symmetric Heegaard splittings of S^3.

THEOREM 10. *Let $Y_g \cup_{\tau\Phi} Y_g'$ be a 2-symmetric 3-manifold. Then the canonical presentations* (i) *and* (ii) *of Lemma* 13 *have the special property that any relators in the basis elements $u_1', \ldots, u_g'$ or $u_1, \ldots, u_g$, if read backwards, remain relators.*

PROOF OF THEOREM 10. The presentations (i) and (ii) of Lemma 10 were obtained after a particular choice of basis elements. These basis elements have the special property that each u_i' and each u_i is mapped to its inverse by the involution T_* (acting on $\pi_1(Y_g \cup_{\tau\Phi} Y_g')$). Since T_* is an automorphism of $\pi_1(Y_g \cup_{\tau\Phi} Y_g')$, our theorem is established.   □

8. Poincaré's conjecture is true for 2-fold coverings of S^3 which are branched over closed 3-braids.

Our object in this section will be to establish

THEOREM 11. *If M is a 2-fold covering of S^3 branched over a link defined by a closed 3-braid, then $\pi_1 M = 1$ implies $M \sim S^3$.*

REMARK. By Lemma 1, every 3-braid knot is a knot of at most 3 bridges, and by Theorem 5 every 2-fold covering of S^3 branched over a 3-bridge knot admits a genus 2 Heegaard splitting. Hence Theorem 11 may be regarded as a

first step in an attempt to establish the "genus 2 Poincaré conjecture".

PROOF OF THEOREM 11. Let β be an element in the 3-string Artin braid group B_3, let $\hat{\beta}$ denote the closed braid determined by β (cf. §2), and let M_β denote the 2-fold covering of S^3 which is branched over $\hat{\beta}$. Let $A_\beta(x)$ denote the Alexander polynomial of $\hat{\beta}$. If M_β is simply-connected, then by a result of Fox [13, p. 179] it must be true that $A_\beta(-1) = 1$. We will delineate the class of 3-braids which have this property, and show that the only possibility, if $\pi_1 M_\beta = 1$, is that $M_\beta \sim S^3$.

We consider two homomorphic images of the group B_3. The first, $\widetilde{B}_3$, is the Burau matrix group [9], [6], [21], a group of 2×2 matrices over the group ring of an infinite cyclic group, say $\langle x \rangle$. The homomorphism $\eta\colon B_3 \to \widetilde{B}_3$ may be defined by

$$(33) \qquad \widetilde{\sigma}_1 = \eta(\sigma_1) = \begin{bmatrix} -x & 0 \\ x & 1 \end{bmatrix}, \quad \widetilde{\sigma}_2 = \eta(\sigma_2) = \begin{bmatrix} 1 & 1 \\ 0 & -x \end{bmatrix},$$

where σ_1 and σ_2 are standard generators of B_3 (cf. §2). It was established by Burau [9] that these matrices are closely related to Alexander matrices of closed braids; explicitly, if $\widetilde{\beta} = \eta(\beta)$, then

$$(34) \qquad A_\beta(x) = \det|\widetilde{\beta} - I|/(1 + x + x)^2, \quad .$$

where I is the 2×2 identity matrix.

Now, we are interested in $A_\beta(-1)$, hence we may define a second homomorphism $\xi\colon \widetilde{B}_3 \to \hat{B}_3$, defining $\hat{B}_3$ to be the matrix group obtained by replacing the indeterminate "x" by "-1". The group $\hat{B}_3$ is easily recognized to be the group of 2×2 matrices with integral entries and determinant $+1$, for on setting $x = -1$ in the matrices $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ we obtain

$$(35) \qquad \hat{\sigma}_1 = \xi(\widetilde{\sigma}_1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \hat{\sigma}_2 = \xi(\widetilde{\sigma}_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which are known to generate the latter group [10, p. 85]. Defining relations in $\hat{B}_3$ are [10]:

$$(36) \qquad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2,$$

$$(37) \qquad (\sigma_1 \sigma_2)^6 = 1.$$

Since relation (36) lifts to the single defining relation in B_3, and since $(\sigma_1 \sigma_2)^6$ is known to belong to the center of B_3, it then follows that

$$(38) \qquad \ker(\xi\eta) = \text{cyclic subgroup of } B_3 \text{ generated by } (\sigma_1 \sigma_2)^6.$$

To delineate the class of 3-braids which have the property that $A_\beta(-1) = 1$, suppose that

(39) $\hat{\beta} = \xi\eta(\beta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc = 1, a, b, c, d$ integers.

Equation (34) then implies that

(40) $A_\beta(-1) = \begin{vmatrix} a - 1 & b \\ c & d - 1 \end{vmatrix} = 2 - a - d.$

Since $A_\beta(x)$ is only defined up to multiplication by an arbitrary power of x, we must consider two cases, namely $2 - a - d = \pm 1$, or $a + d = 1$ or 3. It is proved in [21, Lemma 2.3] that if trace $\hat{\beta} = 1$ or 3, then $\hat{\beta}$ is conjugate to one of the elements $\sigma_1\sigma_2$, $\sigma_2^{-1}\sigma_1^{-1}$, or $\sigma_2\sigma_1^{-1}$. Using equation (38) above, we then find that $A_\beta(-1) = 1$ if and only if β is conjugate in B_3 to one of the elements $(\sigma_1\sigma_2)^{6m+1}$, $(\sigma_1\sigma_2)^{6m-1}$, $\sigma_2\sigma_1^{-1}(\sigma_1\sigma_2)^{6m}$, $m = 0, \pm 1, \pm 2, \ldots$. Since conjugate braids determine equivalent closed braids (in the sense of link equivalence), we have thus proved that $A_\beta(-1) = 1$ implies

(41)
$$\beta = \beta_m^{(1)} = (\sigma_1\sigma_2)^{6m \pm 1} \quad \text{or} \quad \beta_m^{(2)} = (\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{6m},$$

$$m = 0, \pm 1, \pm 2, \ldots.$$

To conclude the proof, we examine the class of links determined by the closed braids enumerated above. A simple picture shows that if $m \neq 0$, the knots defined by the braids $\beta_m^{(1)}$ are torus knot types, hence their groups have nontrivial centers. By a recent result of Gordon and Heil [15], if M is a simply-connected 3-manifold which is represented as a covering of S^3 of finite degree branched over a knot which has a group with a nontrivial center, then in fact M must be S^3. On the other hand, suppose $\beta = \beta_m^{(2)}$, $m \neq 0$. Then, an argument given by J. Montesinos in [25] shows that the 2-fold covering of S^3 branched over the knot determined by $\beta_m^{(2)}$ is homeomorphic to the three-manifold obtained by cutting out a "twist knot" from S^3 and sewing it back differently. By results of Bing and Martin [35], a simply-connected 3-manifold cannot be obtained from S^3 by surgery on a nontrivial twist knot. Therefore the only case where we might find a homotopy sphere which is not a 3-sphere is if $\beta = \beta_0^1$ or β_0^2, but in this case $\beta =$ is easily seen to define the trivial knot type, so that in fact M is S^3. This concludes the proof of Theorem 10.   □

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