

Holonomic and Legendrian parametrizations of knots

Joan S. Birman *

e-mail jb@math.columbia.edu

Nancy C. Wrinkle

e-mail wrinkle@math.columbia.edu

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Abstract

Holonomic parametrizations of knots were introduced in 1997 by Vassiliev, who proved that every knot type can be given a holonomic parametrization. Our main result is that any two holonomic knots which represent the same knot type are isotopic in the space of holonomic knots. A second result emerges through the techniques used to prove the main result: strong and unexpected connections between the topology of knots and the algebraic solution to the conjugacy problem in the braid groups, via the work of Garside. We also discuss related parametrizations of Legendrian knots, and uncover connections between the concepts of holonomic and Legendrian parametrizations of knots.

1 Introduction:

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ periodic function with period 2π . Following Vassiliev [Va97], use f to define a map $\tilde{f} : S^1 \rightarrow \mathbb{R}^3$ by setting $\tilde{f}(t) = (-f(t), f'(t), -f''(t))$. Let π be the restriction of \tilde{f} to the first two coordinates. We call π the *projection* of $K = \tilde{f}(S^1)$ (onto the xy plane). It turns out that with some restrictions on the choice of the defining function K will be a knot and π will yield a knot diagram with some very pleasant properties, which we now begin to describe. We highlight our assumptions about f with bullets, and describe their consequences:

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(1) We assume that f is chosen so that $\tilde{f}(S^1)$ is a knot $K \subset \mathbb{R}^3$, i.e.

- There do not exist distinct points $t_1, t_2 \in [0, 2\pi)$ such that $(-f(t_1), f'(t_1), -f''(t_1)) = (-f(t_2), f'(t_2), -f''(t_2))$.

Note that this implies that double points in the projection which are off the x axis are transverse. For, at a double point $(-f(t_1), f'(t_1)) = (-f(t_2), f'(t_2))$. The double point is transverse if the tangent vectors to the projected image are distinct at t_1 and t_2 . The tangent vectors are $(-f'(t_1), f''(t_1))$ and $(-f'(t_2), f''(t_2))$. They are distinct because \tilde{f} is an embedding, which implies that $f''(t_1) \neq f''(t_2)$.

(2) The reasoning used in (1) above shows that the tangent to $\pi(K)$ at an instant when $\pi(K)$ crosses the x axis is vertical. We observe that this implies that if a double point occurred at an axis crossing, then it would necessarily be a point where the two branches of $\pi(K)$ had a common tangent. We rule out this behavior, which is not allowed in a 'regular' knot diagram, by requiring that f be chosen so that all double points are away from the x axis, i.e.

- If $(-f(t_1), f'(t_1)) = (-f(t_2), f'(t_2))$ then $f'(t_1) \neq 0$.

(3) The singularities in a regular knot diagram are required to be at most a finite number of transverse double points. To achieve that we need one more assumption, i.e.

- There do not exist distinct points $t_1, t_2, t_3 \in [0, 2\pi)$ such that $(-f(t_1), f'(t_1)) = (-f(t_2), f'(t_2)) = (-f(t_3), f'(t_3))$.

Vassiliev observed that these conditions hold for generic f . A simple example is obtained by taking $f(t) = \cos(t)$, giving the unknot which is pictured in Figure 1. Two additional examples are given in Figure 2, which show the projections of the knots which are defined by the functions $f_{\pm}(t) = \cos(t) \pm \sin(2t)$.

There is an immediate suggestion of a closed braid in this parametrization, for the following reasons. In the half-space $y > 0$, we know that $f'(t) > 0$, so $-f(t)$ is decreasing. Similarly, in the half-space $y < 0$, we have that $-f(t)$ is increasing. Since f is assumed to be generic, if K is crossing from the half-space $y < 0$ to the half-space $y > 0$ at t_0 then $z = -f''(t_0) < 0$ and if it is crossing from the half-space $y > 0$ to the half-space $y < 0$ at t_0 then $z = -f''(t_0) > 0$. Thus the projected image of K on the xy plane winds continually in an anticlockwise sense (anticlockwise because the x coordinate is $-f(t)$). The only reason it may not already be a closed braid is that there may not be a single point on the x axis which separates all of the axis-crossings with $f''(t) > 0$ from the crossings where $f''(t) < 0$. An example of a holonomic knot which is not in braid form is pictured in Figure 3.

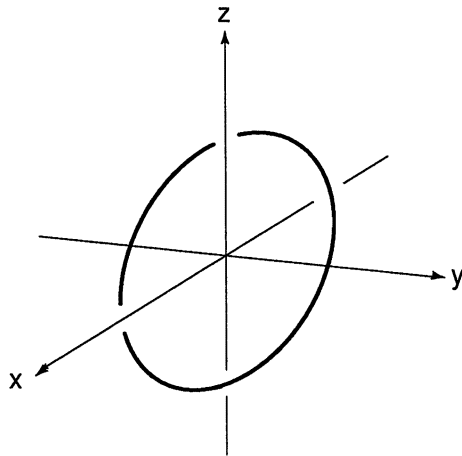


Figure 1: Three-space view of the unknot which is defined by $f(t) = \cos(t)$.

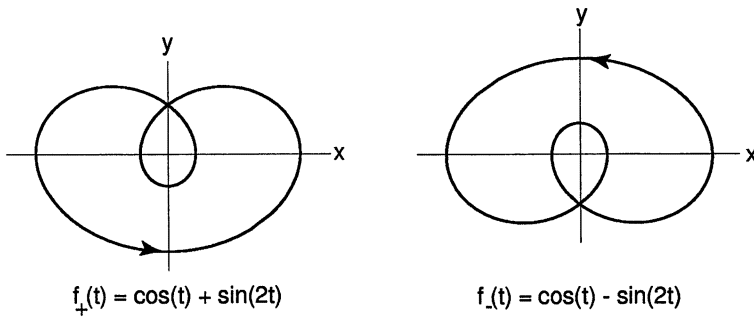


Figure 2: Two additional representations of the unknot

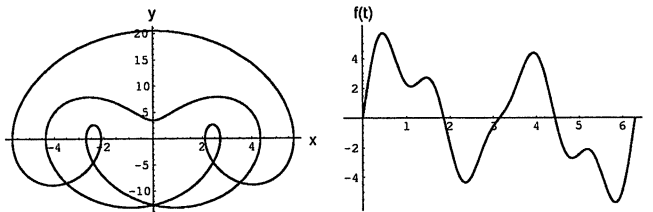


Figure 3: The function $f(t) = \sin(t) + 4\sin(2t) + \sin(4t) + 1.5\sin(5t)$ defines a holonomic trefoil that winds continually anticlockwise but does not have a single point about which it winds.

(4) Analyzing the difficulty, we see that the number of zeros in the x coordinate, i.e. in the graph of $f(t)$, is smaller than the number of zeros in the y coordinate, i.e. in the graph of $f'(t)$. We add one more requirement:

- The number of zeros in one cycle of f is the same as the number of zeros in one cycle of f' .

When $f(t)$ is chosen to satisfy (1)-(4) our parametrization gives a closed braid. The braid index is then one-half the number of zeros in one cycle of f (or of f'). When all these conditions are satisfied our knot is said to have a *holonomic parametrization*.

There is more to be learned from elementary observations. Consult Figure 4(a), which shows four little arcs in the projection of a typical $\mathcal{K} = \tilde{f}(S^1)$ onto the xy plane. The

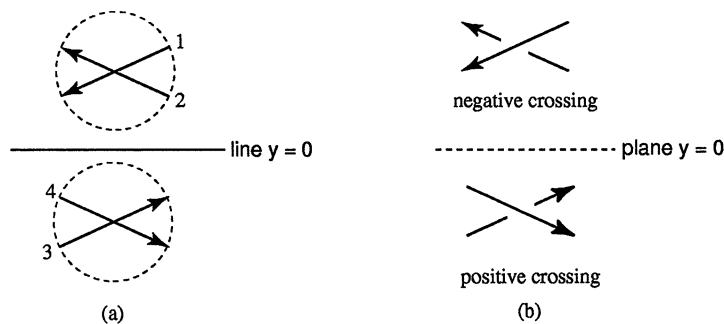


Figure 4: Determining the sign of a crossing: Figure (a) shows the projected images onto the xy plane; Figure (b) show their lifts to 3-space.

four strands are labeled 1,2,3,4. First consider strands 1 and 2. Both are necessarily oriented in the direction of decreasing x because they lie in the half-space defined by $f'(t) > 0$. Since f' is decreasing on strand 1, it follows that f'' is negative on strand 1, so $-f''$ is positive, so strand 1 lies above the xy plane. Since f' is increasing on strand 2, it also follows that strand 2 lies below the xy plane. Thus the crossing associated to the double point in the projection must be negative, as in the top sketch in Figure 4(b), and in fact the same will be true for *every* crossing in the upper half-plane. For the same reasons, the projected image of every crossing in the lower half of the xy plane must come from a positive crossing in 3-space. Thus, up to Reidemeister II moves, \mathcal{K} is a closed braid which factorizes (up to cyclic permutation) as a product NP , where the open braid N (resp. P) represents some number of negative (resp. positive) crossings. Moreover, the type of any such knot is completely defined by its singular projection onto the xy plane.

As an example, the single double point in the left sketch in Figure 2 lifts to a negative crossing in 3-space, whereas that in the right sketch lifts to a positive crossing. Thus \tilde{f}_+ defines the 2-braid representative σ_1^{-1} of the unknot and \tilde{f}_- gives the representative σ_1 .

A different example is given in Figure 5, which shows a holonomic parametrization of the positive trefoil knot as the 2-braid σ_1^3 . The graph of the defining function $f(t)$ is also illustrated. It has 2 local maxima and 2 local minima and 4 zeros, so the knot is defined by a holonomic 2-braid. The parametrization in Figure 5 was found by

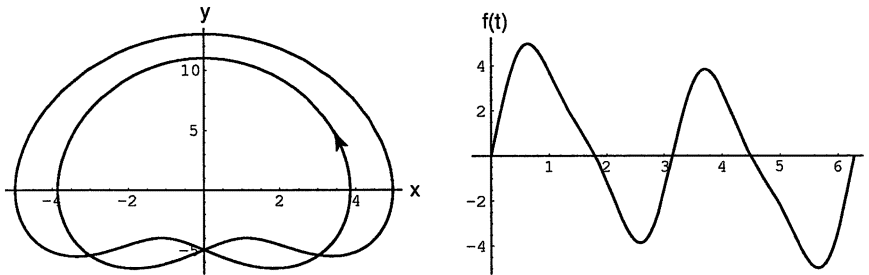


Figure 5: The function $f(t) = \sin(t) + 4\sin(2t) + \sin(4t)$ determines a holonomic positive 2-braid trefoil.

John Buetti, Michael Kinnally, and Felix Tubiana during an undergraduate summer research project ¹ at Columbia University. Notice that the graph of $f(t)$ suggests a sawtooth. They were able to generalize this example to 2-braid representatives of other type $(2, q)$ torus knots by the use of truncated Fourier approximations of certain sawtooth functions. However, their partial results are a little bit complicated to describe, and at this writing they have not proved that their functions work for all q . It is clear that much remains to be done.

Note that our holonomic knots are special cases of Kauffman’s *Fourier* knots and Trautwein’s *harmonic* knots, both of which are parametrized by three distinct truncated Fourier series in \mathbb{R}^3 [Kau97], [Tr95].

These knots were introduced into the literature in [Va97], where they appeared as a special case of the n -jet extension of $f : C^k \rightarrow \mathbb{R}$, where C^k is the disjoint union of k copies of S^1 , i.e., the map $\tilde{f}_n : C^k \rightarrow \mathbb{R}^n$ which is defined by $\tilde{f}_n(t_1, \dots, t_k) = (f(t_1, \dots, t_k), f'(t_1, \dots, t_k), \dots, f^{(n-1)}(t_1, \dots, t_k))$. (Remark: We have changed Vassiliev’s conventions slightly because the 3-jet extension, as he used it, results in sign conventions which will be confusing for knot theorists.) Vassiliev called his n -knots *holonomic knots* and studied them. One of his results for $n = 3$ was:

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Theorem [Va97]: *Every tame knot type in \mathbb{R}^3 can be represented by a holonomic closed braid of some (in general very high) braid index.*

In [Va97] Vassiliev asked the question: “Is it true that any two holonomic knots in \mathbb{R}^3 represent the same knot type if and only if they are isotopic in the space of holonomic knots?” Figure 6 gives an example of a holonomic and a non-holonomic isotopy.

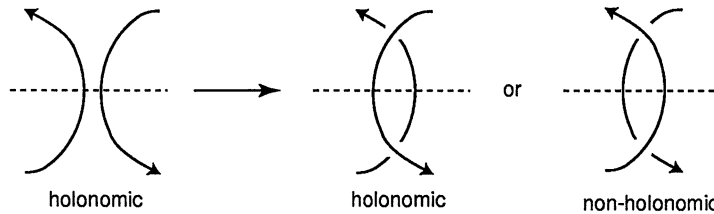


Figure 6: Holonomic and non-holonomic isotopies on fragments of a knot diagram.

The main results in this note are a very simple new proof of a sharpened version of Vassiliev’s Theorem and an affirmative answer to his question:

Theorem 1 *Every tame knot type in \mathbb{R}^3 can be represented by a holonomic closed braid \mathcal{H} . Moreover the braid index of \mathcal{H} can be chosen to be the minimal braid index of the knot type.*

Theorem 2 *If two holonomic knots with defining functions f_0 and f_1 represent the same knot type, then there is a generic holonomic isotopy $F : S^1 \times I \rightarrow \mathbb{R}^3$ with $F(t, 0) = f_0(t)$ and $F(t, 1) = f_1(t)$. Thus the study of holonomic knot types is equivalent to the study of ordinary knot types.*

Here is an outline of the paper. In §2 we set up essential background. We will also state and prove Theorem 0, which may be of interest in its own right. In §3 we prove Theorems 1 and 2. In §4 we discuss parametrizations of Legendrian knots. Since the literature on Legendrian knots may not be well-known to knot theorists, we will discuss them in fairly simplistic terms in §4.1, using insights which we gained as we struggled to understand the constraints placed by the requirement that Legendrian knots are tangent to the standard tight contact structure on \mathbb{R}^3 . Propositions 1 and 2 of §4.2 uncover relationships between holonomic and Legendrian parametrizations of knots.

Remark: The proofs of the results in this paper make very heavy use of the work of Garside [Gar69] and the related work in [Ad86], [Ep92], [E-M94]. Indeed, it appeared to us as we worked out details, that many of Garside’s subtler results seemed to be

designed explicitly for holonomic knots! We note that this is the first time that we have encountered a direct natural connection between the algebraic solution to the word and conjugacy problems in the braid group, via the work of Garside and others, and the geometry of knots. This will become clear after the statement and proof of Theorem 0.

Acknowledgments: We thank Oliver Dasbach, Sergei Chmutov and Serge Tabachnikov for stimulating discussions and helpful remarks.

2 Background and notation:

We summarize below the main facts we will need from the published literature. The reader is referred to [Bir74] for background material on the Artin braid groups $\{\mathbf{B}_n; n = 1, 2, \dots\}$. We will use the standard elementary braid generators $\sigma_1, \dots, \sigma_{n-1}$, where σ_i denotes a positive crossing of the i^{th} and $(i + 1)^{\text{st}}$ braid strands. Defining relations in \mathbf{B}_n are:

(A1). $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$ for all $1 \leq i, j \leq n - 1$.

(A2). $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, where $1 \leq i \leq n - 1$.

We shall use the symbols:

X, Y, Z, \dots for words in the generators of \mathbf{B}_n .

$\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \dots$ for the elements they represent in \mathbf{B}_n .

$\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \dots$ for the associated cyclic words.

$[\mathcal{X}], [\mathcal{Y}], [\mathcal{Z}], \dots$ for their conjugacy classes in \mathbf{B}_n .

P, P_1, P_2, \dots for positive words in the generators of \mathbf{B}_n .

N, N_1, N_2, \dots for negative words in the generators of \mathbf{B}_n .

$X = Y$ if X and Y represent the same element of \mathbf{B}_n .

$H = NP$ if H can be so represented, with P positive and N negative.

$\mathcal{H} = N|P = P|N$ if the cyclic word \mathcal{H} has such a representation.

The vertical bar in the notation $\mathcal{H} = N|P = P|N$ can be interpreted geometrically as separating the closed braid into *Negative* (above the xy plane) and *Positive* (below the xy plane) pieces.

The contributions of Vassiliev. We shall use the following results from [Va97]:

- (V1). If a knot is represented by a diagram in the xy plane which has only negative (resp. positive) crossings in the upper (resp. lower) half-plane, then it may be modified by isotopy to a knot which has a holonomic parametrization.
- (V2). Every holonomic knot may be modified by a holonomic isotopy to a holonomic closed braid, i.e., a closed braid which splits as a product $N|P$. (Vassiliev calls them *normal* braids, but we prefer the term *holonomic* closed braid.)
- (V3). The following modifications in a holonomic closed braid $N|P$ are realized by holonomic isotopy:
 - (a) Positive (resp. negative) braid equivalences in P (resp. N).
 - (b) If $P = N_1P_1$ in \mathbf{B}_n , where N_1 is negative and P_1 is positive, replace $N|P$ by $NN_1|P_1$, with a similar move at the other interface. A special case of this move occurs when we add or delete $\sigma_j^{-1}\sigma_j$ at the interface.
 - (c) Insert $\sigma_n^{\pm 1}$ at either interface of the n -braid $N|P$ to obtain an $(n+1)$ -braid, or the inverse of this move.

The contributions of Garside. Let \mathbf{B}_n^+ be the semigroup of positive words in \mathbf{B}_n which is generated by $\sigma_1, \dots, \sigma_{n-1}$, with defining relations (A1) and (A2). If X, Y are words in \mathbf{B}_n^+ we write $X \doteq Y$ to indicate that X and Y are equivalent words in \mathbf{B}_n^+ . In [Gar69] F. Garside proved that the natural map from $\mathbf{B}_n^+ \rightarrow \mathbf{B}_n$ is an embedding, i.e., if P_1, P_2 are positive words in the generators of \mathbf{B}_n then $P_1 = P_2$ if and only if $P_1 \doteq P_2$. The same is true for negative words and negative equivalences.

In [Gar69] Garside introduced the n -braid Δ , a ‘half-twist’ which is defined by the word:

$$\Delta = \Delta_n = (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)(\sigma_1)$$

and uncovered some of its remarkable properties. A *fragment* of Δ is any initial subword of one of the (many) positive words which are representatives of Δ .

- (G1). For each $i = 1, 2, \dots, n - 1$ the weak commutativity relation $\Delta_n\sigma_i \doteq \sigma_{n-i}\Delta_n$ holds.
- (G2). For each $i = 1, 2, \dots, n - 1$ there are fragments of Δ , say U_i and V_i , such that $\Delta \doteq U_i\sigma_i \doteq \sigma_i V_i$, or equivalently $\sigma_i^{-1} = \Delta^{-1}U_i = V_i\Delta^{-1}$.
- (G3). ([Gar69], [Ad86], [Ep92]) For any $\mathbf{X} \in \mathbf{B}_n$ and any X which represents \mathbf{X} there exists a systematic procedure for converting X to a unique *normal form* $\Delta^k P_1 \cdots P_r$. In the normal form each P_i is a fragment of Δ , also each P_i is a longest possible fragment of Δ in the class of all positive words which are positively equal to P_i . Finally, k is maximal and r is simultaneously minimal for all such representations. If one starts with $X = \Delta^i Q$, where Q is positive,

one finds the normal form by repeatedly ‘combing out’ powers of Δ from Q , i.e. using the fact that if $k > i$, then $X = \Delta^{i+1}Q_1$ where $Q \doteq \Delta Q_1$. A finite number of such combings yields P . An examination of all positive words which represent the same element of \mathbf{B}_n as P produces the decomposition $P \doteq P_1 \cdots P_r$.

- (G4). ([Gar69] and [E-M94]) For any conjugacy class $[\mathcal{X}] \in \mathbf{B}_n$ and any braid X whose closed braid represents $[\mathcal{X}]$, there exists a systematic procedure for converting X to a related normal form $\Delta^{k'}P'_1 \cdots P'_{r'}$, where k' is maximal and r' is minimal for all such representations of words in the same conjugacy class. Call this a *summit form* for $[\mathcal{X}]$. The integers k' and r' are unique but the positive braid $P'_1 \cdots P'_{r'}$ is not unique. However there is a finite collection of all such positive braids and it is unique.
- (G5). There is a systematic procedure for finding a summit form: Assume that $X \in \mathbf{B}_n$ is in the normal form of (G3). Garside proves that there exists a positive word W which is a product $A_1A_2 \cdots A_z$, where each A_i is a fragment of Δ , such that $W^{-1}XW = X'$, where X' is a summit form. He also shows how to find A_1, \dots, A_z . Let $W_i = A_1A_2 \cdots A_i$. Then each $W_i^{-1}XW_i = \Delta^{k_i}P_{1,i}P_{2,i} \cdots P_{r_i,i}$ where $k_i \geq k_{i-1}$ and $r_i \leq r_{i-1}$, also $r = r_1$ and $r' = r_z$.
- (G6). If $X' = \Delta^{k'}P'_1 \cdots P'_{r'}$ and $X'' = \Delta^{k''}P''_1 \cdots P''_{r''}$ are both summit forms of X , then X' and X'' are related by a series of positive conjugacies, as in (G5), with each $k_i = k'$ and each $r_i = r'$.

We pause in our review of the background material to point out a connection between Garside’s work and holonomic isotopy.

Theorem 0: *Given any open braid H which is in holonomic form NP , the following hold:*

1. *The open braid NP may be brought to Garside’s form $\Delta^{-q}Q$, where q is a non-negative integer and $Q \in \mathbf{B}_n^+$, by a holonomic isotopy.*
2. *A further holonomic isotopy converts the open braid $\Delta^{-q}Q$ to its unique normal form $\Delta^kP_1P_2 \cdots P_r$.*
3. *A final holonomic isotopy converts the associated closed braid to one of its summit forms $\Delta^{k'}P'_1 \cdots P'_{r'}$.*

Proof of Theorem 0:

1. We are given $H = NP$. If $N = \emptyset$ there is nothing to prove, so assume that $N \neq \emptyset$. Using only negative equivalences, comb out powers of Δ^{-1} to write N in the form $N = \Delta^{-k}\sigma_{j_1}^{-1} \cdots \sigma_{j_{s-1}}^{-1}\sigma_{j_s}^{-1}$ where the word $\sigma_{j_1}^{-1} \cdots \sigma_{j_{s-1}}^{-1}\sigma_{j_s}^{-1}$ is not equivalent to any word which contains a power of Δ^{-1} . By (V3) this

move can be realized by a holonomic isotopy. If $s = 0$ the theorem is true, so assume that $s \geq 1$. By (G2) we may find a fragment U_{j_s} of Δ such that $\sigma_{j_s}^{-1} = \Delta^{-1}U_{j_s}$. Therefore by (V3) our closed braid is holonomically equivalent to $\Delta^{-k}\sigma_{j_1}^{-1} \cdots \sigma_{j_{s-1}}^{-1} \Delta^{-1}U_{j_s}P$, which by (G1) is holonomically equivalent to $\Delta^{-k-1}\sigma_{n-j_1}^{-1} \cdots \sigma_{n-j_{s-1}}^{-1}U_{j_s}P$. Induction on s completes the proof.

2. To change $\Delta^{-k}Q$ to normal form for its word class, first comb out powers of Δ from the positive part Q . Then put the new positive part into Garside's normal form. Both of these steps are achieved by positive braid equivalences, so by (V3) this part of the work is realizable by a holonomic isotopy.
3. Following (G5) we move the normal form just achieved to a summit form and check that this move is holonomic. When we consider conjugates of H in (G5), Garside tells us that it is enough to work with conjugates $W^{-1}HW$, where W is positive. If H was holonomic before conjugating by W , then the positivity of W keeps the new braid holonomic after conjugation. ||

The contributions of Markov. We will need to use Markov's Theorem (see [Bir74] or [Mo86]) in our work. We state it in the form in which it will be most useful in this paper:

Markov's Theorem Let $X \in \mathbf{B}_p$ and $X' \in \mathbf{B}_q$ be two braids whose associated closed braids $\mathcal{X}, \mathcal{X}'$ define the same oriented knot type. Then there is a sequence of conjugacy classes in the braid groups $\{\mathbf{B}_n, 1 \leq n < \infty\}$:

$$[\mathcal{X}] = [\mathcal{X}_0] \longrightarrow [\mathcal{X}_1] \longrightarrow [\mathcal{X}_2] \longrightarrow \cdots \longrightarrow [\mathcal{X}_{r-1}] \longrightarrow [\mathcal{X}_r] = [\mathcal{X}']$$

where $[\mathcal{X}_j] \subset \mathbf{B}_{n_j}$, and there are open braid representatives $X_{j,1}$ for each $1 \leq j \leq r$ and $X_{j,2}$ for each $0 \leq j \leq r - 1$ of the conjugacy class $[\mathcal{X}_j]$ such that either:

- (M1). $n_{j+1} = n_j + 1$ and $X_{j+1,1} = X_{j,2}\sigma_{n_j}^{\pm 1}$ (adding a trivial loop), or
- (M2). $n_{j+1} = n_j - 1$ and $X_{j+1,1}\sigma_{n_{j+1}}^{\pm 1} = X_{j,2}$ (deleting a trivial loop).

3 The proofs:

Proof of Theorem 1. By a well-known theorem of Alexander (see [Bir74] for example), any knot type may be represented as a closed braid. In the collection of all closed braid representatives of a given knot type, let us suppose that \mathcal{K} is the closure of the braid $\sigma_{\mu_1}^{\epsilon_1} \cdots \sigma_{\mu_r}^{\epsilon_r}$, where each $\epsilon_j = \pm 1$. Use (G2) to replace each $\sigma_{\mu_i}^{-1}$ by $\Delta^{-1}V_{\mu_i}$. Then use (G1) to push the Δ^{-1} 's to the left. This replaces the given representative

by a new closed braid \mathcal{H} which is in the desired form $N|P$. But then, by (V1), \mathcal{H} is isotopic to a holonomic closed braid.

If we had chosen the braid representative of our knot \mathcal{K} to have minimum braid index (which we can do without loss of generality) then the holonomic braid \mathcal{H} will also have minimum braid index because the changes which we introduced to achieve $N|P$ form do not change the number of braid strands. ||

Remark: This proof differs from Vassiliev’s proof in the following way. We both begin with an arbitrary knot diagram. He modifies the given diagram by a move which eliminates positive (resp. negative) crossings in the upper (resp. lower) half-plane, at the expense of adding some number of anticlockwise loops which encircle points on the x -axis. He then changes the resulting holonomic knot to a holonomic closed braid by using holonomic Reidemeister II moves, as in our Figure 6. In our proof, we modify the original diagram to a closed braid, and then use a very small part of Garside’s work, without changing the number of braid strands, to complete the proof. We will see this theme expanded in the proof of Theorem 2.

Proof of Theorem 2. We begin our proof of Theorem 2 with two holonomic knots which, by (V1) and (V2), can be assumed to be holonomic closed braids. Thus $\mathcal{H} = N|P$ and $\mathcal{H}' = N'|P'$. By hypothesis, our closed braids define the same oriented link types in \mathbb{R}^3 . Markov’s Theorem then gives us a chain of conjugacy classes of braids which connects them:

$$[\mathcal{H}] = [\mathcal{X}_0] \longrightarrow [\mathcal{X}_1] \longrightarrow [\mathcal{X}_2] \longrightarrow \cdots \longrightarrow [\mathcal{X}_{r-1}] \longrightarrow [\mathcal{X}_r] = [\mathcal{H}'].$$

Let’s consider the passage from the class $[\mathcal{X}_j]$ to the class $[\mathcal{X}_{j+1}]$. By Markov’s Theorem, we must choose representative open braids $X_{j,1}, X_{j,2}$ of the conjugacy class $[\mathcal{X}_j] \subset \mathbf{B}_{n_j}$ and show that in either of the two cases (M1), (M2) our representatives and the Markov moves between them are holonomic.

- In the situation of (M1) the braid $X_{j,2} \in \mathbf{B}_{n_j}$ is not necessarily holonomic. We change the closure of $X_{j,2}$ to a holonomic closed braid $H_{j,2}$, if necessary, using Garside’s methods: $H_{j,2} = \Delta_{n_j}^{-p}P$ where $p \geq 0$ and $P \in \mathbf{B}_{n_j}^+$. Then $\Delta_{n_j}^{-p}P\sigma_{n_j}^{\pm 1} = \mathcal{H}_{j+1,1}$ is holonomic for both choices of the exponent of σ_{n_j} . By (V3) the passage $\mathcal{H}_{j,2} \longrightarrow \mathcal{H}_{j+1,1}$, which adds a trivial loop at the interface between the positive and negative parts of $\mathcal{H}_{j,2}$, can be realized by a holonomic isotopy.
- In the situation of (M2) the braid $X_{j+1,1} \in \mathbf{B}_{n_j-1}$ is in general not holonomic, but we may replace it as above with a holonomic $(n_j - 1)$ -braid $H_{j+1,1}$ which is in the form $N|P$. But then $\mathcal{H}_{j,2}$ is also holonomic, for both choices of the exponent of σ_{n_j-1} , because the ambiguously signed letter is at the interface

between the positive and negative parts of $\mathcal{H}_{j,2}$. The passage $\mathcal{H}_{j,2} \longrightarrow \mathcal{H}_{j+1,1}$, which deletes the trivial loop, can clearly be realized by a holonomic isotopy.

The initial and final braids $\mathcal{H}_{0,1}$ and $\mathcal{H}_{r,2}$ have not yet been defined; we take them to be the given representatives $N|P$ and $N'|P'$ of $[\mathcal{H}]$ and $[\mathcal{H}']$ respectively.

Thus we have produced a sequence of conjugacy classes of holonomic braids which joins $[\mathcal{H}]$ to $[\mathcal{H}']$, and in each conjugacy class we have two holonomic representatives $\mathcal{H}_{j,1}$ and $\mathcal{H}_{j,2}$, and for each $j = 1, 2, \dots, r - 1$ we go from $\mathcal{H}_{j,2}$ to $\mathcal{H}_{j+1,1}$ via a holonomic isotopy.

The only point which remains to be proved is that, in each conjugacy class $[\mathcal{X}_j] \subset \mathbf{B}_{n_j}$ in the sequence, there is a holonomic isotopy between the two chosen holonomic representatives. The proof will be seen to be independent of j , so we simplify the notation, setting $n = n_j$. Assume that we have two holonomic representatives $\mathcal{H}, \mathcal{H}'$ of the same conjugacy class in \mathbf{B}_n . Our task is to prove that they are holonomically isotopic.

In view of Theorem 0, we may assume without loss of generality that \mathcal{H} and \mathcal{H}' are summit forms. By (G6), two summit forms in the same conjugacy class are related by conjugations by positive words which are fragments of Δ using only positive conjugacy. Again, positive conjugation by a positive word sends holonomic braids to holonomic braids. Thus we have the desired holonomic isotopy from \mathcal{H} to \mathcal{H}' and the proof of Theorem 2 is complete. ||

4 Legendrian knots

Theorem 2 was in some ways an unexpected result, because it is well-known that the analogous theorem fails in the case of Legendrian knots. With the goal of understanding this situation better we investigate, briefly, parametrizations of Legendrian knots. We begin by introducing Legendrian knots, via their parametrizations.

4.1 Parametrizing Legendrian knots

Introduce coordinates (x, v, z) in \mathbb{R}^3 . A knot is *Legendrian* if it is represented by a smooth embedding of $S^1 \rightarrow \mathbb{R}^3$ whose image is everywhere tangent to the planes of the standard contact structure on \mathbb{R}^3 . The *standard* contact structure on \mathbb{R}^3 is the nonintegrable field of planes defined by the differential 1-form $\alpha = zdx - dv$. A curve will be everywhere tangent to these 2-planes if it is in the kernel of α , so if a knot can be parametrized by $(x(t), v(t), z(t))$, where x, v, z are real-valued periodic functions with period (say) 2π , then it is tangent to this contact structure if and only

if $z(t) = dv/dx = v'(t)/x'(t)$ for all $t \in [0, 2\pi]$.

A simple example is given by the parametrization $(-\cos(t), -\sin^3(t), -3\sin(t)\cos(t))$. See Figure 7. In this example the projection onto the xv plane (the so-called *front projection*) has cusps, but they do not represent points of non-smoothness in the space curve. Indeed, the front projection of a Legendrian knot always contains $2m$ cusps for some $m \geq 1$. Notice that the z coordinate is the slope of the tangent to the curve in the front projection. This makes these projections similar to our xy projections of holonomic knots, because the signs of the crossings in the front projection are completely determined without further data. Here is a simple rule for finding the signs of the crossings: The front projection has no vertical tangencies, because $z(t)$ would be undefined at a vertical tangency. Therefore at a double point both branches intersect a vertical line through the double point transversally. Crossings are positive (respectively negative) if the two branches at a double point intersect a vertical line through the double point from opposite (respectively the same) directions.

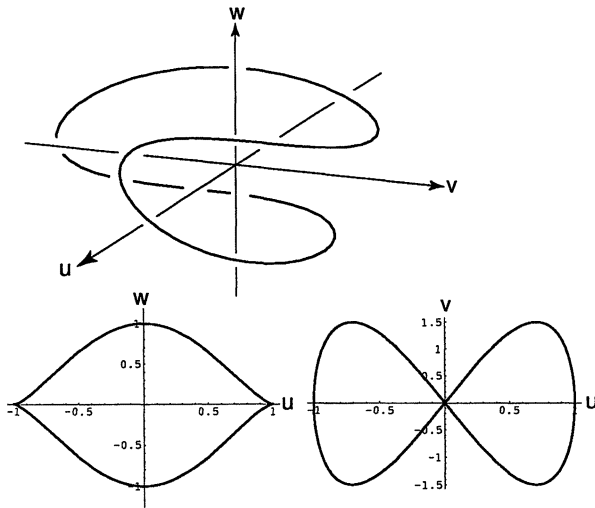


Figure 7: Legendrian unknot $(-\cos(t), -\sin^3(t), -3\sin(t)\cos(t))$.

It is simple to construct other examples (indeed all possible examples) of parametrized Legendrian knots. Let $x(t)$ and $z(t)$ be real-valued smooth periodic functions with period 2π . Then $x(t), z(t)$ have Fourier expansions:

$$x(t) = \sum_i a_i \sin(it) + b_i \cos(it), \quad z(t) = \sum_j c_j \sin(jt) + d_j \cos(jt).$$

The third coordinate $v(t)$ has a similar expansion, determined up to an additive

constant from the first two by the condition $v'(t) = x'(t)z(t)$. Of course we need to be sure that $x(t)$ and $z(t)$ are generic (i.e. that they satisfy restrictions analogous to the bulleted assumptions in §1).

4.2 The Legendrian cousins of holonomic knots

Holonomic knots have a parametrization $(x(t), y(t), z(t))$ where $x(t) = -f(t)$, $y(t) = f'(t)$, and $z(t) = -f''(t)$, and f is a function chosen as in section 1. Following a suggestion of S. Chmutov, we consider now the differential 1-form to whose kernel this parametrization corresponds. The holonomic parametrization imposes the following conditions on the three coordinates: $-y(t) = x'(t)$ and $-z(t) = y'(t)$. If we assume $y(t) \neq 0$, then $z = (ydy)/dx$. Hence the 1-form β whose kernel contains a holonomic knot is given by $\beta = zdx - ydy$. For the 1-form β to define a contact structure there is an additional condition it must satisfy: it must have a non-vanishing volume form $\beta \wedge d\beta$. Here $\beta \wedge d\beta = dx \wedge ydy \wedge dz$, so $\beta \wedge d\beta$ is only non-zero off the plane $y = 0$, and our earlier assumption that $y \neq 0$ is an essential part of the story. By construction our holonomic knots are tangent to this contact structure, and we show further that this contact structure is isomorphic to the standard one on \mathbb{R}^3 :

Proposition 1 *Every holonomic knot is tangent to a contact structure on $\mathbb{R}^3 - \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$ which is isomorphic to the standard contact structure on \mathbb{R}^3 in the complement of the plane $\{(x, v, z) \in \mathbb{R}^3 : v = 0\}$.*

Proof: We map the upper half-space $U_+ = \{(x, y, z) \in \mathbb{R}^3 : y > 0\}$ to the upper half-space $V_+ = \{(x, v, z) \in \mathbb{R}^3 : v = y^2/2 > 0\}$ and the lower half-space $U_- = \{(x, y, z) \in \mathbb{R}^3 : y < 0\}$ to V_+ with the same transformation $y \rightarrow y^2/2$. This gives us transformations sending the two half-spaces of $\mathbb{R}^3 - \{y = 0\}$ to the upper half-space V_+ of \mathbb{R}^3 . In V_+ we have the new relations $dv = ydy$ and $dv = zdx$, so our 1-form $\beta = zdx - ydy$ becomes the 1-form $\alpha = zdx - dv$. The contact structure defined by α on V_+ is exactly the restriction of the standard one on \mathbb{R}^3 . We can extend it from V_+ to all of \mathbb{R}^3 then, although on all of \mathbb{R}^3 the transformation from β to α is not invertible. (It is not invertible along the plane $v = 0$, where $z = dv/dx$ goes to infinity.) ||

In view of Proposition 1, a simple modification of our holonomic parametrizations suggests itself as a method for parametrizing a related class of Legendrian knots. Let \mathcal{H} be any knot type in \mathbb{R}^3 . Using Theorem 1, choose a smooth periodic function $f(t)$ such that $(-f(t), f'(t), -f''(t))$ is a holonomic parametrization $H(t)$ of \mathcal{H} . Let $(-f(t), f'(t)^3, -3f'(t)f''(t))$ be a parametrization $L(t)$ of a new knot \mathcal{L} . We call \mathcal{L} the *Legendrian cousin* of the holonomic knot \mathcal{H} . We already gave one example in Figure 7, which illustrated the Legendrian cousin of the unknot of Figure 1. A different example is given in Figure 8, which shows the projection onto the xy -plane of the Legendrian cousin of the holonomic trefoil of Figure 5.

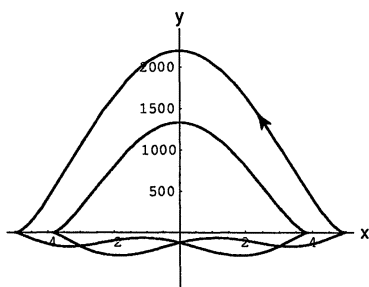


Figure 8: Legendrian cousin of the holonomic trefoil of Figure 5.

Proposition 2 *Let \mathcal{H} be any knot type in \mathbb{R}^3 . As above, let $H(t)$ be its holonomic parametrization and $L(t)$ the parametrization of \mathcal{L} , its Legendrian cousin. Then:*

1. *The parametrized curve $L(t)$ is Legendrian, relative to the standard contact structure $zdx - dv$ on $\mathbb{R}^3 - \{(x, v, z) \in \mathbb{R}^3 : v = 0\}$.*
2. *The curves $H(t)$ and $L(t)$ are closed braids with respect to the z axis. Their projected images onto the xy plane define equivalent immersions, i.e., there is a 1 – 1 correspondence between the singularities of the two projections and an isotopy from one projection to the other induced by the isomorphism of \mathbb{R}^2 taking coordinates $(x, y) \rightarrow (x, v)$.*
3. *All crossings in the front projection for \mathcal{L} are negative.*
4. *Whereas any knot type \mathcal{H} is represented by a holonomic $H(t)$, the knot type \mathcal{L} which is represented by its Legendrian cousin $L(t)$ is always a fibered knot, i.e. its complement fibers over the circle with fiber a closed orientable surface F , with $\partial F = L$. In fact, \mathcal{L} belongs to a very special class of fibered knots which can be represented by closed negative braids.*

Proof: To prove (1), notice that

$$-3f'(t)f''(t) = \frac{d((f'(t))^3/dt}{-f'(t)}.$$

To prove (2), let $D(H)$ and $D(L)$ be the parametrized immersed curves $(-f(t), f'(t))$ and $(-f(t), f'(t)^3)$ in the xy plane so that $D(H)$ is a holonomic projection for \mathcal{H} and $D(L)$ is a front projection for \mathcal{L} . Then the diagrams $D(L)$ and $D(H)$ are the same immersions, up to the isomorphism of \mathbb{R}^2 which changes coordinates from (x, y) to (x, v) . For observe that

$$(-f(t_1), f'(t_1)) = (-f(t_2), f'(t_2)) \iff (-f(t_1), f'(t_1)^3) = (-f(t_2), f'(t_2)^3).$$

Notice that the cusps in $D(L)$ occur precisely at the points where $D(H)$ crosses the x axis. The assertion about closed braids is then clear, from our work in §1 of this paper.

(3) The statement about the signs of the crossings in \mathcal{L} follows from the fact that, using the rule given in Section 4.1, the signs of the crossings in $D(L)$ are always negative. However, in $D(H)$ they are negative in the upper half-plane and positive in the lower half-plane. It's easy to understand why this is the case. At a double point in both $D(h)$ and $D(L)$ we have distinct values t_1, t_2 with $f(t_1) = f(t_2)$ and $f'(t_1) = f'(t_2)$. The signs of the crossings are determined by the z -values on the two branches, that is by $-f''(t_1)$ and $-f''(t_2)$ in $H(t)$, but by $-f''(t_1)f'(t_1)$ and $-f''(t_2)f'(t_2)$ in $L(t)$. From this it follows that at a double point in the upper half-plane, where $f'(t_1) = f'(t_2) > 0$, the signs of the crossings will be the same in $H(t)$ and $L(t)$, whereas at a double point in the lower half-plane, where $f'(t_1) = f'(t_2) < 0$, the signs of the crossings in $L(t)$ are opposite to those in $H(t)$. From this it follows that all crossings in $L(t)$ are negative. An example can be seen in Figures 8 and 5. The holonomic knot in Figure 5 is a positive type (2,3) torus knot, whereas the Legendrian knot in Figure 8 is a negative type (2,3) torus knot.

(4) Finally, it is well-known (see, for example, [St78]) that knots which can be represented by closed negative or positive braids are a very special subclass of fibered knots.||

Proposition 2 shows that the failure of the contact structure defined by β to extend across the plane $y = 0$ has profound consequences. By Theorem 2 an arbitrary isotopy between holonomic knots can always be deformed to a holonomic isotopy, but it should now be clear that we cannot expect any such result for the Legendrian cousins of holonomic knots. Indeed, our proofs of Theorems 1 and 2 above depended heavily on our ability to move across the plane $y = 0$ via holonomic isotopies, but for Legendrian cousins we are restricted to Legendrian isotopies in either half-space, so entirely new methods would be needed to prove an analogous theorem for Legendrian cousins.

Remark: The Legendrian cousins of Proposition 2 can be generalized to an infinite family of Legendrian cousins of $H(t)$, with parametrizations

$$L_k(t) = (-f(t), f'(t)^{2k+1}, -(2k + 1)f'(t)^{2k-1}f''(t)).$$

They share a common knot diagram, so they all represent the same knot type. In fact, they represent the same Legendrian knot type. An explicit Legendrian isotopy $L(t, s)$ from $L_k(t) \rightarrow L_m(t)$ was suggested by Oliver Dasbach. It begins at $L(t, 0) = L_k(t)$ and ends at $L(t, 1) = L_m(t)$ and is defined by $L(t, s) =$

$$(-f(t), sf'(t)^{2k+1} + (1-s)f'(t)^{2m+1}, -s(2k+1)f'(t)^{2k-1}f''(t) - (1-s)(2k+1)f'(t)^{2m-1}f''(t))$$

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- Joan S. Birman, Math. Dept, Barnard College of Columbia University, 604 Mathematics, 2990 Broadway, New York, N.Y. 10027.
- Nancy C. Wrinkle, Math. Dept., Columbia University, 408 Mathematics, 2990 Broadway, New York, N.Y. 10027.