BRAIDED CHORD DIAGRAMS

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Abstract

The notion of a braided chord diagram is introduced and studied. An equivalence relation is given which identifies all braidings of a fixed chord diagram. It is shown that finite-type invariants are stratified by braid index for knots which can be represented as closed 3-braids. Partial results are obtained about spanning sets for the algebra of chord diagrams of braid index 3.

Keywords: knots, finite type invariants, chord diagrams, braids

The first author dedicates her work in this paper to Professor James Van Buskirk of the University of Oregon. She thanks Jim for his friendship and his many kindnesses over the years.

1 Introduction

The motivation for this work was the classification of links which are closed 3-braids by Birman and Menasco in [3] and the classification of finite type invariants for knots which have braid index 2 in [12]. Our goal was to introduce the notion of braid index of chord diagrams, and then to study finite-type invariants restricted to closed 3-braids. In particular, knowing exactly which closed 3-braids represent non-invertible knots, and having a precise classification theorem for all knots which are closed 3-braids, we hoped to find examples which would prove that finite-type invariants detect non-invertibility.

We call the conjecture that such examples exist the NIC (non-invertibility conjecture). The essential difficulty in finding examples to prove the NIC is that present methods of calculation do not allow one to study weight systems which have order larger than 9 [4]. A similar problem has arisen in other conjectures in this area. Since it is known that every quantum invariant gives rise to an infinite family of finite type invariants, it had been conjectured [4] that all finite-type invariants of a knot come from classical Lie Algebras, i.e. types A,B,C,D. That conjecture was based upon the known data, i.e. weight systems of order up to 9. However, P.

Vogel proved that conjecture to be false [14] when he found a counterexample (of order 32). We conjecture that a similar situation exists with regard to the NIC.

We remark that it has been proved by G. Kuperberg [9] that if finite-type invariants contain enough information to completely classify prime *unoriented* knot types, then finite-type invariants are also strong enough to classify *oriented* knots. Thus the question we are studying is intimately tied to the larger question of whether finite-type invariants are a complete set of algebraic invariants of knots.

This paper develops a method and contains partial results toward the goal of finding examples which could settle the NIC. We begin in §2 by proving some general results about braided chord diagrams and braid index. We show how two braidings of the same diagram are related (Theorem 2.3) and give an algorithm for computing the braid index of a diagram (Theorem 3.2). The notion of amalgamating fans makes the computation of braid index significantly easier (see Corollary 3.5) than it would be without such a notion. Proposition 3.6 gives a lower bound for the braid index of a diagram which (as will be shown in §5) essentially allows us to characterize chord diagrams of braid index 3.

A fundamental question is whether the value of a finite-type invariant on a closed b-braid be computed from its values on diagrams of braid index at most b. This can't be true if one uses Kontsevich's universal Vassiliev invariant in the form in which it was presented in [5] for the computation, because the universal Vassiliev invariant evaluated on the trefoil involves diagrams with braid index greater than 2. Thus we turned to the actuality tables of [13] and [2]. We say that finite-type invariants are *stratified* by braid index if their values on b-braids can be computed from a completed actuality table, using only diagrams of braid index at most b. The case for $b = 2$ was completed in [12], where finite-type invariants restricted to closed 2-braids were characterized. The difficulty for higher braid index is showing that the crossing changes necessary to compute $v(K)$ can be done without increasing the braid index. To show this when K is a closed 3-braid, we begin by showing in Theorem 4.1 that 3-braid representatives of n-diagrams are essentially unique. We then use what we call standard singular knot representatives to fill in the actuality tables. Since 3-braid representatives are essentially unique, this choice is well-defined. Finally we use the algebraic structure of the braid group to prove that finite-type invariants on closed 3-braids are stratified by braid index (Theorem 4.4). Thus, by the end of §4 we have shown that our program is sound: it is possible to develop a completed actuality table for the computation of Vassiliev invariants of arbitrary order, which computes such invariants for knots that are closed 3-braids and which uses only singular knots and n-diagrams of braid index 3. Since the chief difficulty in collecting numerical data on Vassiliev invariants is that the number of unknowns and the number of linear equations which one must solve grows unmanageably large as one increases n [4], and since both the number of diagrams and the number of equations is necessarily much smaller if one restricts to braid index ≤ 3, it then follows that in principle one should be able to compute higher order invariants in the 3-braid case than in the general case.

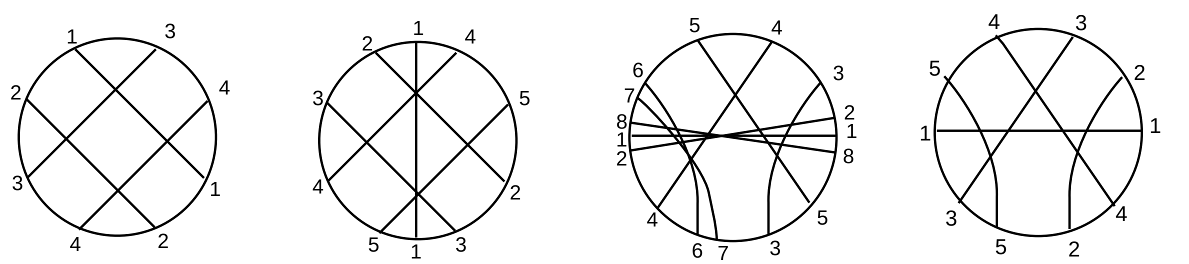
Unfortunately, our results fell short of that goal. We were able to determine all 4-term and 1-term relations in which all of the chord diagrams have braid index 3, and also some more general linear relations between them, but we do not at this time have a full understanding of all of the linear relations which hold between chord diagrams of braid index 3. In §5 we give strong forms of earlier theorems of Bar Natan [6] and also Lin [10], which take into account the braid forms of the 4-term relations *and also* cyclic permutations of closed braided chord diagrams. We give some examples which show why a better understanding of what we call "stabilization" is necessary to complete the picture.

2 Braidings of Diagrams

In this section we give a quick review of the background material. Then we introduce the notion of braid index. It is shown that every diagram admits a braiding. We show that any two braidings of the same diagram are related by a certain type of commutativity, cyclic permutation and 'stabilization'.

A *singular knot* is the image of a circle S^1 in S^3, under an immersion $\iota : S^1 \to S^3$ whose singularities are at most finitely many transverse double points. Recall [2] that any numeric knot invariant v can be extended to singular knots via the Vassiliev skein relation: $v(K_p) = v(K_{p_+}) - v(K_{p_-})$, where K_p is a singular knot with a transverse double point at p and K_{p_+} (resp. K_{p_-}) is the singular knot with one less singular point which is obtained from K_p by 'resolving' the singular point at p to a positive (resp. negative) crossing. All knot invariants will be considered to be extended to singular knot invariants in this manner. An invariant v is of *finite-type* if it vanishes on singular knots with more than n double points, for some n. It has *order* n if n is the smallest such integer. It follows immediately that an order-n invariant evaluated on a singular knot K^n with n double points depends only on the double points of the knot (because changing crossings in K^n changes the value of v by its value on a singular knot with $n+1$ double points, which is zero).

A *chord diagram* is an n-diagram, for some n, i.e. an oriented circle with n chords. Examples are given in Figure 2.1. The numbers around the outer circle label the chords, relative to an arbitrary starting point and an anticlockwise orientation on the circle. We describe the diagram by a *name*, i.e. the string of $2n$ integers obtained by recording the labels of the chords as they are passed while traversing the outer circle, as indicated in the examples in Figure 2.1. Different choices of labeling and different starting points will yield names which are cyclically equivalent after relabeling the chords. Given a name, one may easily reconstruct the diagram D. Thus if two n-diagrams have the same name, they are the same. An n-diagram can be thought of as the addition of extra structure to the pair (ι, S^1), vis: think of S^1 as a planar circle and of the endpoints of the chords as the points which are identified under ι. A singular knot K^n which has a given n-diagram D as its preimage is said to *respect* D.



(a) 12342143 (b) 1234513254 (c) 1234567812467358 (d) 1234513524

Figure 2.1: Examples of chord diagrams

If a knot invariant v is of order n, then $v(K^n)$ depends only on the diagram D which K^n respects. The values which an invariant v of order n takes on n-diagrams are determined by a system of linear equations whose unknowns are in one-to-one correspondence with the collection of all possible n-diagrams. The linear relations in this system are known as the *4-term* and *1-term* relations [2],[4]. A linear functional v on n-diagrams is called a *weight system* if it satisfies the 4 and 1-term relations. Kontsevich [8] proved that any weight system on

n-diagrams extends to an invariant of order n on knots, and also that the extension is unique modulo invariants of lesser order. Thus the study of finite-type invariants can be reduced to the study of weight systems on n-diagrams. Moreover, if the functional v on n-diagrams satisfies the four-term relation but not the 1-term relation, there is a canonical way to modify v so that it satisfies the one-term relation as well [4].

If we replace our planar circle S^1 by the closure of the standard m-braid representative $\sigma_{m-1}^{-1}\sigma_{m-2}^{-1}\cdots\sigma_1^{-1}$ of the unknot, and choose the chords of D to be n radial segments (relative to the braid axis) in this representation, then we say that we have an *m-braid representative* or a *braiding* of the n-diagram. Cutting open the closed braid, we obtain a singular m-braid which represents the diagram and has all of its chords as horizontal line segments. Two examples are given in Figures 2.2(a), where we have constructed two braidings of the diagram 12342143 of Figure 2.1(a). The associated open braids are illustrated in Figure 2.2(b). Clearly we may recover the closed braids by identifying the end points of the strands in the singular braid.

The chord algebra C_m is defined as having generators $A(i,j)$ $1 \leq i < j \leq m$ and relations

$$[A(i,j), A(k,l)] = A(i,j)A(k,l) - A(k,l)A(i,j) = 0$$

for i,j,k,l distinct. The geometric interpretation of a generator $A(i,j)$ is a chord joining the i^{th} and j^{th} strand of the identity m-braid, and multiplication in C_m is concatenation. While the braids we are discussing are singular braids, we shall simply refer to them as 'braids'.

A word in the generators of C_m can be closed in many ways to give a braided chord diagram, hence a convention must be made. If $W \in C_m$ is a monomial, then the *closure* of W is the braided chord diagram $\mathbf{W}$ obtained by adjoining the standard m-braid representative of the unknot to W and forming the closure of the resulting singular braid by identifying the top and bottom of each singular braid strand. See Figure 2.2 for examples. The 4-diagram in Figure 2.2(a) (which has the same name as that in Figure 2.1(a)) is the braided chord diagram associated to the closure of the singular 4-braid $A(1,3)A(1,2)A(1,4)A(1,2)$ of Figure 2.2(b). The same chord diagram is also defined by the closure of the singular 3-braid of Figure 2.2(b).

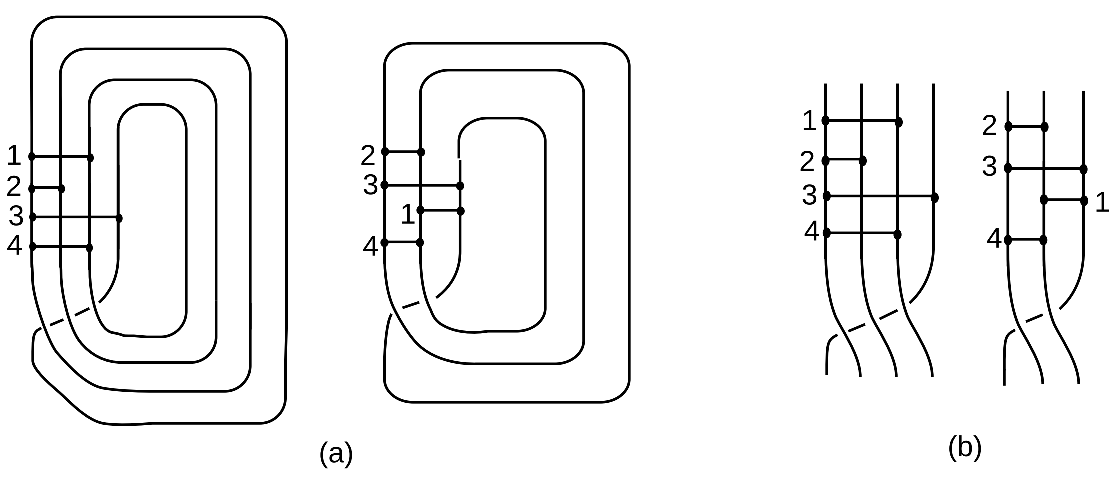


Figure 2.2: Braidings of chord diagrams

There is a natural injection $i : C_l \to C_m$ for $l \leq m$ which acts as the identity on C_l. Geometrically this injection is realized by adding $m - l$ unused strands. The reader should note that if $W \in C_l$ then $\mathbf{W}$ and $\mathbf{i}(\mathbf{W})$ represent the same n-diagram since the unused strands in $i(W)$ can be unkinked in $\mathbf{i}(\mathbf{W})$. Thus we adopt the convention that if W is said to be in C_m, then W contains at least one generator of the form $A(i, m)$. This convention is unnecessary, but it simplifies some of the arguments which follow.

Lemma 2.1. *Any n-diagram has a braided chord diagram representative.*

Proof. Let N be a name for D, so that each integer r for $1 \leq r \leq n$ occurs twice in N. Let i_r be the position of the first occurrence of r in N, and let j_r be the position of the second. We claim that the word $W = A(i_1, j_1)A(i_2, j_2)\cdots A(i_n, j_n)$ closes to a representative of D.

Before proving this claim, we observe that $W \in C_{2n}$ and every strand in W has exactly one endpoint on it. These properties turn out to characterize braidings of D obtained in this way (see Lemma 2.2).

Now let's prove that $\mathbf{W}$ is a braiding of D. To do so we simply label the chord $A(i_r, j_r)$ of W by r. Now begin reading the name for $\mathbf{W}$ on the left-most strand. Note that by the way we labeled the chords of W, the label on the k^{th} strand is the k^{th} entry in N. Thus N is a name for $\mathbf{W}$ and $\mathbf{W}$ is a braid representative for D. $\|$

Note that the proof of the lemma gives an easy algorithm for constructing a $2n$-braid representative from a name for an n-diagram. This motivates the following definition: Given a name N for an n-diagram D, the braid representative $\mathbf{W}_N$ of D obtained as in the proof of Lemma 2.1 will be called the *canonical braiding* of D relative to N. Note that in a canonical braiding the labeling of the chords determines the order of the generators representing those chords. Thus relabeling the chords alters the canonical braiding by commutativity relations in C_{2n}. Changing the starting point for reading N alters the canonical braiding by what will be called a *cyclic permutation.*

One further definition is needed. Let W be a word in the chord monoid which closes to a braid representative of D. We can obtain a name for D from W by labeling the generators in W in the order they occur, and choosing the upper left-most strand as a starting point. The resulting name for D will be called the *standard name* for D from W, and the process of obtaining the standard name from W will be called *reading a name* for D from W.

Before showing how braidings of a fixed diagram are related, we characterize canonical braidings:

Lemma 2.2 *Let D be an n-diagram and $\mathbf{W}$ a braid representative of D. Then $\mathbf{W}$ is a canonical braiding of D with respect to a name N if and only if $W \in C_{2n}$ and each strand of W has exactly one endpoint on it.*

Proof. We have observed the only if portion of this lemma in the proof of Lemma 2.1. Conversely, let $W = A(i_1, j_1)A(i_2, j_2)\cdots A(i_n, j_n)$ have the desired properties. Let N be the standard name for D obtained by reading W. Then the k^{th} entry of N is the label on the k^{th} strand of $\mathbf{W}$, implying that $\mathbf{W}$ is the canonical braiding of D with respect to N. $\|$

Recall that Markov's theorem implies that any two braid words with isotopic closures are related by relations in B_m, conjugation in B_m, and stabilization (where B_m is the m-strand braid group [1]). The corresponding theorem for braided chord diagrams is similar. The commutativity relations $[A(i, j), A(k, l)] = 0$ in C_m play the role of the relations in B_m. To interpret conjugation in the chord diagram setting, consider the closure $\mathbf{W}$ of a word $W \in C_m$. The final generator $A(i, j)$ in the word W can be slid around the closure portion of $\tilde{W}$ at the expense of increasing the indices i and j by one. Algebraically, we have that the closed braid associated to $W'A(i, j)$ is equal to the closed braid associated to $A(i + 1, j + 1)W'$ for

$1 \leq i < j < m$, also the same for $W'A(i,m)$ and $A(1,i+1)W'$. Thus we make one more definition: The operation of replacing the word $W'A(i,j) \in C_m$ with $A(i+1,j+1)W'$ (with appropriate modification if $j = m$) is called *cyclic permutation* in C_m. The operation which takes C_m to C_{m+1} by replacing $WA(i,m) \in C_m$ with $WA(i,m+1) \in C_{m+1}$, as in Figure 2.3, is called *stabilization.* We shall also use the term stabilization for the inverse of this operation.

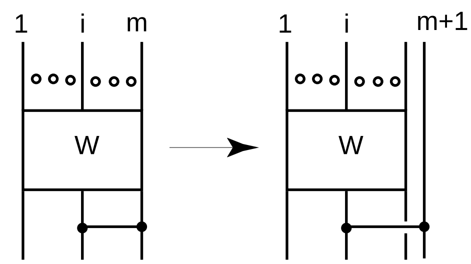


Figure 2.3: Stabilization

The notions of cyclic permutation and stabilization can be combined to yield an operation that will be called generalized stabilization. Generalized stabilization is adding or deleting a trivial loop anywhere in the word W. In the case of adding a trivial loop, this amounts to the following algebraic operation. Let $W = UA(i,j)V$ and replace it with $W' = U'A(i,j+1)V'$, where U' is U with all indices greater than j increased by one and V' is V with all indices $k \geq j$ increased by one. This corresponds to cyclically permuting W until the generator $A(i,j)$ is last and has the endpoint corresponding to j on the final strand, stabilizing, and then un-permuting to get W'. See Figure 2.4. Deleting a trivial loop can occur when $W = UA(i,j)V$ with U void of the index j and V void of the index $j-1$. The word $W' = U'A(i,j-1)V'$ is again a braiding of the same diagram as W, where U', V' are U, V with indices greater than $j-1$ reduced by one. Either of these processes is a generalized stabilization. We will call adding a trivial loop an *increasing stabilization*, as the braid index increases under this operation. Similarly, deleting a trivial loop will be called a *decreasing stabilization.*

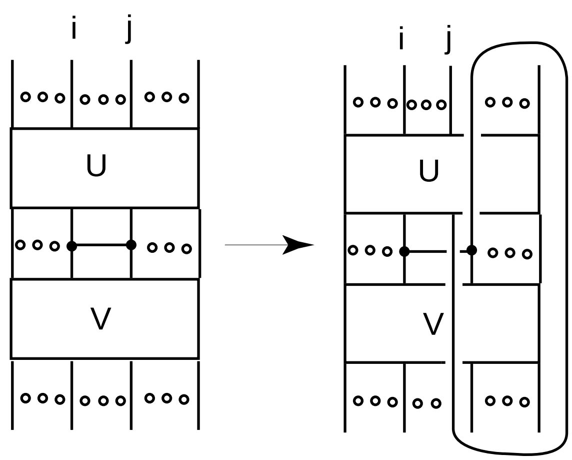


Figure 2.4: Generalized stabilization

Using these definitions, we describe an equivalence relation on braidings which identifies all braidings of a fixed diagram:

Theorem 2.3. *Let* **V** *and* **W** *be two braidings of a fixed diagram D. The words V and W are related by a sequence of the following moves:*

(i) *Commutativity relations in C_m,*

(ii) *Cyclic permutation in C_m, and*

(iii) *Stabilization.*

Proof. We first show (Lemma 2.4) that any two canonical braidings are equivalent under moves (i)-(iii) above. Then we show (Lemma 2.5) that these moves can be used to change an arbitrary braid representative into a canonical braiding. Lemmas 2.4 and 2.5 prove the theorem.

Lemma 2.4. *If $\mathbf{B}_N, \mathbf{B}_M$ are canonical braidings of D relative to the names N and M, then $\mathbf{B}_N, \mathbf{B}_M$ are equivalent under moves of type (i) and (ii).*

Proof. In order to prove the lemma we recall how two names for a diagram are related. There are two choices to be made when determining a name for a diagram; namely, the labels on the chords and the choice of starting point for reading the labels. Fixing the starting point but changing the labels amounts to a reordering of the generators in the canonical braid representative. Since all the generators commute in a canonical braid representative, this is accomplished by using type (i) moves. Fixing the labels and changing the starting point changes the name by a cyclic permutation, which changes the canonical braid representative by moves of type (ii). This proves the lemma. ||

Lemma 2.5 *If* **W** *is a braiding of the diagram D then* **W** *is equivalent under moves (i)-(iii) of Theorem 2.3 to a canonical braiding for D.*

Proof. Let **W** be a braid representative for D, where $W \in C_m$. First we can assume that all the indices $1 \leq r \leq m$ are used in the word W, or that every strand of W has an endpoint on it. If not, then a finite number of decreasing stabilizations will reduce W to a braiding of D without empty strands. As decreasing stabilizations are a consequence of moves (i) and (ii), we make this simplifying assumption.

Let us assume, then, that every strand of W has endpoints on it. If D is an n-diagram and each strand of W has exactly one endpoint on it, then $W \in C_{2n}$ and Lemma 2.2 implies that W is a canonical braid representative for D. We are reduced to the case where W has at least one strand with more than one endpoint on it. Note, however, that if an increasing stabilization is performed just before the last endpoint on a strand, then the number of endpoints on the given strand is reduced by one and a new strand with a single endpoint is introduced. See Figure 2.5. Thus a finite number of increasing stabilizations will create a word W' in which each strand has a single endpoint on it. Thus W' is a canonical braiding for D by Lemma 2.2, proving the lemma and the theorem. ||

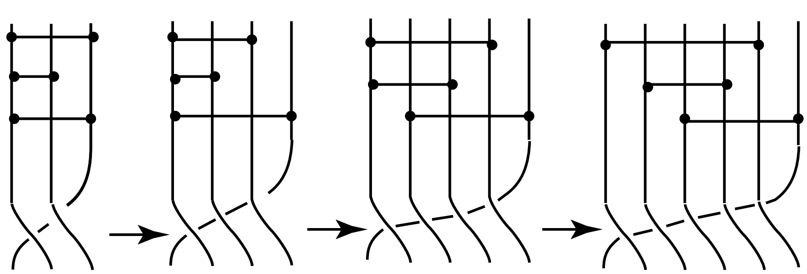


Figure 2.5

The reader who is familiar with [2] or [4] will notice that up to this point we have not considered the 4 or 1-term relations at all. Theorem 2.3 does not consider equivalence modulo four-term relations, merely equivalence of braidings of a single diagram. The algebra

$$A^{hor} = \oplus_{m=0}^{\infty} C_m / \{4 \ term \ relations\}$$

is considerably more complex than C_m. Since we have lots of work to do before we get to A^{hor} we defer our discussion of the 4 and 1-term relations to §5.

3 Braid index

In this section we define and study the braid index of chord diagrams. One reason braid index is of interest lies in the study of finite-type invariants. When studying weight systems on n-diagrams, restricting to diagrams of a fixed braid index may simplify calculations. We give an algorithm for computing the braid index of a chord diagram, and determine the effect of certain diagram characteristics on braid index.

Definition 3.1. *The braid index of an n-diagram D is the least number of strands used in any braid representative for D.*

One might think that a more natural definition of braid index would be the minimum braid index of any singular knot which respects D. It turns out that this definition is equivalent to Definition 3.1. Indeed, given a braid representation of D of minimum braid index one can obtain a singular knot which respects D and has the same braid index. Merely replace each chord $A(i,j)$ with the generator A_{ij} of the pure braid group with one of its crossings smashed (see [1] for a definition of the A_{ij} and Figure 3.1). The singular knot which results from this operation will be called the *standard knot* respecting $\mathbf{W}$. This is well-defined as smashing different crossings in the full twist of A_{ij} results in singular knots related by flypes.

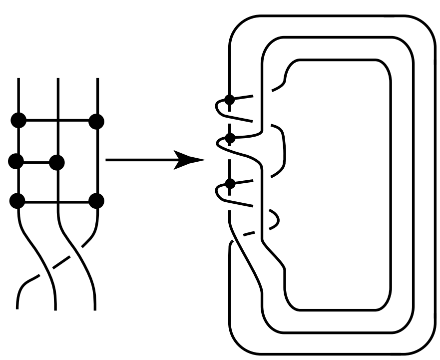


Figure 3.1: The standard knot respecting the closure of a singular braid

Conversely, given a braided singular knot K which respects D of minimum braid index one can construct a braid representative of D with the same braid index. To see this, let K have braid index b, and number the strands of K by $1, \ldots, b$. To create a braid representative for D from K note that any braid which closes to a knot must have an associated permutation that is a cycle. Since every b-cycle is conjugate to $(12\ldots b)$, it must be possible to cut the singular braid K open somewhere and arrange it so that the permutation on the cut-open braid is the

b-cycle $(12\dots b)$. Let β be the cut-open braid and x the non-singular braid $\sigma_{b-1}\sigma_{b-2}\dots\sigma_1$. Then $(\beta x^{-1})x$ closes to K and the associated permutation for βx^{-1} is the identity. To finish constructing the braid representative for D, begin with the identity braid on b strands. Reading down the singular braid K, each time a double point involves strands i and j connect strands i and j with a chord in the candidate for a braiding for D. See Figure 3.2. After you pass completely through the singular braid, you have the structure of the double points on the b-strand identity braid. Closing the word in C_b yields a braiding of D. Thus the topological and algebraic definitions of braid index agree.

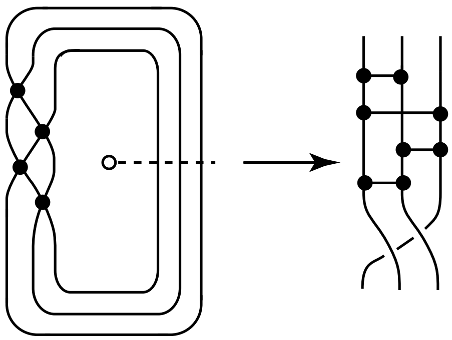


Figure 3.2: From a singular knot to a braided diagram

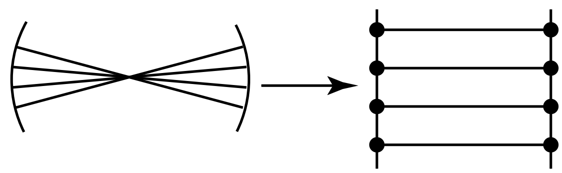


Figure 3.3: The braid form for a fan

We now give an algorithm for computing the braid index of an n-diagram D. To do so, we first recall the processes of increasing and decreasing stabilization. Recall that increasing stabilization is the operation of replacing $W = UA(i,j)V$ with $W' = U'A(i,j+1)V'$, where U' is U with all indices greater than j increased by one and V' is V with all indices $k \geq j$ increased by one. The inverse of this process is a decreasing stabilization. Recall that one can easily recognize from a word $W' = U'A(i,j)V'$ in the chord monoid C_m when a braid index reducing stabilization is possible. If the index $j-1$ does not occur in V' and the index j doesn't appear in U', then a braid index reducing stabilization is possible. It has the effect of replacing the word $W' = U'A(i,j)V' \in C_m$ with the word $W = UA(i,j-1)V \in C_{m-1}$, where U is U' with all indices greater than j reduced by one, and V is V' with all indices at least as great as j reduced by one See Figure 2.4. Decreasing stabilzations and canonical braid representatives will be the essential ingredients of our algorithm for computing the braid index. The algorithm is as follows:

Step 1. Fix any chord of D, label it 1, and choose one endpoint of chord 1, labeling it $*$.

Step 2. For each labeling of the remaining chords:

i. Form the name resulting from beginning to read the labels at $*$.

ii. Construct the canonical braiding of D relative to that name.

iii. Perform all possible decreasing stabilizations to the canonical braiding, and record the braid index of the resulting braid.

Theorem 3.2. *The minimum of the braid indices found using the above algorithm is the braid index of D.*

Proof. Let **W** be a braiding of D of minimal braid index. We will show that W is cyclically equivalent to a word which is obtained using only decreasing stabilizations from one of the names in step 2 above. As cyclic permutations don't change the braid index, this implies that the minimum braid index is found in part (iii) of step 2.

Cyclically permute W until the generator corresponding to chord 1 is first and of the form $A(1, j_1)$, where the endpoint corresponding to $*$ is on the first strand. Call the result W again, and label the remaining chords of D in the order in which their corresponding generators occur in W. Now, as in the proof of Lemma 2.5, use increasing stabilizations to achieve a canonical braiding for D relative to some name N. Since increasing stabilizations do not change the order in which generators occur, and since generators in canonical braidings occur in the order given by the labeling of the chords, we have the following. The canonical braiding which W increases to is precisely the one found in part (ii) of step 2 using the ordering of chords dictated by the order of generators in W. $\|$

Once the first chord is chosen, there are $(n-1)!$ ways of labeling the remaining chords. Thus this algorithm grows factorially in n. Intuitively this algorithm is very simple. Just place the double points in a cycle, and connect corresponding double points with edges always traveling counterclockwise. The result is a closed braid representative of D, so record its braid index. There are $(n-1)!$ cyclic orderings of n points, and this algorithm just checks each ordering.

We now discuss some techniques which help in determining braid index. First we will discuss amalgamating fans, which allows one to relate an n-diagram to a diagram with (possibly) fewer chords and the same braid index. Then we obtain a lower bound on the braid index as being one more than the greatest number of parallel chords of a diagram. These techniques will be used to characterize 2- and 3-braid diagrams.

Definition 3.3. *Given a chord diagram D, a fan F in D is a collection of chords satisfying two properties:*

(i) *Every chord in F crosses every other chord in F,*

(ii) *There are two arcs α_1 and α_2 on the outer circle of D such that each chord in F has one endpoint in each arc and no other chords in D have endpoints in the α_i. See Figures 3.3 and 2.1 (c) for illustrations of fans.*

We remark that fans are easily recognized in names for D, they result identical strings of integers occurring twice in the name. A maximal fan is a fan which is not contained in any other fan. It is clear that any chord in D is contained in a unique maximal fan, possibly consisting of the chord alone. The process of replacing a fan F of chords in D with a single chord, and labeling that chord with a weight equal to the number of chords in F, will be called amalgamating the fan F. Let D_a denote the diagram obtained from D by amalgamating all maximal fans. One retrieves D from D_a by simply replacing each chord weighted w with a fan of w chords. The reason for discussing fans at this point is that fans do not affect the braid index (see Proposition 3.4 and its corollary which follows). Thus to compute the braid index of a diagram D it suffices to compute that of D_a, ignoring the weights on the chords. When we speak of braid representatives of D_a, then, we mean the unweighted diagram D_a.

Proposition 3.4.

(i) *If* $W_a = A(i_1, j_1) \dots A(i_n, j_n)$ *closes to represent* D_a*, then the word* $W = A(i_1, j_1)^{w_1} \dots A(i_n, j_n)^{w_n}$ *closes to represent* D.

(ii) *Any minimum braid index representative* W *of* D *is equivalent, using commutativity relations in* C_m *and cyclic permutation, to the closure of a word* $A(i_1, j_1)^{w_1} \dots A(i_n, j_n)^{w_n}$*, where* $W_a = A(i_1, j_1) \dots A(i_n, j_n)$ *closes to braid representative for* D_a.

Proof. To prove (i) it suffices to check that raising a generator $A(i_r, j_r)$ to a power w_r has the effect of replacing the chord represented by $A(i_r, j_r)$ in D by a fan of weight w_r.

To see this we will compare the name N_a obtained from W_a, that is $A(i_1, j_1) \dots A(i_n, j_n)$ to the name N obtained from $W = A(i_1, j_1)^{w_1} \dots A(i_n, j_n)^{w_n}$. Increasing the power of a generator leads to parallel chords, and since the name N is obtained by reading down the strands in W, this leads to identical sequences of integers (see the example in Figures 2.1 (c) and (d)). Hence replacing $A(i_r, j_r)$ in W_a with $A(i_r, j_r)^{w_r}$ has the effect of replacing both occurrences of r in N_a with a sequence of w_r integers. By the remark preceding this proposition, this is tantamount to replacing chord r in D_a by a fan of w_r chords.

Now let us prove (ii). Let $\mathbf{W}$ be a braiding of D, and choose a starting point $*$ on the outer circle of D which is not interior to any fan. Label the first chord after $*$ by 1, and cyclically permute W until the first generator corresponds to chord 1. Further permute until the first generator is $A(1, j_1)$, and the endpoint on the first strand corresponds to the endpoint of 1 which occurs immediately after $*$ on D. We will call this cyclic permutation W as well. (In other words, we permute W until the name of D obtained by reading W is the same as if we start at $*$). We would like to show that using commutativity relations in C_m alters W to a word of the form $A(1, j_1)^{w_1} \dots A(i_n, j_n)^{w_n}$, where each maximal fan of D is represented in W by the power of a generator. If this is the case, then removing the exponents is equivalent to amalgamating fans and $W_a = A(i_1, j_1) \dots A(i_n, j_n)$ is a braiding of D_a.

First we will show that each chord in a maximal fan is represented in W by the same generator, then that relations in C_m can be used to put W in the desired form. Let F be a fan in D, and r, s adjacent chords in F. We claim that r and s must be represented by the same generator in W. Since r and s are adjacent chords in F, the name for D is of the form $N = \dots rs \dots rs \dots$. If r and s are represented by different generators $A(i_r, j_r)$ and $A(i_s, j_s)$, then $i_r \leq i_s$ and $j_r \leq j_s$ (this follows from the fact that N is obtained by reading W). Since $N = \dots rs \dots rs \dots$, no endpoints can occur between the first r and first s in W. If $i_r < i_s$, then a reducing stabilization would be possible. This contradicts the fact that W is of minimal braid index, hence $i_r = i_s$. Similarly one sees that $j_r = j_s$. Since adjacent chords in F must be represented by the same generator in W, all chords in F must be represented by the same generator.

Now, if all the generators representing chords in F are not in the same syllable, then other generators must be between them. However, since F corresponds to identical sequences of integers in N, the indices of the intermittent generators must be distinct from those corresponding to chords in F. Thus the commutativity relations in C_m can be used to make the portion of W corresponding to F into a single syllable. ||

Corollary 3.5. *The braid indices of* D *and* D_a *are the same.*

Proof. Let $b(D)$ and $b(D_a)$ be the respective braid indices. Part (i) of the previous proposition implies that $b(D) \leq b(D_a)$, while part (ii) implies the reverse inequality. ||

Thus to find the braid index of an n-diagram, one can first amalgamate all maximal fans and work with D_a. This technique will be used in proving that 3-braid representatives of chord diagrams are unique up to cyclic permutation, for the most part.

We now establish some natural bounds on the braid index of a diagram.

Proposition 3.6. *If the set of chords of the diagram D has a subset of p parallel chords, then $b(D) \geq p+1$.*

Proof. We show that if D_p is a diagram with p non-intersecting chords then $b(D_p) = p+1$. The proposition follows by noting that if D_p is the subdiagram of D consisting of the p parallel chords, then $b(D) \geq b(D_p)$.

If $p = 1$ then D_p consists of a single chord and $b(D_p) = 2$. Now suppose that $b(D_p) = p+1$ for all diagrams with p non-intersecting chords, and let D_{p+1} be a diagram with $p+1$ parallel chords. Choose an outermost chord of D_{p+1}, i.e. a chord which has all other chords on one side of it and no endpoints on the other, and call it chord $p+1$. Let D_p be the diagram obtained from D_{p+1} by removing the chord $p+1$, and let $\mathbf{W}$ be a minimal braid representative for D_{p+1}. Since one side of chord $p+1$ has no endpoints on it, one can cyclically permute $\mathbf{W}$ until the generator representing chord $p+1$ is of the form $A(1,2)$. Moreover, by removing the generator $A(1,2)$ one introduces a decreasing stabilization, hence $b(D_{p+1}) = b(D_p)+1$. By the induction hypothesis, $b(D_p) = p+1$, and we are done. $\|$

Figure 2.1(b) gives an example which shows that the lower bound of Proposition 3.6 is not sharp. By Proposition 3.6 the braid index of the diagram in Figure 2,1(b) is at least 3. It is easy to see that there is a 4-braid representative, so in fact it is at most 4. For 3-braids we shall see in §5 that, modulo the 4-term relations there is a simple way to recognize diagrams of braid index 3. Indeed, by Proposition 5.3, the braid index cannot be 3, so it is in fact 4. The bound obviously is sharp in the case of 2-braids. In spite of much effort we were unable to find a combinatorial property of diagrams (such as that given by Proposition 3.6) which would allow us to compute the braid index by inspection, in the general case. As a weak substitute we mention that there is a trivial upper bound:

Lemma 3.7. *If D is an n-diagram, then $b(D) \leq n+1$.*

Proof. Think of D as being built by adding n chords to a circle. Each time you add a chord, you increase the braid index by at most one. Since the braid index of the circle is 1, $b(D) \leq n+1$. $\|$

4 Closed 3-braids

In this section we address the question of whether or not finite-type invariants are stratified by braid index. We give an affirmative answer in the case of closed 2 and 3 braids. A more precise statement of the question we address follows: Can one choose representatives for chord diagrams in such a way that if K is a (honest-to-goodness) knot of braid index b and v is any finite-type invariant, then $v(K)$ is determined by its values on chord diagrams of braid index at most b?

If one restricts their attention solely to the universal Vassiliev invariant, the answer is clearly negative. Computations made in [5] show that the universal Vassiliev invariant of the trefoil includes diagrams of braid index higher than 2. Thus one has to change their point of view and consider completed actuality tables as in Vassiliev's initial work [Va] and [2]. Our goal is to choose a singular knot which respects each chord diagram in such a way that finite-type invariants are stratified by braid index in this sense.

Recall [2] that if one is given a completed actuality table for a Vassiliev invariant v and wants to determine $v(K)$ for some knot K, one uses the Vassiliev skein relation and crossing changes. More precisely, one changes crossings of K until it is the unknot and using skein relation writes $v(K)$ in terms of the value of v on the unknot and on singular knots with one double point. The value of v on the unknot is known, and one changes crossings in the singular knots until the value of v on them is in terms of known values and values on knots with more double points. The process continues until you are working with singular knots with n double points, where n is the order of v. At each level, the known values of v are the values of v on the choices of singular knots respecting chord diagrams. One can think of this process as writing the original knot K as a sum of singular knots with increasing double points, where the singular knots in the sum are those on which the value of v is known (see [11] for a development of Vassiliev's theory along these lines). The question then is can we choose singular knots respecting chord diagrams at each level so that for any closed b-braid K, this sum contains only singular knots of index at most b.

To show this when K is a closed 3-braid, we first show that 3-braid representatives of n-diagrams are essentially unique. We then use the standard singular knot representative of $\mathbf{W}$ to fill in the actuality tables (if 3-braid representatives are not unique, then one has to show that this choice is well-defined). Finally we use the algebraic structure of the braid group to prove that finite-type invariants on closed 3-braids are stratified by braid index.

Before beginning we make some notational simplifications. The generators for C_3 are $A(1,2)$, $A(2,3)$, and $A(1,3)$ which we denote by a, b, and c respectively. See Figure 4.1.

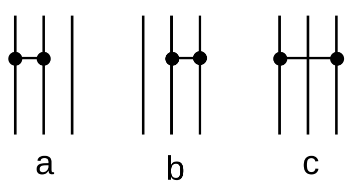


Figure 4.1: Generators for C_3

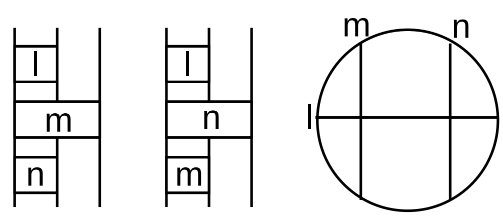


Figure 4.2: Distinct 3-braid representatives for the same (l+m+n)-diagram

Thus if D is a 3-braid diagram, it can be represented by $\mathbf{W}$ where W is a word in a, b, and c. Recall that in braid index considerations one can consider the amalgamated diagram D_a instead. Since all fans in D_a consist of a single chord, the powers of generators must be one in any word W which closes to a representative of D_a. In what follows we will usually consider the diagram D_a, returning to our original diagram D as necessary. Note that in C_3 all generators are cyclically equivalent. In particular, under cyclic permutation we have $a \to b \to c \to a$.

We now turn our attention to the question of uniqueness of 3-braid representatives of chord diagrams. We will show that the words $a^l c^m a^n$, $a^l c^n a^m \in C_3$ close to represent the same chord diagram, but are not cyclically equivalent. See Figure 4.2. We will also show that this is the only case in which 3-braid representatives of a diagram are not unique up to cyclic permutation.

Theorem 4.1. *Let D be a 3-braid diagram, and let $V, W \in C_3$ be words which close to represent D. Then either:*

(i) *V and W are cyclically equivalent, or*

(ii) *V and W are cyclically equivalent to the words $a^l c^m a^n$, $a^l c^n a^m \in C_3$.*

Proof. Let D_a be the amalgamation of D. As $V, W \in C_3$ close to represent D, Proposition 3.4 (ii) implies they are cyclically equivalent to words in which all maximal fans appear as powers of generators. Thus we can cyclically permute V, W until one obtains a braid representative for D_a from each simply by changing the exponents of generators to one. We call the representatives of D_a obtained in this way V_a and W_a respectively. Note that cyclic permutations of V_a and W_a lift to permutations of V and W, so when we permute V_a and W_a we will consider the words V and W permuted as well.

We would like some more structure, so we pick a labeling $1, \ldots, n$ of the chords of D_a and a starting point $*$, which we choose just before one of the endpoints of chord 1. Let N be the name for D_a obtained from this labeling and choice of starting point. Now further permute V_a and W_a until the first generator in each corresponds to chord 1 of D_a and has the endpoint immediately following $*$ on the first strand. Thus V_a and W_a begin with an a or a c generator, and reading either V_a or W_a yields the name N. There are now three cases to consider, one of which immediately reduces to one of the other two. Our goal in each case is to show that $V_a = W_a$, except in case (ii) of the theorem. If $V_a = W_a$, it follows that $V = W$. Recall that V and W are obtained from V_a, W_a by replacing single chords with fans (increasing the powers of exponents). Given that $V_a = W_a$, the only way that we could have inequality in V and W is if the same generator in V_a, W_a corresponded to different maximal fans in D. This would mean that the same generator in $V_a = W_a$ corresponded to different chords in D_a. This can't happen since reading either word yields the name N.

Once we show that $V_a = W_a$, then, we will know that V and W are cyclically equivalent. The situation where $V_a \neq W_a$ happens rarely, and we show that it happens only in situation (ii) of the theorem. Recall that V_a and W_a must have either an a or a c generator, and we study the following cases.

case 1: Both V_a and W_a begin with a c generator.

If this is the case, change the chosen endpoint $*$ of chord 1 and cyclically permute the words once. You are then in case 2.

case 2: Both V_a and W_a begin with an a generator.

We show that V_a and W_a are identical except in a few cases relying heavily on the fact that reading either of the words gives the name N for D_a.

Since the first generator in V_a, W_a is an a, and since the power of each generator is 1, the second generator in each word is either a b or c. If the second generators are identical we move on to the third, so assume they are different. Specifically, assume $V_a = acV'$ and $W_a = abW'$, and that the b generator in W_a corresponds to chord i in D_a while the c in V_a corresponds to chord j. Reading N from W_a we have $N = 1 \ldots 1i \ldots i \ldots$, and from V_a we have $N = 1j \ldots 1 \ldots j \ldots$. Combining these partial names we see that j must follow the first 1 and i the second in N. Further, the second occurrence of j in N happens after the first i; therefore, we have $N = 1j \ldots 1i \ldots i \ldots j \ldots$ or $N = 1j \ldots 1i \ldots j \ldots i \ldots$. Lets consider each possibility separately.

Suppose that $N = 1j \ldots 1i \ldots i \ldots j \ldots$, and recall that $V_a = acV'$. From N we see that chord i is parallel to both chords 1 and j. The fact that i is parallel to 1 and its generator follows that for 1 in V_a implies that the generator corresponding to i in V_a must be a b. However, a chord represented by a b generator in V' must intersect j, contradicting the fact that i is parallel to both. Hence this case cannot occur.

Now suppose $N = 1j \ldots 1i \ldots j \ldots i \ldots$. Since each generator in V occurs to the first power, the third generator must be either an a or a b. If $V_a = acaV''$, and the second a corresponded to the chord k, then reading V_a would give $N = 1jk \ldots 1k \ldots j \ldots$. Hence this case cannot occur and $V_a = acbV''$. Moreover, name considerations imply that the b generator must correspond to the chord i. This yields that $N = 1j \ldots 1i \ldots ji \ldots$. Similar considerations with the word W_a yields $N = 1j \ldots 1ij \ldots i \ldots$. Taken together, these imply that $N = 1j \ldots 1iji \ldots$, and that $V_a = acbV''$ and $W_a = abaW''$ See Figure 4.3.

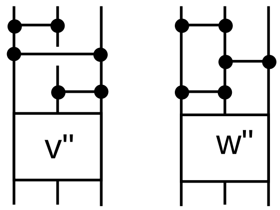


Figure 4.3: Possible V_a and W_a

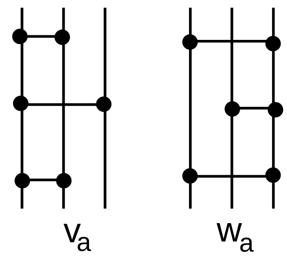


Figure 4.4: V_a and W_a in case 3

Since no endpoints can occur between the first i and second j, the middle strand of V'' must be void of endpoints and V'' consists entirely of c generators. However, if V'' were non-empty the c generator would form a fan with the original a generator in V_a. Since V_a represents D_a this cannot happen, and V'' is empty. Since V_a and W_a both represent D_a, it follows that W'' is also empty.

So far we have that if V_a, W_a both begin with a generators, but have different second generators, then $V_a = acb$ and $W_a = aba$ (see Figure 4.2 for the diagram D_a). The words $V_a = acb$ and $W_a = aba$ are cyclically equivalent; however, the cyclic permutation taking W_a to V_a does not preserve the initial point $*$ or the chords. (this arises because of the symmetry of the diagram D_a in Figure 4.2). However preservation of the point $*$ is essential if we consider the original diagram D again. If chords $1, i, j$ are weighted n, m, l respectively, then we have $W = a^n b^m a^l$ and $V = a^n c^l b^m$. These are both representatives of the diagram in Figure 4.2, but are not cyclically equivalent as the ordered triples of exponents are not. These words, however, are cyclically equivalent to those in part (ii) of the theorem, as it is easy to check.

Thus, if V_a, W_a both begin with an a generator and have different second letters, then we are in case (ii) of the theorem. We now have to consider what happens if the first two generators of V_a and W_a are identical, then the first three, etc. Luckily we needn't argue as above, but we prove a lemma and an easy induction takes care of this case.

Lemma 4.2. *Let V_a, W_a be three-braid representatives of D_a, cyclically permuted to start at the same point, and such that $V_a = UV'$ and $W = UW'$ with $l(U) \geq 2$. Then V' and W' are identical.*

Proof. Since $l(U) \geq 2$ at least two types of generators occur in U. Suppose U ends with an a generator, then $N = u_1 a \ldots u_2 a \ldots u_3 \ldots$ where the u_i are the portions of N coming from U and the $\ldots$ are the portions coming from V' or W'. We wish to show that the first generator in V' and W' are identical. Since U ends in an a, V' and W' must begin with either a b or a c. If V' begins with a b, then $N = u_1 a \ldots u_2 a k \ldots u_3 k \ldots$ where k is the chord represented by the first generator in V'. If W' begins with a c generator, we have that $N = u_1 a l \ldots u_2 a \ldots u_3 l \ldots$, contradicting the fact that k follows u_3. Hence W' begins with a b as well. A simple induction proves the lemma in this case. All other cases are similar. ||

We can now consider
case 3: V_a and W_a begin with different generators.

Recall that neither V_a nor W_a can begin with a b generator, so assume that V_a begins with an a generator and W_a with a c. Hence $V_a = aV'$ and $W_a = cW'$. Recall that the first generator in each word corresponds to chord 1 of D_a, and reading each word gives the name N. Thus $N = 1\ldots_1 1\ldots_2$ where $\ldots_1$ contains all endpoints on the first strand of V' and all those on the first and second strand of W'. See Figure 4.4. This implies that no b generators can occur in V', otherwise we would have $N = 1\ldots 1\ldots i\ldots i\ldots$, and both i endpoints would have to be on the third strand of W'. Since W_a is a braiding for D_a, this can't happen. Similarly, one shows that there are no a generators in W'. Thus we have $V_a = acaca\ldots a$ and $W = cbcbc\ldots c$ (V_a can't end in a c generator, as that would create a fan in D_a. For the same reason W_a ends in a c generator).

Recall that the first generator in V_a and W_a corresponds to chord 1 in D_a. The name obtained from reading V_a is then $N = 1c_1a_1\ldots c_na_n1a_1a_2\ldots a_nc_1c_2\ldots c_n$ where the number a_i (resp. c_i) is the label on the chord that the i^{th} a (resp. c) generator in V_a corresponds to. Similarly, reading W_a we have $N = 1c_1c_2\ldots c_nb_1\ldots b_n1b_1c_1b_2c_2\ldots b_nc_n$. There is no labeling of the a_i, b_i, c_i which make these names identical unless $n = 1$ (note that the c_i in V_a could be different than the c_i in W_a). For $n > 1$, this contradicts the fact that reading V_a gives the same name for D_a as reading W_a. Thus the only case in which V_a and W_a could start with different generators is the case where $V_a = aca$ and $W_a = cbc$.

Again, even though these words are cyclically equivalent, the original words V and W may not be. Both V_a and W_a represent the diagram D_a of Figure 4.2, and we have already seen that if V and W are not cyclically equivalent then we are in situation (ii) of the theorem.

Thus, in the case where both V_a and W_a begin with the same generator, we have that

1. $V_a = W_a$ and V, W are cyclically equivalent, or
2. V_a, W_a are short and either cyclically equivalent or we are in case (ii) of the theorem.

In the case where V_a and W_a begin with different generators we've seen that V_a, W_a are short and either cyclically equivalent or examples of part (ii) of the theorem.

The proof is complete.||

Now that 3-braid representatives of chord diagrams are essentially unique, we have a well-defined method for choosing singular knots which respect 3-braid diagrams. Recall that the standard knot respecting a braided diagram **W** is the one obtained from the word W by replacing each $A(i, j)$ with the corresponding generator A_{ij} of the pure braid group, and smashing one of the crossings. To fill an actuality table, one must choose singular knots which respect all chord diagrams. For 3-braid diagrams, choose the standard knot corresponding to a braiding for each diagram. We have the following

Corollary 4.3. *The choice of standard knots to respect 3-braid diagrams is well-defined.*

Proof. One must show that standard knots respecting different braidings of the same diagram are isotopic. We do this by picture. First, cyclically equivalent braidings yield isotopic knots, as seen in Figure 4.5.

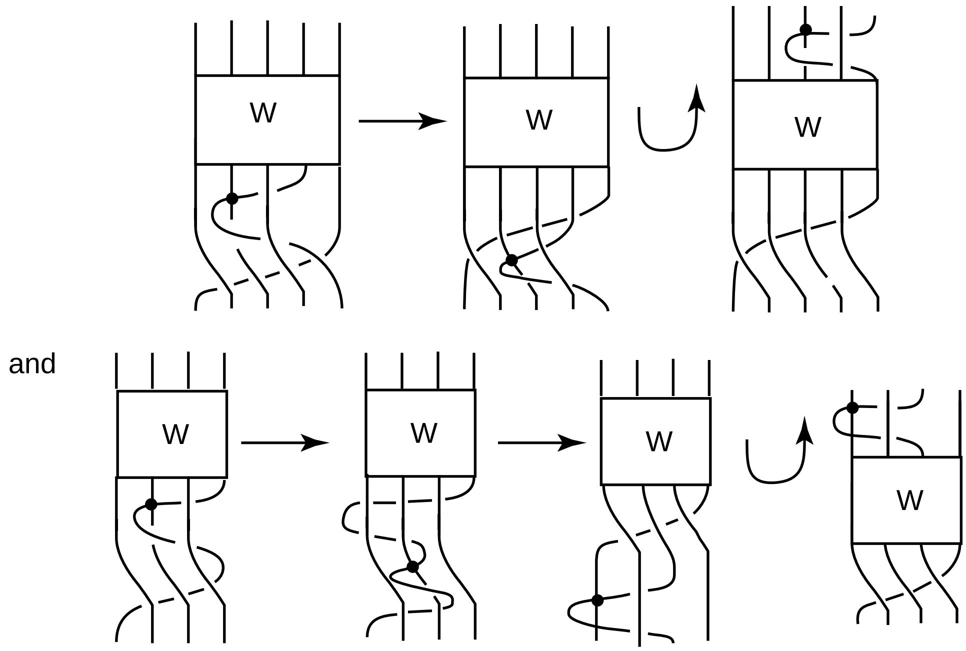


Figure 4.5: Isotopic standard knots

Secondly, one shows that the standard knots coming from the words $a^l c^m a^n$ and $a^l c^n a^m$ are isotopic via a braid-preserving flype (note the similarity with the case of links in [3]. See Figure 4.6. ||

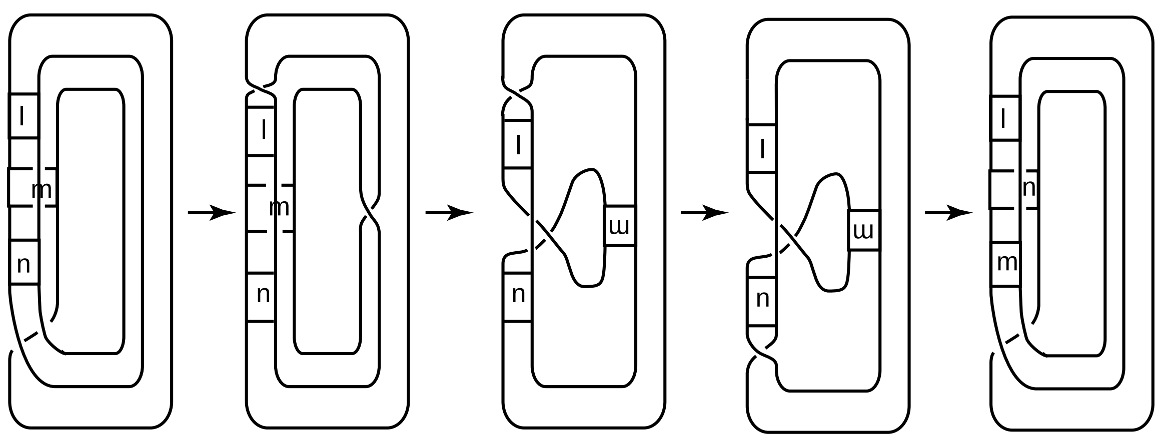


Figure 4.6: A braid-preserving flype

We can now prove the main theorem:

Theorem 4.4. *Finite-type invariants are stratified by braid index for closed 3-braids.*

Proof. We must show that if we're given a closed 3-braid and intend to evaluate a finite-type invariant on it, then we can perform the necessary crossing changes without ever increasing the braid index. It is well known [1] that every element $\beta \in B_n$ can be written uniquely in the form

$$\beta = \beta_2 \beta_3 \dots \beta_n \pi_\beta,$$

where π_β is a permutation braid and each β_j belongs to the subgroup of B_n generated by the set $\{A_{ij} : i < j\}$. We are concerned here with the case $n = 3$, and choose the permutation braid $\pi_\beta = \sigma_2^{-1}\sigma_1^{-1}$. Thus every knot which is a closed 3-braid can be combed so that it's in the form $\beta_2\beta_3\sigma_2^{-1}\sigma_1^{-1}$. Since the permutation braid is the one we've chosen to close our braided chord diagrams, it is also the one appearing at the end of every standard knot obtained from a braiding for a chord diagram. Thus switching crossings to undo the "pure braid" part will result in the standard knots we've chosen to respect our 3-braid diagrams, and we are done. ||

The difficulty with extending this proof to higher braid index is that Corollary 4.3 no longer holds. Then one can't be assured that merely switching crossings that exist in the braid diagram will yield the singular knot chosen to respect a given chord diagram. If one has to introduce other crossings, there is no guarantee that braid index can be preserved.

5 Four-term relations

In this section we make some concluding remarks about four-term relations in the braid setting and discuss, briefly, the work which needs to be done to achieve the goal we set forth at the beginning of this paper, i.e. to compute finite-type invariants for knots which are closed 3-braids, using weight systems which are restricted to chord diagrams of braid index 3.

First, as is noted by many authors (e.g. [5], [10]) the braid form of four-term relations is

$$[A(i,j), A(j,k) + A(i,k)] = 0.$$

Using this relation it has been noted that you can comb elements in $C_m/\{Four\ term\}$ to have the form shown in Figure 5.1 (e.g. see [6]). That is, there is a spanning set for $C_m/\{Four\ term\}$ whose elements have the special form $W_m W_{m-1} \dots W_2$, where W_j is a word in the generators $A(1,j), A(2,j), \dots, A(j-1,j)$.

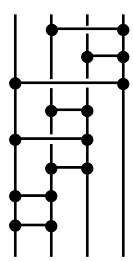


Figure 5.1: A combed element in C_4

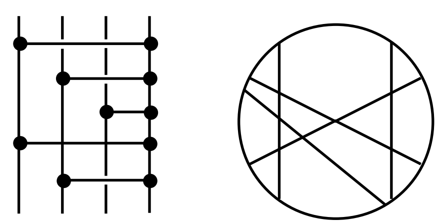


Figure 5.2: A special chord

We now show that there is a significant improvement in this normal form when we allow cyclic permutations in addition to the four-term relations. (i.e. ask what happens if we "close" words in C_m). Letting $A_m = C_m/\{four\ term\ \&\ cyclic\ permutation\}$ we have the following result:

Proposition 5.1. *A_m is spanned by words W_m in the generators $\{A(i,m),\ i = 1, \dots, m-1\}$ of C_m.*

Proof. Since we're working modulo four-term relations, we can assume the words we begin with are already combed. Now consider the final generator in a combed word $W \in A$. If

$W = W'A(i,m)$ then we're done since W is combed. Suppose that $W = W'A(i,j)$ with $j < m$, and let us call each collection of generators in W with the same last endpoint a "block" of W. Then the last block in the combed W must involve the j^{th} strand.

Cyclically permute W to $A(i+1,j+1)W'$. This braid is not combed. We can comb $A(i+1,j+1)W'$ using, the four-term and commutativity relations in C_m. The result is a sum of combed words ΣW_i with the property that the final blocks of the W_i's have one less generator than the original W. A finite number of such cyclic permutations and combings (now with each W_i) replaces W with a sum of combed words in which the last block uses the $(j+1)^{st}$ strand. If $j+1 = m$, we're done: otherwise, repeat the above procedure until all words in the sum have the desired form. ||

Remark 5.2: Words in the generators $A(i,m)$ in C_m close to diagrams which have a nice property. Before describing that property, recall that in [7] it is proved that, modulo the 4-term and 1-term relations, one can choose a basis of chord diagrams which have a chord that intersects every other chord (a 'special' chord). The diagrams obtained by closing "one-block" words in C_m either have a special chord or a single chord could be added that intersects every other chord. The key observation is that if W is a one-block word, then $A(1,m)W$ closes to a diagram with a special chord. This is clear since every chord in W has exactly one endpoint on the last strand, hence the endpoints are on either side of the endpoints of the initial $A(1,m)$. To see this, begin reading the braid at the top of the m^{th} strand. See Figure 5.2. Thus the above proposition is similar to the theorem of [7], yet not as precise. On the other hand, the one-term relation was not needed for the proof of Proposition 5.1, where it is definitely needed to prove the theorem of [7].

One might think that the reason that Proposition 5.1 is weaker than the result of Chmutov and Dhuzhin [7] is that we have not used the 1-term relations, but more is at issue than that. Let's look at the special case $m = 3$. Proposition 5.1 implies that A_3 is spanned by words in the generators b and c for C_3 (see Figure 4.1). If the word W begins with a b generator, then the special chord has weight zero (i.e. the word cW represents a diagram with a special chord). After amalgamating fans, the cases where there is no special chord are thus the words $(bc)^k$. Applying the methods used to prove the Main Theorem in [7] to express these special diagrams $(bc)^k$ as a linear combination of diagrams with special chords (see Figure 5.3 for the cases $k = 2,3,4$) we find that $k-1$ of the terms on the right hand side involve diagrams with 3 parallel chords, which (by Proposition 3.6) have braid index at least 4. Of course it is entirely possible that there is some *other* linear combination of diagrams of braid index 3 which will do the same job, we do not know. We note that the examples in Figure 5.3 *do* use the 1-term relation. Clearly more work needs to be done to understand this situation.

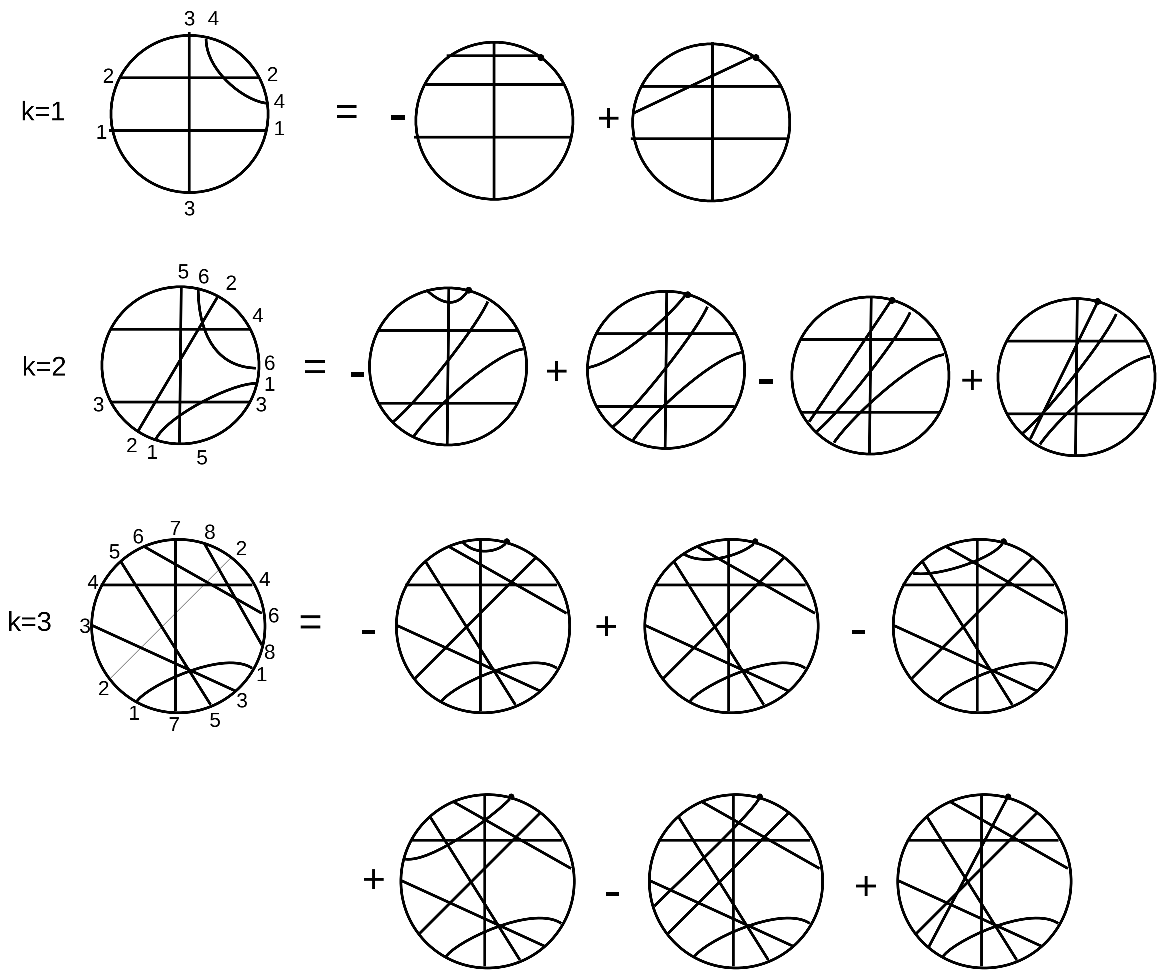


Figure 5.3

We continue our investigations. We next show that 3-braid diagrams with special chords are easily recognized. First note that a diagram with special chord (possibly of weight zero) admits a particular type of name obtained as follows: begin reading at the special chord, numbering the chords as they are encountered. The name so obtained is of the form $N = 123\dots n\sigma(1)\sigma(2)\dots\sigma(n)$, where n is the number of chords in D and σ is a permutation on n letters. Given an diagram D with special chord, call the permutation σ a permutation associated to D. Note that in general there may be several permutations associated to one diagram. At the very least, beginning to read at different ends of the special chord yields the permutations σ and σ^{-1}. Recall that a descent is a place where $\sigma(i) > \sigma(i+1)$. We now prove

Proposition 5.3. *A diagram D with special chord has braid index 3 if and only if an associated permutation for D has a single descent.*

Proof. First suppose that D is a 3-braid diagram, and let $\mathbf{W}$ be a 3-braid representative for D. We must show that there is a permutation σ associated to D with a single descent. Cyclically permute W until the generator corresponding to the special chord is a c generator at the beginning of the word, and call the result W again. Thus $W = cW'$, and since c intersects every other chord W' is a word in the generators b and c. Number the chords by the order they appear in W and begin reading the name for D at the top of the third strand. Reading down the third strand yields the sequence $123\dots n$, while continuing on the first strand yields an increasing sequence consisting of the numbers of the c generators. Finally, reading the second strand yields a second increasing sequence coming from the b generators. Hence the name obtained in this manner has a single descent.

Now suppose a name $N = 12\dots n\sigma(1)\sigma(2)\dots\sigma(n)$ for D has a single descent. We construct a 3-braid representative for D as follows. The sequence $\sigma(1)\dots\sigma(n)$ consists of two increasing sequences. Associate a c generator to each number in the first sequence and a b generator to each in the second, then construct W by placing the b and c generators in numerical order. Reading the word W beginning at the top of the third strand yields the original name N. $\|$

Concluding Remarks: From Remark 5.2 and the example given there it is clear that a key obstacle to understanding the four-term relations in the braid setting remains: to understand the interplay between the 4-term and 1-term relations and 'stabilization'. We need to learn all of the relations between chord diagrams of braid index 3 which are consequences of 4-term relations in which chord diagrams of higher braid index appear. The spanning set of Proposition 5.1, and that of [7], are too large to be a basis for weight systems on chord diagrams of braid index 3, so unknown additional relations clearly exist.

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