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Transactions of the American Mathematical Society is currently published by American Mathematical Society.

BRAIDS, LINK POLYNOMIALS AND A NEW ALGEBRA

JOAN S. BIRMAN AND HANS WENZL

ABSTRACT. A class function on the braid group is derived from the Kauffman link invariant. This function is used to construct representations of the braid groups depending on 2 parameters. The decomposition of the corresponding algebras into irreducible components is given and it is shown how they are related to Jones' algebras and to Brauer's centralizer algebras.

In [J,3] Vaughan Jones announced the discovery of a new polynomial invariant of knots and links, which bore many similarities to the classical Alexander polynomial, but was seen to detect properties of a link which could not be detected by the Alexander invariants. The discovery was a real surprise, one of those exciting moments in mathematics when two seemingly unrelated disciplines turn out to have deep interconnections. The discovery came about in the following way. Jones' earlier contributions in the area of Operator Algebras had produced, in [J,1], a family of algebras $A_n(t)$, $t \in \mathbf{C}$, indexed by the natural numbers $n = 1, 2, 3, \ldots$, and equipped with a trace function $\tau\colon A_n(t) \to \mathbf{C}$. His algebra $A_n(t)$ was a quotient of the well-known Hecke algebra of the symmetric group, which we denote by $\mathscr{H}_n(l, m)$ to delineate our particular 2-parameter version of it. Jones had discovered, in [J,2], that there were representations of Artin's braid group B_n in the algebra $A_n(t)$, in fact there were maps

$$B_n \xrightarrow{\phi} \mathscr{H}_n(l, m) \xrightarrow{p_2} A_n(t)$$

from B_n into the multiplicative group of $A_n(t)$ which factored through $\mathscr{H}_n(l, m)$.

Links enter the picture via braids. Each oriented link L in oriented S^3 can be represented by a (nonunique) element β in some braid group B_n. There is an equivalence relation on $B_\infty = \coprod_{n=1}^{\infty} B_n$, known as Markov equivalence, which determines a 1-1 correspondence between equivalence classes $[\beta] \in B$ and isotopy types of the associated oriented links L_β. Jones' discovery was that with a small renormalization his trace function on $A_\infty(t) = \coprod_{n=1}^{\infty} A_n(t)$ could be made into a function which lifted to an invariant on Markov classes

in B_∞. That modified trace, described in [J,3] and in more detail in [J,4], is the Jones polynomial $V_L(t)$. (It becomes a polynomial when the parameter t is regarded as an indeterminate.)

The polynomial $V_L(t)$ was quickly generalized in a six-author paper [FYHLMO], to a 2-variable polynomial $P_L(l,m)$. One of the authors was A. Ocneanu. Ocneanu's interpretation of $P_L(l,m)$, as described in [FYHLMO] and [J,4], is via a lift of the Jones trace to the Hecke algebra $\mathscr{H}_n(l,m)$. Others, notably Lickorish and Millett [L-M] and Hoste [H], had discovered the identical polynomial $P_L(l,m)$ by combinatorial methods which had little to do with braids, traces or algebras.

On the heels of $P_L(l,m)$ came two additional polynomials, the Kauffman polynomial $K_L(l,m)$ and a precurser later identified as $K_L(l,m)$. References are [B-L-M and K,1]. These were proved to be well defined by various combinatorial methods, including not only generalizations of the methods used in [L-M and H], but also new techniques in [K,1], which again bypassed braids. The polynomials $P_L(l,m)$ and $K_L(l,m)$ were shown to be independent, with each distinguishing links the other could not distinguish. On the other hand (see [L]), $V_L(t) = K_L(t^{-3/4}, -(t^{1/4} + t^{-1/4})) = P_L(it^{-1}, -i(t^{1/2} - t^{-1/2}))$. (We shall have more to say about this curious fact in §4 below.)

The purpose of this note is to reverse the process begun by Jones. We will use the existence of $K_L(l,m)$, and apply the methods used to construct it in [K,1] to construct a new two-parameter family of finite-dimensional algebras, $\{\mathscr{C}_n(l,m); n = 1,2,3,\dots\}$, complete with trace, such that $K_L(l,m)$ is, after appropriate renormalization, that trace, just as $P_L(l,m)$, renormalized, was shown by Ocneanu to be the trace on $\mathscr{H}_n(l,m)$.

In §1 below, we will review the background. In §2 we define our algebra by generators and relations, and explain our motivation, which is based upon Kauffman's work. In §3 we study the algebra $\mathscr{C}_n(l,m)$. We prove that $\mathscr{C}_\infty(l,m) = \coprod_{n\in\mathbb{Z}}^\infty \mathscr{C}_n(l,m)$ supports a nondegenerate trace. The existence of this trace is a direct consequence of the fact that $K_L(l,m)$ is a link type invariant. We use the existence of the trace to uncover the structure of $\mathscr{C}_n(l,m)$, by techniques which derive from the "basic construction" of Jones in [J,1]. We prove that $\mathscr{C}_n(l,m)$ is semisimple and is a direct sum $\mathscr{H}_n \oplus \mathscr{H}_n'$, where $\mathscr{H}_n = \mathscr{H}_n(l,m)$. The structure of $\mathscr{H}_n'$ as an algebra over $\mathbb{C}[l,\alpha]$, where $m = \alpha + \alpha^{-1}$, is determined inductively from the inclusion pattern $\mathscr{C}_{n-2} \subset \mathscr{C}_{n-1}$. The dimension of $\mathscr{H}_n(l,m)$ is $n!$ (see [J,4] for a proof of this well-know fact). It follows from our structure theorem that the dimension of $\mathscr{C}_n(l,m)$ is $1 \cdot 3 \cdot 5 \cdots (2n-1)$.

The relationship between $\mathscr{C}_n(l,m)$ and the 1-parameter Jones algebra $A_n(t)$ is very interesting. We prove in §4 that there are two distinct homomorphisms from $\mathscr{C}_n(l,m)$ onto $A_n(t)$. One of them factors through $\mathscr{H}_n(l,m)$, and is not unexpected in view of the fact that $\mathscr{C}_n(l,m) = \mathscr{H}_n \oplus \mathscr{H}_n'$, with $\mathscr{H}_n(l,m)$ mapping in a known way onto $A_n(t)$. The other does *not* factor through $\mathscr{H}_n(l,m)$. The manner in which the irreducible representations of $\mathscr{C}_n(l,m)$ go over to

irreducible representations of $A_n(t)$ in the second homomorphism seems quite remarkable. We discovered that there are two homomorphsims in our attempts to understand the curious fact that $V_L(t)$ occurs both as $P_L(it^{-1}, i(t^{1/2} - t^{-1/2}))$ and as $K_L(t^{-3/4}, -t^{1/4} - t^{-1/4})$.

The existence of these two homomorphisms is not the only curious fact about $\mathscr{C}_n(l, m)$. In §5 we discuss an algebra which was studied by R. Brauer in 1937 [Br]. Like $\mathscr{C}_n(l, m)$, Brauer's algebra has dimension $1 \cdot 3 \cdot 5 \cdots (2n - 1)$. It is defined by pictures which bear a striking similarity to our pictures, indeed his pictures give a very easy way to find a basis for our algebra. We will show that our algebras can be regarded as a deformation of Brauer's similarily as $\mathscr{H}_n(l, n)$ is a deformation of CS_n. This was also observed independently by H. Morton and P. Traczyk (see [MT]).

The final section of the paper, §6, discusses a possible application of our work to the question of whether the braid group is a linear group.

1. The braid group B_n, the algebra $\mathscr{H}_n$, and the link polynomial $P_L(l, m)$

Let $\vec{p} = (p_1, \ldots, p_n)$ be an n-tuple of distinct points p_i on the complex plane, which for convenience may be assumed to lie along the x-axis. For present purposes, a *braid* β on n-strands is an equivalence class $[e]$ of level-preserving embeddings of n disjoint copies of the unit interval $I^{(n)} = \coprod_{j=1}^n I_j$ in $C \times I$, where e sends the n copies of $\{0\}$ (respectively $\{1\}$) to $\vec{p} \times \{0\}$ (respectively $\vec{p} \times \{1\}$). The equivalence relation is $e \approx e'$ if e is isotopic to e' rel $\partial I^{(n)}$. Multiplication is by concatenation and rescaling, and the identity element is the constant embedding $e_j(t) = p_j \times t$ for each $t \in I_j$, where e_j is the jth coordinate function, $j = 1, \ldots, n$. This makes the set of all n-braids into a group B_n, *Artin's braid group*. A typical element of B_4 is illustrated in

Figure 1, via its projection onto the $x - t$ plane, but with double points replaced by overpasses and underpasses. From such pictures it is intuitively clear that the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ shown in Figure 2 generate B_n. This was proved by Artin in [Ar], who also proved that defining relations are:

(1) $$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2, \ 1 \leq i, j \leq n - 1;$$

(2) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \leq i \leq n - 2.$$

Note that there is a natural way to orient a braid, determined by the orientation on the unit interval. We orient our braids from top to bottom.
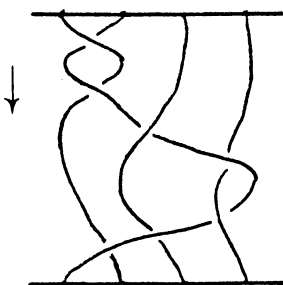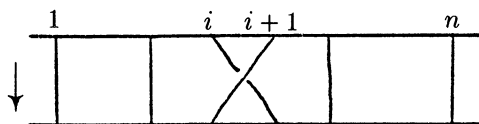
FIGURE 1   FIGURE 2

Links are obtained from braids when the free ends at the top and bottom of a braid are joined up by n disjoint arcs as in Figure 3 to form a *closed braid*. If $\beta \in B_n$, we denote the link so-obtained by L_β. It carries a natural orientation, determined by the orientation on β.

Two oriented links L, L' in oriented 3-space are *equivalent* if L is isotopic to L'. The equivalence class is a *link type*. Alexander proved in [A1] that every oriented link is isotopic to L_β for some (nonunique) element $\beta \in B_\infty$, $B_\infty = \coprod_{n=1}^\infty B_n$. Markov's theorem (see [B]) asserts that the equivalence relation on B_∞ which is generated by the following two moves:

  (i) $\beta \leftrightarrow \alpha \beta \alpha^{-1}$, $\alpha, \beta \in B_n$;
  (ii) $B_n \leftrightarrow B_{n+1}$, by $\beta \leftrightarrow \beta \sigma_n^{\pm 1}$

determines the equivalence class of all elements $\beta \in B$ which close to determine a given link type. Thus the problem of classifying link types is equivalent to the algebraic problem of classifying *Markov classes* $[\beta]$ in B_∞.

In view of Markov's first move ((i) above) a necessary condition for a function with domain B_∞ to be invariant on Markov classes is that it be a class function on each group B_n. We now review how Markov class invariants have been constructed out of class functions on representations of the braid groups.
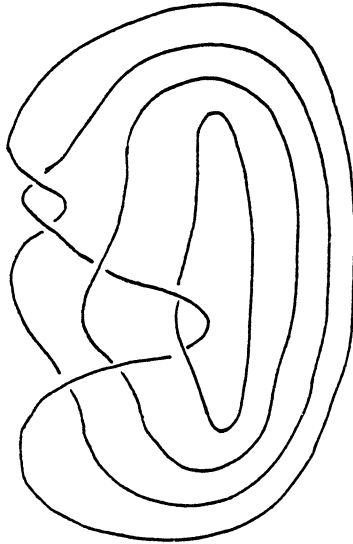
FIGURE 3

The Hecke algebra $\mathscr{H}_n(l, m)$ of the symmetric group is a 2-parameter family of algebras, with parameters $l, m \in \mathbf{C}$, and with $n \in \mathbf{N}$ a fixed natural number. For our purposes the most convenient definition of $\mathscr{H}_n(l, m)$ will be by generators and relations. Our generators will be denoted $\tilde{g}_1, \dots, \tilde{g}_{n-1}$.

Notation. Instead of writing the relation $\tilde{g}_i \tilde{g}_j = \tilde{g}_j \tilde{g}_i$, $|i - j| \geq 2, \dots$, we will write $(1)_{\tilde{g}}$, and similarly $(2)_{\tilde{g}}$ for relation (2) in the variables $\tilde{g}_1, \dots, \tilde{g}_{n-1}$. With this convention, $\mathscr{H}_n(l, m)$ has defining relations $(1)_{\tilde{g}}$, $(2)_{\tilde{g}}$ and

$$(3) \qquad\qquad l\tilde{g}_i + l^{-1}\tilde{g}_i^{-1} = m.$$

The algebra $\mathscr{H}_n = \mathscr{H}_m(l, m)$ has been studied by Ocneanu [O] with a full account of the relevant features of his work given by Jones in [J,4]. The facts which we will need here are summarized by

Proposition 1.1 [J,4]. *The algebras $\{\mathscr{H}_n; n \in \mathbf{N}\}$ are ordered by inclusion $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \mathscr{H}_3 \subset \cdots$, with $\mathscr{H}_n$ the subalgebra of $\mathscr{H}_{n+1}$ generated by $\tilde{g}_1, \dots, \tilde{g}_{n-1}$. The algebra $\mathscr{H}_n$ has dimension $n!$. Each $\mathscr{H}_n$ supports a trace function $\tau: \mathscr{H}_n \to \mathbf{C}$, which is characterized uniquely by the conditions:*

$$(4) \qquad\qquad \tau(a + b) = \tau(a) + \tau(b);$$

$$(5) \qquad\qquad \tau(ab) = \tau(ba);$$

$$(6) \qquad\qquad \tau(1) = 1;$$

$$(7) \qquad \tau(w\tilde{g}_n) = (m^{-1}(l + l^{-1}))^{-1}\tau(w) \quad \text{whenever } w \in \mathscr{H}_n \subset \mathscr{H}_{n+1}.$$

$$L_+ \qquad\qquad L_- \qquad\qquad L_0 \qquad\qquad L_\infty$$

FIGURE 4

Note that (3), (4), (6) *and* (7) *imply that*:

(8) $\qquad \tau(\tilde{g}_i) = \tau(\tilde{g}_i^{-1}) = (m^{-1}(l + l^{-1}))^{-1}, \qquad 1 \le i \le n - 1.$
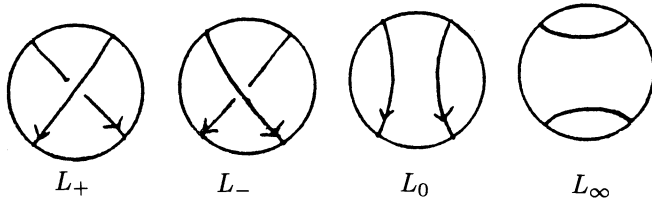
The more common parametrization of $\mathscr{H}_n$ may be recovered by introducing new generators $g_i^* = t^{-1/2}\tilde{g}_i$ where t is a new parameter, with $l = it^{-1}$, $m = i(t^{1/2} - t^{-1/2})$. Then (3) becomes

(3)* $\qquad\qquad\qquad (g_i^*)^2 = (t - 1)g_i^* + t.$

This gives a 1-parameter version of the algebra $\mathscr{H}_n(l, m)$ which we will denote by the symbol $\mathscr{H}_n(t)$. A second parameter z can then be introduced via the trace, if one replaces (7) by

(7)* $\qquad \tau(wg_n^*) = z\tau(w) \quad \text{whenever } w \in \mathscr{H}_n(t) \subset \mathscr{H}_{n+1}(t).$

Equations (3)*, (4), (6) and (7)* imply:

(8)* $\qquad\qquad \tau(g_i^*) = z, \qquad \tau(g_i^{*-1}) = t^{-1}z + t^{-1} - 1$

which is less convenient than (8) because of Markov's second move. This explains our choice of the generators $\tilde{g}_i$.

Since the trace is a class invariant in $\phi(B_n)$, and since (by (7) and (8)) it behaves nicely under the mapping $B_n \leftrightarrow B_{n+1}$ defined by Markov's second move ((ii) above) one sees immediately that the function

(9) $\qquad\qquad P_L(l, m) = (m^{-1}(l + l^{-1}))^{n-1}\tau(\phi(\beta))$

is invariant on the Markov class of each $\beta \in B_\infty$. It is the six-author polynomial of [FYHLMO].

Since the unknot is the closure of the 1-braid $1 \in B_1$, one sees from (9) that for an unknotted circle:

(10) $\qquad\qquad\qquad P_{\text{unknot}}(l, m) = 1.$

Let L_+, L_-, L_0 be links which are defined as the closures of n-braids β_+, β_-, β_0, where the defining braids are products of the elementary braids which are identical except for a single letter, with $\beta_+ = \alpha\sigma_i$, $\beta_- = \alpha\sigma_i^{-1}$, $\beta_0 = \alpha$, $\alpha \in B_n$, $1 \le i \le n - 1$. This means that L_+, L_-, L_0 have link diagrams which are identical everywhere except in a small disc, where they differ in the manner indicated in the first three pictures in Figure 4. It follows immediately from

formula (3) and the linearity of the trace function that their polynomials are related by the *crossing change formula*

$$(11) \qquad l P_{L_+}(l, m) + l^{-1} P_{L_-}(l, m) = m P_{L_0}(l, m).$$

This is Proposition 6.2 of [J,4].

The key idea in the papers of Lickorish and Millett [L-M] and Hoste [H] was to use (10) and (11) as the basis of a *definition* of $P_L(l, m)$, thereby bypassing algebras, traces and all of the attendent machinery and focusing on the combinatorics of link diagrams. Their proof that $P_L(l, m)$ is a well-defined link invariant is equivalent to Jones' proof of Proposition 1.1. Our idea in the next section will be to reverse the procedure, using combinatorics to motivate a way to define a new algebra.

2. THE DEFINITION OF $\mathscr{C}_n(l, m)$

For each natural number n we define a 2-parameter family of algebras $\mathscr{C}_n(l, m)$ with generators $G_1, \ldots, G_{n-1}$ and relations $(1)_G$, $(2)_G$, and others which involve elements $E_1, \ldots, E_{n-1}$ defined by

$$(12) \qquad G_i + G_i^{-1} = m(1 + E_i).$$

The additional relations are:

$$(13) \qquad E_i E_{i\pm 1} E_i = E_i,$$

$$(14) \qquad G_{i\pm 1} G_i E_{i\pm 1} = E_i G_{i\pm 1} G_i = E_i E_{i\pm 1},$$

$$(15) \qquad G_{i\pm 1} E_i G_{i\pm 1} = G_i^{-1} E_{i\pm 1} G_i^{-1},$$

$$(16) \qquad G_{i\pm 1} E_i E_{i\pm 1} = G_i^{-1} E_{i\pm 1},$$

$$(17) \qquad E_{i\pm 1} E_i G_{i\pm 1} = E_{i\pm 1} G_i^{-1},$$

$$(18) \qquad G_i E_i = E_i G_i = l^{-1} E_i,$$

$$(19) \qquad E_i G_{i\pm 1} E_i = l E_i.$$

These imply the further relations:

$$(20) \qquad E_i E_j = E_j E_i \quad \text{if } |i - j| \geq 2,$$

$$(21) \qquad E_i^2 = (m^{-1}(l + l^{-1}) - 1) E_i,$$

$$(22) \qquad G_i^2 = m(G_i + l^{-1} E_i) - 1.$$

In every case the indices are chosen from $1 \leq i, j \leq n - 1$ to be all for which the relation in question makes sense.

Remark. Our choice of l and m as parameters suggests (as we will prove in §3) that $\mathscr{H}_n(l, m)$ is a quotient of $\mathscr{C}_n(l, m)$. The homomorphism $\phi : B_n \rightarrow \mathscr{H}_n(l, m)$ defined earlier will be seen to factor through $\mathscr{C}_n(l, m)$, via $\sigma_i \rightarrow l^{-1} G_i \rightarrow \tilde{g}_i$.

The rest of this section will be devoted to explaining our motivation in writing down these particular defining relations for $\mathscr{C}_n(l, m)$, using ideas from [K,1]. The reader who has not studied that paper may wish to omit the rest of this section on a first reading. The underlying idea is that we want to define our algebra in such a way as to force it to admit a trace, with the Kauffman polynomial a renormalization of that trace.

Recall that in §1 we showed that $P_L(l, m)$ satisfied the crossing-change formula (11). A similar situation exists for $K_L(l, m)$, but it requires a digression. Let L be an oriented link which is defined by a link diagram D, i.e. a regular planar projection. Let ε be the algebraic crossing number of the diagram, where crossings are counted as being positive (or negative) to correspond to the first (or second) picture in Figure 4. Kauffman defines in [K,1] a precurser of the polynomial $K_L(l, m)$, which he calls the " L-polynomial". We denote it by $\widehat{K}_D(l, m)$, defining it by:

$$(23.1) \qquad \widehat{K}_D(l, m) = l^\varepsilon K_L(l, m).$$

Note that $\widehat{K}_D(l, m)$ cannot be a link type invariant if $K_L(l, m)$ is, because the addition of a trivial loop in a link diagram, for example using Markov's second move, must leave $K_L(l, m)$ invariant but must then change $\widehat{K}_D(l, m)$. In fact, it must be true that:

(23.2) . If D and D' are diagrams which are identical except for the addition of a trivial loop in D', which increases the algebraic crossing number, then $\widehat{K}_{D'}(l, m) = l\widehat{K}_D(l, m)$.

Continuing, let D_+, D_-, D_0, D_∞ be 4 link diagrams which are identical except inside a small disc, where they differ in the manner indicated in the four pictures in Figure 4. The $\widehat{K}_D$-polynomials $\widehat{K}_{D_+}, \widehat{K}_{D_-}, \widehat{K}_{D_0}, \widehat{K}_{D_\infty}$ which are associated to these diagrams are related by a crossing change formula, vis:

$$(23.3) \qquad \widehat{K}_{D_+}(l, m) + \widehat{K}_{D_-}(l, m) = m(\widehat{K}_{D_0}(l, m) + \widehat{K}_{D_\infty}(l, m)).$$

Two other properties of $\widehat{K}_D(l, m)$ are:

(23.4)   If D is a planar circle, then $\widehat{K}_D(l, m) = 1$;

(23.5)   If D, D' are diagrams which are related by regular isotopy in the plane (see [K,1]), then $\widehat{K}_D(l, m) = \widehat{K}_{D'}(l, m)$.

The 5 axioms (23.2)–(23.5) suffice to determine $\widehat{K}_D(l, m)$ on all diagrams, and (adding (23.1)) to define $K_L(l, m)$ on all links. It is proved in [K,1] that $K_L(l, m)$ is well defined and a link type invariant, and that $\widehat{K}_D(l, m)$ is well defined on *diagrams*, and is an invariant of equivalence classes of diagrams, under "regular isotopy" of diagrams in the plane (see [K,1] for the definition of regular isotopy).

Our idea is as follows. Let $\beta = \sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$ be a braid which closes to an oriented link $L = L_\beta$. The braid determines a diagram $D = D_\beta$ for the link

L, and associated to this diagram is the polynomial $\widehat{K}_D(l, m)$. Eventually, we will define the trace (see §3) so that

$$\text{trace}(G_{\mu_1}^{\varepsilon_1} \cdots G_{\mu_r}^{\varepsilon_r}) = (m^{-1}(l + l^{-1}) - 1)^{1-n} \widehat{K}_D(l, m).$$

Since $\widehat{K}_D(l, m)$ is well defined on diagrams, the trace will be seen to be well defined on monomials in the algebra. Then, if $g_i = l^{-1} G_i$, $1 \le i \le n - 1$, and if $\varepsilon = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1}$, this is equivalent to

$$\text{trace}(g_{\mu_1}^{\varepsilon_1} \cdots g_{\mu_r}^{\varepsilon_r}) = (m^{-1}(l + l^{-1}) - 1)^{1-n} K_L(l, m),$$

or (cf equation (9) above):

$$K_L(l, m) = (m^{-1}(l + l^{-1}) - 1)^{n-1} \text{trace}(g_{\mu_1}^{\varepsilon_1} \cdots g_{\mu_r}^{\varepsilon_r}).$$

We now return to our algebra $\mathscr{C}_n(l, m)$, and to our attempts to motivate the relations in $\mathscr{C}_n(l, m)$. Recall from the discussion in §1 that if L_+, L_-, L_0 are as defined there, and if L_+ is the closure of a braid $\beta_+ \in B_n$, then it must be true that $\beta_+ = \alpha\sigma_i$, $\beta_- = \alpha\sigma_i^{-1}$, $\beta_0 = \alpha$, for some $\alpha \in B_n$, up to cyclic permutation of the defining braid. There seems to be no such braid interpretations for L_∞, indeed L_∞ cannot be defined by a braid in the usual sense.
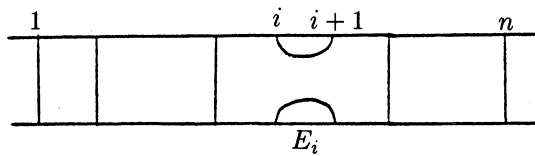
FIGURE 5

We introduce $(n - 1)$ new elements $E_1, \ldots, E_{n-1}$, using (12) as our *definition* of E_i. Clearly E_i ought to have come from a braid-like object in a "braid monoid", namely the embedding on $I^{(n)} = \coprod_{j=1}^n I_j$ in $\mathbf{C} \times I$ which goes with the picture in Figure 5, and it will be helpful to think of it in this way [Bo, Y, K,2 and K,3].

There is no difficulty in composing such generalized braids with one another and with ordinary braids by concatenation. This allows us to interpret relations such as $E_i E_{i+1} E_i = E_i$ (relation (13)) and $G_{i+1} G_i E_{i+1} \approx E_i E_{i+1}$ (relation (14)) by pictures (Figure 6). Note that relations (13), (14), (15), (16), (17) are all associated to pictures which involve only regular isotopy. On the other hand, relations (18) and (19) require something more, because on the passage from the left side of (18) or (19) to the right we must delete a trivial loop, and axiom (23.2) asserts that this must be accounted for by the addition of a multiplicative

$$E_i E_{i+1} E_i \qquad = \qquad E_i$$

$$G_{i+1} G_i E_i \qquad = \qquad E_i E_{i+1}$$
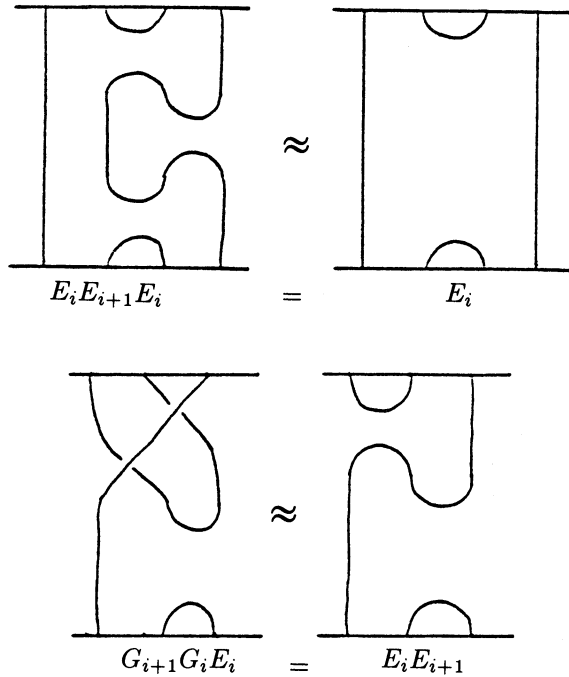
FIGURE 6

factor $l^{\pm 1}$. See Figure 7. The remaining relations (20)–(22) are consequences of the others. Relation (20) follows from $(1)_G$ and the definition of E_i in (12). To obtain (21), multiply each term in (12) by E_i and solve for E_i^2, using (18). To obtain (22), multiply each term in (12) by G_i and solve for G_i^2, using (18). This accounts for all of the relations in $\mathscr{C}_n(l,m)$, and explains how we were led to our definition. In the next section we will study the algebra.

3. PROPERTIES OF $\mathscr{C}_n(l,m)$

**Lemma 3.1.** $\mathscr{C}_n(l,m)$ *is finite dimensional. Its dimension is bounded below by* $n!$. *Each element of* $\mathscr{C}_n(l,m)$ *can be written as a linear combination of elements of the form* $w_1 \gamma w_2$ *with* $\gamma \in \{G_{n-1}, E_{n-1}, 1\}$ *and* w_1, w_2 *monomials in* $1, G_1, E_1, \ldots, G_{n-2}, E_{n-2}$.

Proof. The mapping $\psi : \mathscr{C}_n(l,m) \to \mathscr{H}_n(l,m)$ defined by $G_i \to l g_i$, $E_i \to 0$ is a homomorphism, so $\dim \mathscr{C}_n(l,m) > \dim \mathscr{H}_n(l,m) = n!$.

To prove that $\dim \mathscr{C}_n(l,m) < \infty$, it suffices to prove that each $w \in \mathscr{C}_n = \mathscr{C}_n(l,m)$ is a sum of monomials of the form $w' \gamma w''$ where w', w'' belong to the subalgebra $\widetilde{\mathscr{C}}_{n-1}$ of C_n generated by $1, G_1, \ldots, G_{n-1}$ and where $\gamma = G_{n-1}$ or G_{n-1}^{-1} or E_{n-1} or 1. The statement of the lemma then follows, because we can use (12) to eliminate $G_1^{-1}, \ldots, G_{n-1}^{-1}$.

If $n = 2$ the assertion is trivially true, so assume $n > 2$. Let w be a monomial in $\mathscr{C}_n$. Then $w = w_0 \gamma_0 w_1 \gamma_1 \cdots w_r \gamma_r$, where each $w_j \in \widetilde{\mathscr{C}}_{n-1}$ and

each $\gamma_j = G_{n-1}$, E_{n-1} or 1. If $r = 0$ we are done, so assume $r \geq 1$. Then $w = w_0\gamma_0 w_1 \gamma_1 z$, where $z = w_2\gamma_2 \cdots w_r\gamma_r$. Since $w_1 \in \widetilde{\mathscr{C}}_{n-1}$, induction on r yields $w_1 = v_0\alpha v_1$, where $v_0, v_1 \in \widetilde{C}_{n-2}$ and $\alpha = G_{n-2}$ or E_{n-2} or 1. Then v_0, v_1 commute with γ_0, γ_1 by $(1)_G$ so $w = (w_0 v_0)(\gamma_0\alpha\gamma_1)(v_1 z)$, and it suffices to prove that the product $\gamma_0\alpha\gamma_1$ is a sum of monomials which involve G_{n-1} or E_{n-1} once, for all $\gamma_0, \gamma_1 \in \{G_{n-1}, E_{n-1}\}$ and for $\alpha \in \{1, G_{n-2}, E_{n-2}\}$.

If $\alpha = 1$, then relations (18), (21), (22) suffice to reduce each of these as claimed. Assume $\alpha \neq 1$. Then

$$\gamma_0\alpha\gamma_1 \in \{G_{n-1}G_{n-2}G_{n-1}, \ G_{n-1}G_{n-2}E_{n-1},$$
$$G_{n-1}E_{n-2}G_{n-1}, \ G_{n-1}E_{n-2}E_{n-1}, \ E_{n-1}G_{n-2}G_{n-1},$$
$$E_{n-1}G_{n-2}E_{n-1}, \ E_{n-1}E_{n-2}G_{n-1}, \ E_{n-1}E_{n-2}E_{n-1}\}.$$

Using relations $(2)_G$, (12), (13), (14), (15), (16), (17), and (19) each of these can be reduced to a sum of monomials, each of which contains E_{n-1} or G_{n-1} once. Induction on r completes the proof. $\square$

Remark. See Theorem 3.7 for a precise formula for $\dim \mathscr{C}_n(l, m)$.

We now show that our algebras support a trace function. The trace is most conveniently defined if we rescale the generators, so let $g_i = l^{-1}G_i$, $1 \leq i \leq n - 1$.

Theorem 3.2. *Each $\mathscr{C}_n(l, m)$ supports a trace function $\tau' : \mathscr{C}_n \to \mathbf{C}$ which is characterized by the properties:*

$(4)'$ $$\tau'(a + b) = \tau'(a) + \tau'(b),$$

$(5)'$ $$\tau'(ab) = \tau'(ba),$$

$(6)'$ $$\tau'(1) = 1 \quad \text{where } 1 \in \mathscr{C}_n \text{ for any } n \in N,$$

$(7)'$ $$\tau'(wg_n) = \tau'(wg_n^{-1}) = \tau'(wE_n) = z\tau(w) \text{ if } w \in \text{ subalgebra of}$$
$$\mathscr{C}_n \text{ generated by } g_1, \dots, g_{n-1}, \text{ where } z^{-1} = m^{-1}(l + l^{-1}) - 1.$$

Proof. There is an algebra homomorphism $\Delta : CB_n \to \mathscr{C}_n = \mathscr{C}_n(l, m)$ which sends σ_i to g_i. For any monomial $w = g_{\mu_1}^{\varepsilon_1} g_{\mu_2}^{\varepsilon_2} \cdots g_{\mu_r}^{\varepsilon_r} \in \mathscr{C}_n$, let $W = \sigma_{\mu_1}^{\varepsilon_1} \sigma_{\mu_2}^{\varepsilon_2} \cdots \sigma_{\mu_r}^{\varepsilon_r} \in \Delta^{-1}(w)$. The braid W determines a closed braid $\widehat{W}$ and so a link which we also refer to as $\widehat{W}$, with Kauffman polynomial $K(\widehat{W})$. Let

(24) $$\tau'(w) = z^{n-1}K(\widehat{W}), \quad \text{where } z^{-1} = m^{-1}(l + l^{-1}) - 1,$$

and extend linearly to all of $\mathscr{C}_n(l, m)$.

For later use, note that by (23.1) the link $\widehat{W}$ has diagram polynomial $\widehat{K}(\widehat{W}) = l^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r} K(\widehat{W})$.

Assuming for the moment that τ' is well defined, independently of the choice of $W \in \Delta^{-1}(w)$, we note that τ' satisfies properties $(4)'$–$(7)'$: Property $(4)'$ is

satisfied by definition. Property $(5)'$ holds because the closed braid $\widehat{W}$ depends only on the *cyclic* word W. Property $(6)'$ follows from the fact that $1 \in \mathscr{C}_n$ lifts to $1 \in B_n$, and $\hat{1}$ is the unlink of n-components, with Kauffman polynomial z^{1-n}. Thus $\tau'(1) = z^{n-1}z^{1-n} = 1$. Property $(7')$ follows from the fact that the links defined by closing the braids $W \in B_n$ and $W\sigma_n^{\pm 1} \in B_{n+1}$ are equivalent, and so have the same Kauffman polynomial. It is immediate that these four properties determine τ'.

It remains to prove that τ' is well defined. For this it suffices to show that K takes the value 0 on the kernel of Δ. Recall that CB_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$, with defining relations $(1)_\sigma$ and $(2)_\sigma$, while $\mathscr{C}_n$ is generated by $G_1, \ldots, G_{n-1}$, with defining relations (12)–(19). Set $\xi_1 = m^{-1}(l\sigma_i + l^{-1}\sigma_i^{-1}) - 1 \in B_n$. Then $\Delta(\xi_i) = E_i$. Lift G_i to $l\sigma_i$. Then $\Delta(\sigma_i) = l^{-1}G_i$.

Each of the relations in (13)–(19) lifts to an element in the kernel of Δ, say R_j $(j = 1, \ldots, q)$, and the kernel of Δ is the smallest two-sided ideal in CB_n spanned by the elements $R_1, \ldots, R_q$. Each R_j is a linear combination of monomials in CB_n. For example, one of the relations in (18) yields:

$$R_1 = l^2\sigma_i\xi_i - \xi_i.$$

If we use the symbol $\widehat{WR_j}$ for the sum of the closed braids determined by left-multiplying R_j by an arbitrary monomial $W \in CB_n$, we must prove that $K(\widehat{WR_j}) = 0$. Note that we need only use left multiplication because closed braids belong to cyclic words in B_n.

$$G_iE_i \qquad = \qquad l^{-1}E_i$$
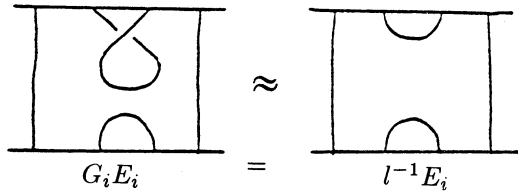
FIGURE 7

We do the calculation in the case of R_1, defined above. Recall that R_1 came from the relation (18), which was motivated by the pictures in Figure 7. We will use Figure 7 in a roundabout way, to be described. First, if we replace ξ_1 in R_1 by its expression as a sum of elementary braids σ_i and σ_i^{-1} we obtain:

$$mR_1 = l^2[l\sigma_i^2 + l^{-1} - m\sigma_i] - [l\sigma_i + l^{-1}\sigma_i^{-1} - m].$$

Therefore, we must prove that

$$l[l^2K(\widehat{W\sigma_i^2}) + K(\widehat{W}) - mlK(\widehat{W\sigma_i})] - [lK(\widehat{W\sigma_i}) + l^{-1}K(\widehat{W\sigma_i^{-1}}) - mK(\widehat{W})] = 0.$$

Equivalently, via (23.1), we must prove that:

$$l[\widehat{K}(\widehat{W\sigma_i^2}) + \widehat{K}(\widehat{W}) - m\widehat{K}(\widehat{W\sigma_i})] - [\widehat{K}(\widehat{W\sigma_i}) + \widehat{K}(\widehat{W\sigma_i^{-1}}) - m\widehat{K}(\widehat{W})] = 0.$$

To do so, we first turn to Figure 4, and recall the crossing-change formula (23.3) which relates the diagram polynomials of four links L_+, L_-, L_0, L_∞ which are defined by diagrams which differ only in a small region, where the difference is as specified in Figure 4. Note that the closed braid diagrams for $\widehat{W\sigma_i^2}, \widehat{W}, \widehat{W\sigma_i}$ make up one such triplet L_+, L_-, L_0 and that $\widehat{W\sigma_i}, \widehat{W\sigma_i^{-1}}, \widehat{W}$ make up another L_+', L_-', L_0'. Therefore if we bring the links L_∞ and L_∞' (with obvious notation) into the picture, we must show that

$$l\widehat{K}(L_\infty) - \widehat{K}(L_\infty') = 0$$

where our links L_∞ and L_∞' are defined by diagrams which are identical everywhere except inside a region where they differ in the manner indicated in Figure 7, the picture on the left corresponding to L_∞ and that on the right to L_∞'. Since Kauffman's axiom (23.2) asserts that $\widehat{K}(L_\infty') = l\widehat{K}(L_\infty)$ as required, and the proof is complete for R_1.

The other relations can be checked in the same way as the previous example. Note that it is enough to show that $\tau'(R_j) = 0$. Indeed, if $W \in B_n$ and R_j is one of our relators, the links corresponding to $\widehat{WR_j}$ are the same everywhere except in a small area in which they differ from each other in the same way as the ones corresponding to $\widehat{R_j}$. So we can show that $K(\widehat{WR_j}) = 0$ using the same crossing change rules coming from the definition of Kauffman's polynomial as we use for showing $K(\widehat{R_j}) = 0$. This implies $\tau'(WR_j) = 0$.

The fact that $\tau'(R_j) = 0$ follows essentially from the definition of our relations which were motivated by pictures coming from the definition of Kauffman's link invariant (see the end of §2). □

We now go on to determine the algebraic structure of $\mathscr{C}_n(l, m)$. Our main tool will be a generalized version of Jones' basic construction in [J,1]. For this, we will first have to extend methods already developed in [J,1 and W]. Our main observation is that the positivity assumptions in those papers are not necessary for our purposes. The only assumption will be that our traces are nondegenerate. This is made precise below.

We will assume throughout this section that $A < B$ are finite dimensional algebras over a field S of characteristic zero. Let $M_k(S)$ be the algebra of all $k \times k$ matrices with entries from S. Assume that $A = \bigoplus A^{(i)}$, $B = \bigoplus B^{(j)}$ with $A^{(i)} \cong M_{a_i}(S)$, $B^{(j)} \cong M_{b_j}(S)$ with $a_i, b_j \in \mathbf{N}$. The vectors $\vec{a} = (a_i)$ and $\vec{b} = (b_j)$ are called the *dimension vectors* of A resp. B.

Let tr be a nondegenerate trace on B, i.e.

(25) $\operatorname{tr}(xy) = 0 \quad \forall y \in B \quad$ implies $x = 0$.

It is well known and easy to check that, in our case, this is equivalent to $\operatorname{tr}(p) \neq 0$ for every minimal idempotent $p \in B$. Let us also recall that if tr is nondegenerate, the map $B \ni b \mapsto \operatorname{tr}(b\cdot) \in B^*$, the dual of B, is an isomorphism between

B and B^* (as usual, $\text{tr}(b\cdot)$, denotes the map $x \mapsto \text{tr}(bx)$), i.e. for any linear map $\varphi : B \to S$ there is a unique $b \in B$ such that $\varphi(x) = \text{tr}(bx)$ for all $x \in B$.

Let us also assume that the restriction of tr onto A is nondegenerate. Then, using the isomorphism above for A and A^*, there exists for every $b \in B$ a necessarily unique $\varepsilon_A(b) \in A$ such that

$$\text{tr}(b\cdot)|_A = \text{tr}(\varepsilon_A(b)\cdot)|_A.$$

It can be checked easily that the map $B \ni b \to \varepsilon_A(b) \in A$ is linear and that

$$(26) \qquad\qquad \text{tr}(b) = \text{tr}(\varepsilon_A(b)),$$

$$(27) \qquad \varepsilon_A(a_1 b a_2) = a_1 \varepsilon_A(b) a_2, \qquad a_1, a_2 \in A, \ b \in B.$$

We call ε_A the trace preserving conditional expectation from B to A. Let us represent B via the left regular representation onto itself. To avoid confusion we write B_ξ if we regard B as the representation space and b_ξ for an element of B_ξ. Let $\mathcal{L}(B_\xi)$ be the set of all linear maps on B_ξ. For any algebra C on B_ξ let

$$C' = \{x \in \mathcal{L}(B_\xi), \ xc = cx, \ \forall c \in C\}.$$

It is well known (and can be checked by explicit matrix multiplications) that C is isomorphic to a direct sum of full matrix algebras iff C' is isomorphic to a direct sum of full matrix algebras. If B (and B_ξ) is finite dimensional it follows that $(B')' = B$ (see for instance [La, XVIII, 3, Theorem 1]).

By our assumptions on the structure of B, we can regard B as a concrete matrix algebra such that with $b \in B$, its transpose, denoted here by b^*, is in B. Moreover, we can choose this matrix representation such that with a also a^* is in $A \subset B$. Note that $^* : B \to B$, $b \mapsto b^*$ is an involution with $(ab)^* = b^* a^*$. We can then define a map $J : B_\xi \to B_\xi$ by $J(b_\xi) = b_\xi^*$. A straightforward computation shows that $JBJ \subseteq B'$. On the other hand, if $b' \in B'$ such that $b'(1_\xi) = b_\xi$, we have $b'(a_\xi) = ab_\xi = (Jb^*J)(a_\xi)$ for any $a \in B$. Hence $JBJ = B'$. As in [J,1], we define $e_A \in L(B_\xi)$ by $e_A(b_\xi) = (\varepsilon_A(b))_\xi$.

Lemma 3.3. *With the above notation, we have*

 (i) $e_A x e_A = \varepsilon_A(x) e_A$ *for $x \in B$.*
 (ii) *Let $x \in B$. Then $x \in A$ iff $e_A x = x e_A$.*
 (iii) $A' = \{B' \cup \{e_A\}\}''$.
 (iv) J *commutes with e_A.*

Proof. See [J, (3.1.4)]. For (ii), replace the statement "ξ is separating" by "the map $B \ni b \to b_\xi \in B_\xi$ is injective".

Obviously, a simple $B^{(j)}$ module is also an A module. Let g_{ij} be the number of simple $A^{(i)}$ modules in its decomposition into simple A modules. The matrix $G = (g_{ij})$ is called the *inclusion matrix* for $A \subset B$. If A and B have the same identity element, we have

$$(28) \qquad\qquad\qquad \vec{b} = G^t \vec{a}.$$

Under these assumptions we can give a precise description of $\langle B, e_A \rangle$, the algebra generated by B and e_A. As in [J,1], we call $\langle B, e_A \rangle$ the *basic construction* for $A \subset B$.

Proposition 3.4. (i) $\langle B, e_A \rangle \cong A'$. *In particular,* $\langle B, e_A \rangle$ *is isomorphic to a direct sum of full matrix algebras.*

(ii) *There exists a canonical 1-1 correspondence between the direct summands of A and $\langle B, e_A \rangle$ such that if q is a minimal idempotent of $A^{(i)}$, $e_A q$ is a minimal idempotent of $\langle B, e_A \rangle^{(i)}$.*

(iii) *The inclusion matrix for $B \subset \langle B, e_A \rangle$ is G^t.*

(iv) *Every element $x \in \langle B, e_A \rangle$ can be written in the form $x = \sum a_i e_A b_i + c$, where $a_i, b_i c \in B$.*

Proof. (i) As $A^* = A$, it follows as in [J,1, (3.1.5)] that $\langle B, e_A \rangle = J A' J$.

(ii)–(iv) follow from [J,1, (3.3.1)]. □

The following theorem is an extension of [W, Proposition (1.2)] to semisimple algebras.

Theorem 3.5. *Let A, B, tr and ε_A be as above. Assume that B is contained in an algebra C and there is an element $e \in C$ such that*

(i) $e^2 = e$,
(ii) $exe = \varepsilon_A(x)e = e\varepsilon_A(x)$ *for all $x \in B$,*
(iii) *The map $a \in A \mapsto ae \in \langle B, e \rangle$ is an injective homomorphism with $e1 = e$.*

Then the 2-sided ideal $\langle e \rangle \subset \langle B, e \rangle$ generated by e is isomorphic to $\langle B, e_A \rangle$ as an algebra. The quotient $\langle B, e \rangle / \langle e \rangle$ is isomorphic to a subalgebra of B and splits as a direct summand in $\langle B, e \rangle$.

Proof. For convenience, let us not distinguish between $B \subset C$ and the isomorphic image of B in the representation on B_ξ. Let us define $\varphi : B \cup \{e\} \to \mathscr{L}(B_\xi)$ by $b \mapsto b. \in \mathscr{L}(B_\xi)$ and $e \mapsto e_A$. We claim that φ extends to a well-defined homomorphsim from $\langle B, e \rangle$ onto $\langle B, e_A \rangle$. The only nontrivial part is to show that φ is well defined. Let $x = \sum a_i e b_i + c = 0$, $a_i, b_i, c \in B$. (It follows from (i)–(iii) that this is the most general form of an element of $\langle B, e \rangle$; see also [J,1].) Let $\tilde{y}. \in B$ and let $\tilde{x} = \sum a_i \varepsilon_A(b_i \tilde{y}) + c\tilde{y}$. Then $0 = eyxe = \varepsilon_A(y\tilde{x})e$ $\forall y. \in B$. By (iii), $\varepsilon_A(y\tilde{x}) = 0$ and therefore also $\text{tr}(y\tilde{x}) = 0$ for all $y. \in B$. Hence $\tilde{x} = 0$ and we obtain

$$\varphi(x)(\tilde{y}\xi) = \tilde{x}\xi = 0.$$

As $\tilde{y}$ was arbitrary, $\varphi(x) = 0$.

On the other hand, let $x = \sum a_i e b_i + c. \in ke\varphi$. Then $0 = \varphi(x)(\tilde{y}\xi) = (\sum a_i \varepsilon_A(b_i \tilde{y}) + c\tilde{y})\xi$ $\forall \tilde{y} = B$. Hence $\sum a_i \varepsilon_A(b_i \tilde{y}) + c\tilde{y} = 0$ and in particular,

(*) $x\tilde{y}e = 0$ $\forall \tilde{y} \in B$.

But then we also have

$$\text{tr}\left(y\left(\sum a_i\varepsilon_A(b_i\tilde{y}) + c\tilde{y}\right)\right) = \text{tr}\left(\sum \varepsilon_A(ya_i)\varepsilon_A(b_i\tilde{y}) + yc\tilde{y}\right)$$
$$= \text{tr}\left(\sum \varepsilon_A(ya_i)b_i + yc)\tilde{y}\right) = 0$$

for all $y, \tilde{y}. \in B$.

So, as tr is nondegenerate, we obtain $\sum \varepsilon_A(ya_i)b_i + yc = 0$ and in particular

$$(**) \qquad\qquad eyx = 0 \qquad \forall y. \in B.$$

Hence if $x \in \ker \varphi$ and $y \in BeB$, we have $xy = yx = 0$ by $(*)$ and $(**)$. In particular, if $z \in \ker \varphi \cap BeB$, z is annihilated both by BeB and $\ker \varphi$.

As $\varphi(BeB) = \langle B, e_A \rangle$ (see Proposition 3.4, (iv)), BeB and $\ker \varphi$ generate $\langle B, e \rangle$. So $z = 1z = 0$, as by (iii) $1 \in B$ is also the multiplicative identity in $\langle B, e \rangle$. It follows from this that $BeB \cong \langle B, e_A \rangle$, which is a direct sum of full matrix algebras.

By (i)–(iii), $\langle e \rangle \cong BeB$. Obviously, the quotient $\langle B, e \rangle / \langle e \rangle$ is already generated by the image of B, hence it is isomorphic to a subalgebra of the semisimple algebra B. It is well known that in this case

$$\langle B, e \rangle \cong \langle e \rangle \oplus \langle B, e \rangle / \langle e \rangle.$$

Lemma 3.6. *Let* $e_n = zE_n$, *where* $z = 1/(m^{-1}(l + l^{-1}) - 1)$ *and assume that* τ' *is nondegenerate on* C_{n-1} *and* C_n. *Then* e_n *has properties* (i)–(iii) *of Theorem 3.5 for* $C_{n-1} \subset C_n$ *and* C_{n-1} *is isomorphic to the subalgebra of* C_n *generated by* $G_1, \ldots, G_{n-2}, E_1, \ldots, E_{n-2}$.

Proof. Using Theorem 3.2, $(7)'$ and the fact that τ' is nondegenerate we can show easily that $\mathscr{C}_{n-1} \ni x \mapsto xe_n \in \mathscr{C}_{n+1}$ is injective. From this follows the last statement of the lemma and property (iii). By Lemma 3.1, any element of $\mathscr{C}_n(l, m)$ can be written as a linear combination of elements of the form $a\chi b$ with $a, b \in \mathscr{C}_{n-1}(l, m)$ and $\chi \in \{1, G_{n-1}, E_{n-1}\}$.

By Theorem 3.2, $(7)'$ we obtain for χ as above and $c \in \mathscr{C}_{n-1}(l, m)$

$$\tau'((a\chi b)c) = \tau'(\chi)\tau'(abc).$$

Hence $\varepsilon(a\chi b) = \tau'(\chi)ab$ where

$$\varepsilon : \mathscr{C}_n(l, m) \to \mathscr{C}_{n-1}(l, m)$$

is the trace preserving conditional expectation. On the other hand

$$e_n(a\chi b)e_n = z^2 aE_n\chi E_n b = \begin{cases} z^2 aE_n b & \text{if } \chi = E_{n-1}, \\ lz^2 aE_n b & \text{if } \chi = G_{n-1}, \end{cases}$$
$$= \tau'(\chi)abe_n = \varepsilon(a\chi b)e_n.$$

This shows property (ii). Property (i) follows from (21). □

We can now determine the structure of C_n as an algebra over an appropriate field.

Theorem 3.7. *Let us regard* $C_{n+1} = C_{n+1}(l, m)$ *as an algebra over* $\mathbf{C}(l, \alpha)$ *where* $m = \alpha + \alpha^{-1}$. *Then* $C_1 \cong \mathbf{C}(l, \alpha)$, $C_2 \cong \mathbf{C}(l, \alpha)^3$ *and*

$$C_{n+1} \cong H_{n+1} \oplus \langle C_n, e_{C_{n-1}} \rangle,$$

where $H_{n+1} = H_{n+1}(l, \alpha + \alpha^{-1})$ *and the second summand is Jones' basic construction for* $C_{n-1} \subset C_n$, *which is semisimple. In particular* C_{n+1} *is semisimple, it has dimension* $1 \cdot 3 \cdot 5 \cdots (2n+1)$ *and* τ', *the trace as defined in Theorem 3.2, is nondegenerate.*

Proof. Clearly, $C_1 \cong \mathbf{C}(l, \alpha)$. Let z be the rational function in l and $m = \alpha + \alpha^{-1}$ as defined in the previous lemma and let

$$p_1 = zE_1, \qquad p_2 = \frac{G_1 - \alpha}{\alpha^{-1} - \alpha}(1 - p_1) \quad \text{and} \quad p_3 = \frac{G_1 - \alpha^{-1}}{\alpha - \alpha^{-1}}(1 - p_1).$$

Then it follows from (12), (18) and (21) that $p_i^2 = p_i$, $p_i p_j = 0$ for $i, j = 1, 2, 3$, and $i \neq j$ and that $p_1 + p_2 + p_3 = 1$. Using property $(7)'$ of Theorem (3.2) one sees that $\tau'(p_i) \neq 0$ for $i = 1, 2, 3$ which shows the assumptions for C_2.

It has already been pointed out earlier that the additional relation $E_i = 0$ reduces the generating relations of C_n to the defining relations of the Hecke algebra H_n. We thus obtain a homomorphism ψ from C_n onto H_n. By definition of ψ, its kernel is equal to the ideal I generated by the E_i's. By a theorem of Benson and Curtis (see [Lu, Theorem (3.1)])

$$(*) \qquad\qquad C_n / \ker \psi \cong H_n \cong \mathbf{C}(l, \alpha) S_n,$$

where S_n denotes the symmetric group on n letters.

Using relation (13) repeatedly, we obtain

$$E_i = E_i E_{i+1} \cdots E_{n-1} E_n E_{n-1} \cdots E_{i+1} E_i.$$

Hence I coincides with the ideal $\langle E_n \rangle \subset C_{n+1}$ generated by E_n. Using relations (13), (16) and (17), we show as in Lemma 3.1 that $\langle E_n \rangle$ is the linear span of products of generators of the form $w_1 E_n w_2$ with $w_1, w_2 \in C_n$. It follows from this, Theorem 3.5 and Lemma 3.6 that

$$\ker \psi = \langle E_n \rangle = C_n E_n C_n \cong \langle C_n, e_{C_{n-1}} \rangle.$$

We shall show at the end of this section by a simple dimension argument that $\langle C_n, e_{C_{n-1}} \rangle = \langle C_n, e_{C_{n-1}} \rangle''$. So by Proposition (3.4), (i) $\ker \psi$ is semisimple which is also true for the quotient $C_{n+1} / \ker \psi$ by $(*)$. Hence

$$C_{n+1} \cong H_{n+1} \oplus C_n E_n C_n.$$

It remains to show that τ' is nondegenerate. Let p be a minimal projection of C_{n-1}. Then $\tau'(e_n p) = z^2 \tau'(p) \neq 0$ by induction assumption. So τ' is nondegenerate on $C_n E_n C_n$ by Proposition (3.4), (ii) and the remark below (25).

Note that if $l = 1$ and $\alpha = i$, $z(1, i) = 0$ and $\tau'(p_2)(1, i) = \tau'(p_3)(1, i) = 1/2$. By the induction assumption $\tau'(p)(1, i) = 0$ for any minimal projection in $C_{n-2}E_{n-2}C_{n-2}$ and $\tau'(p)(1, i) \neq 0$ for any minimal projection in the H_{n-1} summand of C_{n-1}. It follows as in the previous paragraph that $\tau'(p)(1, i) = 0$ for any minimal projection in $C_n E_n C_n$. So τ' can be considered as a trace on H_{n+1} having Markov property $(7)'$. These traces are uniquely determined by $q = -\alpha^2$ and $\eta = (\operatorname{tr}(\tilde{g}_i) + 1)/(1 + q)$. For our special choice of values we obtain $q = 1$ and $\eta = 1/4$. But then it follows from Ocneanu's classification of positive traces on H_∞ that $\tau'(p)(1, i) > 0$ for all minimal projections in H_{n+1} (see [W, proof of Theorem (3.5), (i), α]). Hence $\tau'(p) \neq 0$ for any minimal projection p in the H_{n+1}-summand of C_{n+1}. This shows that τ' is nondegenerate on C_{n+1}.

We next describe how to use "Bratteli diagrams" to determine the structure of $C_n(l, \alpha)$. These diagrams allow one to compute the irreducible representations of these algebras by an iterative procedure. We will first assume that $\langle C_n, e_{C_{n-1}} \rangle = \langle C_n, e_{C_{n-1}} \rangle''$, where the right-hand side is semisimple by Proposition 3.4. We will then show by computing dimensions that the left-hand side cannot be a proper subalgebra of the right-hand side. An example will be given after we describe the basic setup. Related constructions are to be found in [J,2] (for the algebra we called $A_n(t)$ in the introduction) and in [W] (for $\mathscr{H}_n(t)$). See also [J,4], where the underlying construction for the group algebra CS_n of the symmetric group by "Young diagrams" is described in a particularly lucid manner.

The Bratteli diagram for $\mathscr{C}_n(l, m)$ is a graph which we now describe. See Figure 8. The vertices are arranged in horizontal rows, those in the nth row from the top being in 1-1 correspondence with the irreducible representations of $\mathscr{C}_n(l, m)$, $n = 1, 2, 3, \ldots$. A vertex in row n is joined to one in row $n - 1$ by k edges if the restriction of the corresponding representation of $\mathscr{C}_n$ to $\mathscr{C}_{n-1}$, under the natural inclusion $\mathscr{C}_{n-1} \hookrightarrow \mathscr{C}_n$, contains that irreducible representation of $\mathscr{C}_{n-1}$ k times. The vertices are labeled by integers which denote the dimension; the dimension of a representation of $\mathscr{C}_n$ is the sum of the dimensions of all those representations of $\mathscr{C}_{n-1}$ which are joined to it by edges. Note that $\mathscr{C}_n = \mathscr{H}'_n \oplus \mathscr{H}_n$ by Theorem 3.7, where $\mathscr{H}'_n$ is $\mathscr{C}_n E_n \mathscr{C}_n$. We now describe how to construct row n of the diagram from row $n - 1$, $n \geq 3$. One first reflects that part of the diagram which consists of the vertices in rows $n - 2$ and $n - 1$ and all edges joining them about the horizontal line corresponding to the representations of $\mathscr{C}_{n-1}$. One then inserts the dimension of the representations of $\mathscr{C}_{n-1}$, and assigns to each representation of $\mathscr{C}_n$ the sum of the dimensions of the representations lying above it. See Figure 8. This determines that part of $\mathscr{C}_n$ which we have called $\mathscr{H}'_n$. One then augments it by adding more vertices, one for each irreducible representation of $\mathscr{H}_n$, using the conventions in [W] (or in [J,2]; see especially §3 of [J,2]). This makes sense

because each $\mathscr{C}_j$, $j < n$, has already been constructed as $\mathscr{H}_j' \oplus \mathscr{H}_j$, so that the Bratteli diagram for $\{\mathscr{H}_n; n = 1, 2, 3, \ldots\}$ is embedded as a subdiagram of that for $\{\mathscr{C}_n; n = 1, 2, 3, \ldots\}$. Finally, we note that the inductive construction begins with the unique 1-dimensional trivial representation of $\mathscr{C}_1$, and three 1-dimensional representations of $\mathscr{C}_2$, one belonging to $\mathscr{H}_2'$ and the other two to $\mathscr{H}_2$.

Figure 8 illustrates the Bratteli diagram through $n = 4$, and shows the construction of $\mathscr{H}_5'$ from that of $\mathscr{C}_4$ and $\mathscr{C}_3$. Later, in §6, we will use this diagram to obtain explicit matrix representations of B_3 and B_4.
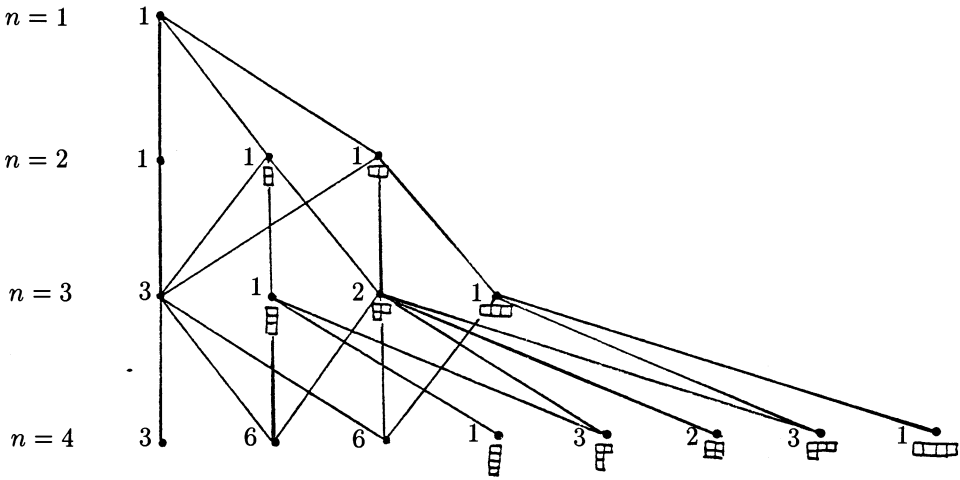
(a) The construction of $H_n' \oplus H_n$, $1 \le n \le 4$
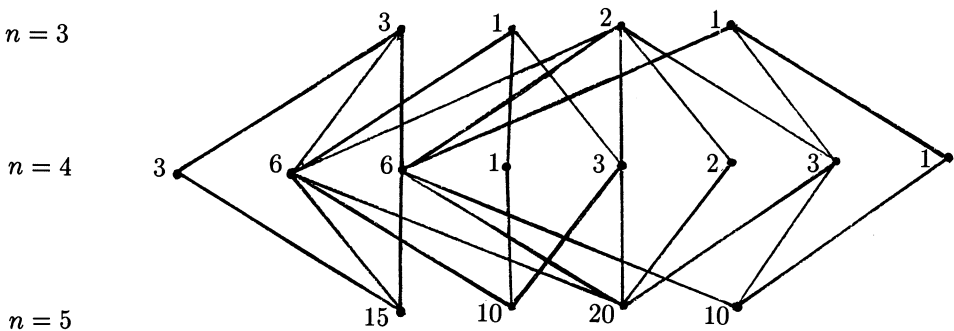
(b) The construction of H_n', $n = 5$

FIGURE 8

We now show the missing part in the proof of Theorem 3.7, namely that $\langle C_n, e_{C_{n-1}} \rangle = \langle C_n, e_{C_{n-1}} \rangle''$. It is a well-known combinatorial fact that the algebras constructed by this procedure (i.e. with $\langle C_n, e_{C_{n-1}} \rangle''$ instead of $\langle C_n, e_{C_{n-1}} \rangle$) have dimension $1 \cdot 3 \cdots (2n + 1)$ (see for instance [S, Lemma 8.7]). On the other hand, we shall see in §5 that for a special choice of (slightly modified) parameters $C_{n+1}(l, \alpha)$ specializes to Brauer's algebra $D_{n+1}(x)$ which has dimension $1 \cdot 3 \cdots (2n + 1)$. So also the dimension of $C_{n+1}(l, \alpha)$ has to be at least that number. This shows again by induction on n that $\langle C_n, e_{C_{n-1}} \rangle = \langle C_n, e_{C_{n-1}} \rangle''$. $\square$

4. Homomorphisms onto $A_n(t)$

Let $A_n(t)$ denote the one-parameter family of algebras studied by Jones in [J,2], with idempotent generators $e_1, \ldots, e_{n-1}$ and defining relations:

$$(29) \qquad e_i e_{i \pm 1} e_i = (t^{1/2} + t^{-1/2})^{-2} e_i,$$

$$(30) \qquad e_i^2 = e_i,$$

$$(31) \qquad e_i e_j = e_j e_i \quad \text{if } |i - j| \geq 2.$$

Jones shows that $A_n(t)$ is semisimple, and that its irreducible representations are in 1-1 correspondence with those irreducible representations of $\mathscr{H}_n(l, m)$ which are associated to Young diagrams having one or two rows. Thus $A_n(t)$ is a homomorphic image of $\mathscr{H}_n(l, m)$. He also shows that there are homomorphisms from B_n into $A_n(t)$ which map the elementary braid σ_i to an appropriate linear combination of e_i and 1, which we will denote by $\hat{g}_i$ here.

We now show that there are two inequivalent homomorphisms ρ and ψ from $\mathscr{C}_n(l, m) \to A_n(t)$. Inequivalent means that there cannot be an isomorphism i of $\mathscr{C}_n(l, m)$ with $\rho = \psi \circ i$.

(i) Define $\rho_1 : \mathscr{C}_n(l, m) \to \mathscr{H}_n(l, m)$ by $\rho_1(E_i) = 0$, $\rho_1(g_i) = \tilde{g}_i$, and $\rho_2 : \mathscr{H}_n(l, m) \to A_n(t)$ by $\rho_2(\tilde{g}_i) = t^{1/2} \hat{g}_i$, and $(l, m) = (it^{-1}, i(t^{1/2} - t^{-1/2}))$. Let $\rho = \rho_2 \circ \rho_1$. The homomorphism ρ was studied in detail in [J,2]. Using the Bratteli diagram for $\mathscr{C}_n(l, m)$ which is described in Figure 8 above, the map ρ_1 may be realized by deleting those representations which belong to vertices which are not labeled by Young diagrams, while ρ_2 deletes those from among the remaining representations in which the Young diagrams have more than two rows.

(ii) It has been shown by Lickorish (see [Li]) that the Jones polynomial can be obtained from the Kauffman polynomial for the special choice of parameters $(l, m) = (t^{-3/4}, -t^{1/4} - t^{-1/4})$. Hence, by factoring over the ideal I_n defined by

$$I_n = \{a \in C_n(t^{-3/4}, -t^{1/4} - t^{-1/4}), \ \tau'(ab) = 0$$
$$\text{for all } b \in C_n(t^{-3/4}, -t^{1/4} - t^{-1/4})\},$$

we obtain a well-defined surjective homomorphism

$$\Psi : C_n(t^{-3/4}, -t^{1/4} - t^{-1/4}) \mapsto C_n(t^{-3/4}, -t^{1/4} - t^{-1/4})/I_n \cong A_n(t).$$

This homomorphism can be described explicitly by (see also [J,2, §3])

$$\Psi(G_i) = t^{-1/4}((1+t)e_i - 1),$$

which implies

$$\Psi(E_i) = -(t^{1/2} + t^{-1/2})e_i.$$

The last equality shows that Ψ does not factor through H_n. An explicit computation shows that τ' annihilates the idempotent p_3 as defined in the proof of Theorem 3.7 for the special choice $\alpha = -t^{1/4}$ and $l = t^{-3/4}$. Hence the hypotheses of Theorem 3.5 are no longer satisfied. In particular, we can no longer construct Bratteli diagrams as in Theorem 3.7. Similar phenomena with Brauer's centralizer algebras (see next section and [W,2]) suggest that C_3 (and also higher C_n's) will no longer be semisimple for this special choice of l and m. Thus the kernel of Ψ sits in a complicated way in $C_n(l, m)$ completely different from our first kernel.

5. Brauer's centralizer algebras

In 1937 R. Brauer introduced, in [Br], a family of algebras which we denote by $\{\mathscr{D}_n(x)\}$, where n is a positive integer and x a parameter or indeterminate. His algebras are ordered by inclusion, $\mathscr{D}_1(x) \subset \mathscr{D}_2(x) \subset \cdots$. We will show that our algebras are perturbations or deformations of $\mathscr{D}_n(x)$ in much the same way as the Hecke algebra $\mathscr{H}_n(t)$ of type A_{n-1} is a deformation or perturbation of CS_n.

First, we explain what we mean about $\mathscr{H}_n(t)$ and CS_n. The group algebra CS_n is generated by elements $s_1, \ldots, s_{n-1}$, where s_i denotes the transposition $(i, i+1)$. It has defining relations $(1)_s, (2)_s$ and $s_i^2 = 1$. On the other hand, the algebra $\mathscr{H}_n(t)$ is generated by $g_1^*, \ldots, g_{n-1}^*$, with defining relations $(1)_{g^*}, (2)_{g^*}$ and $(g_i^*)^2 = (t-1)g_i^* + t$. In particular, this last relation tells us that $\mathscr{H}_n(1)$ is isomorphic to CS_n. Even more, we can use similar pictures to describe monomials in $\mathscr{H}_n(t)$ and in CS_n. Recall the geometric definition of a braid in §1 above as an equivalence class of embeddings of $I^{(n)}$ in $\mathbf{C} \times I$. Similarly, a permutation $\alpha \in S_n$ may be regarded as an equivalence class of immersions of $I^{(n)}$ in $\mathbf{R} \times I \subset \mathbf{C} \times I$. See Figure 9. Multiplication is by concatenation and rescaling, as for braids. There are $n!$ possible equivalence classes of embeddings, and CS_n is the free algebra on these $n!$ possible patterns.

Pictures similar to those we have just given in CS_n were used by R. Brauer in [Br] to describe another algebra $\mathscr{D}_n(x)$ which contains CS_n as a subalgebra. One has the same $2n$ base points, i.e. n in $\mathbf{R} \times \{0\}$ and n in $\mathbf{R} \times \{1\}$, only now the immersions of $I^{(n)}$ in $\mathbf{R} \times I$ are allowed to join these $2n$ points in pairs in an arbitrary fashion, that is we no longer require that the arcs join a point

on $\mathbf{R} \times \{0\}$ to a point on $\mathbf{R} \times \{1\}$. One then obtains a set of $1 \cdot 3 \cdot 5 \cdots (2n - 1)$ pictures because the first point can be joined to $2n - 1$ others, the second to $2n - 3$ others, etc. Multiplication is as before by concatenation, erasure of the middle line and rescaling, but there is a new phenomenon. If S, T are two such graphs, let U be the graph obtained by concatenation and erasure as above. Note that on concatenation there may be closed loops which arise in the middle. Let $m(S, T)$ be the number of such loops. Then $ST = (x)^{m(S,T)} U$, where x is a parameter. The free algebra on the $1 \cdot 3 \cdot 5 \cdots (2n - 1)$ generalized patterns with this product rule, is Brauer's algebra $\mathcal{D}_n(x)$. It is easy to see that it is generated by $2n - 2$ elements $s_1, \ldots, s_{n-1}, e_1, \ldots, e_{n-1}$ where s_i and e_i are the patterns shown in Figure 9.
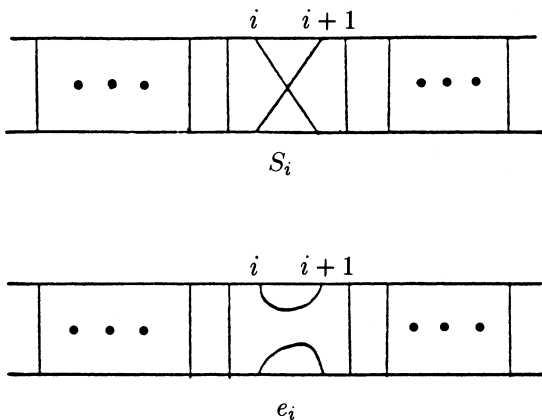

FIGURE 9

Note that for the point $l = i$, $m = 0$ in the parameter space of our algebra E^2 cannot be determined by relation (21). We will avoid this singularity by choosing different parameters (which results in "blowing up" the point $(i, 0)$ to a line). After these substitutions it will be easy to see how our algebras are connected with Brauer's. So let us define elements $\widehat{G}_j$, $\widehat{E}_j$ and parameters $\hat{l}$ and x by

$$G_j = i\widehat{G}_j, \quad E_j = -\widehat{E}_j, \quad l = -i\hat{l} \quad \text{and} \quad m = i\frac{\hat{l}^{-1} - \hat{l}}{x + 1}$$

or, if we solve for x

$$x = m^{-1}(l + l^{-1}) - 1.$$

It is immediate that whenever x and m are well defined, the algebra $\widehat{\mathscr{C}}_n(\hat{l}, x)$ generated by $\widehat{G}_1, \widehat{E}_1, \ldots, \widehat{G}_{n-1}, \widehat{E}_{n-1}$ is equal to $\mathscr{C}_n(l, m)$. Note that relations (13)–(20) also hold for $\widehat{G}_j$ and $\widehat{E}_j$ instead of G_j and E_j, while (21) becomes $\widehat{E}_j^2 = x\widehat{E}_j$. If we set $\hat{l} = 1$, we obtain from (22) $\widehat{G}_j^2 = 1$. One may now check (by pictures) that $\widehat{G}_j \to s_j$ and $\widehat{E}_j \to e_j$ can be extended to a homomorphism from $\widehat{\mathscr{C}}_n(1, x)$ onto $\mathscr{D}_n(x)$. Comparing the dimensions, it follows that

$\widehat{\mathscr{C}}_n(1,x)$ is isomorphic to $\mathscr{D}_n(x)$. So Brauer's algebra can be considered as a special case of our algebra. We finally remark that this does not determine the structure of $\mathscr{D}_n(x)$. As we have seen in §4, the algebras at special values of the parameters can have quite a different structure. The above mentioned problem is studied in a separate paper [W,2] which uses the connections of Brauer's algebra with the representation theory of orthogonal groups.

6. MATRIX REPRESENTATIONS OF B_n

A classical open problem is whether the braid groups $\{B_n; n = 1, 2, 3, \dots\}$ are linear groups. For $n = 1, 2, 3$ the answer is known to be "yes", but for $n > 3$ the problem is unsolved. Since our work gives new linear representations of B_n it seems worthwhile to give some of the new representations explicitly.

There are three mutually nonisomorphic 1-dimensional representations of C_2 belonging to the idempotents p_1, p_2 and p_3 as defined in the proof of Theorem 3.7.

We now turn to B_3. It follows directly from the defining relations that $E_1, E_2 E_1$ and $G_2 E_1$ form a basis of a minimal left ideal of $C_2 E_2 C_2$. Using this basis, we obtain the following new matrix representations of B_3:

$$\sigma_1 \mapsto \begin{bmatrix} l^{-1} & m & 0 \\ 0 & m & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad \sigma_2 \mapsto \begin{bmatrix} 0 & 0 & -1 \\ 0 & l^{-1} & l^{-1}m \\ 1 & 0 & m \end{bmatrix}.$$

Theorem 3.7 determines the structure of $\mathscr{C}_3(l, m)$ inductively. See Figure 8, or proceed as follows. We know that

$$\mathscr{C}_2 \cong \mathscr{H}_2 \oplus \mathscr{C}_1 E_1 \mathscr{C}_1 \cong \mathbf{C}(l, \alpha)^3.$$

The inclusion matrix G for $\mathscr{C}_1 \subset \mathscr{C}_2$ is a 3×1 matrix given by

$$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Using (28) and Proposition 3.4, (iii) we obtain that

$$\mathscr{C}_2 E_2 \mathscr{C}_2 \cong M_3(\mathbf{C}(l, \alpha)).$$

This shows that the matrices above generate the full matrix algebra of 3×3 matrices. On the other hand $\mathscr{H}_3 \cong \mathbf{C}[l, \alpha]S_3$, the group algebra of the symmetric group on 3 letters. Hence

$$\mathscr{H}_3 \cong \mathbf{C}(l, \alpha)^2 \oplus M_2(\mathbf{C}(l, \alpha)).$$

Similarly, we can determine

$$\mathscr{C}_3 E_3 \mathscr{C}_3 \cong M_6(\mathbf{C}(l, \alpha)) \oplus M_6(\mathbf{C}(l, \alpha)) \oplus M_3(\mathbf{C}(l, \alpha)).$$

Then $\mathscr{H}_4'$ is also $\mathscr{C}_3 E_3 \mathscr{C}_3$, a direct sum of two six-dimensional representations and one 3-dimensional representation. Since the representations of B_4 in $\mathscr{H}_4$

are known to be determined by the Burau representation of B_4 (see [J,2]), we focus on $\mathcal{H}_4'$. The 3-dimensional representation of B_4 factors through B_3, and so it cannot be faithful, but the 6-dimensional ones stand a chance. We therefore describe one of them in detail.

As above, let $m = \alpha + \alpha^{-1}$, and let

$$e_i = \frac{1}{m^{-1}(l + l^{-1}) - 1} E_i.$$

Let $v = (G_1 - \alpha)(1 - e_1)$. Then, a basis for the 6-dimensional part which we investigate is vE_3, E_2vE_3, G_2vE_3, $E_1E_2vE_3$, $G_1E_2vE_3$, and $G_1G_2vE_3$. Allowing the generators G_1, G_2, G_3 of $\psi(B_4)$ to act on this basis on the left, and using the defining relations for C_4, we obtain the following two parameter matrix representations of B_4:

$$\sigma_1 \to \begin{bmatrix} \alpha^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & l^{-1} & ml^{-1} & \alpha ml^{-1} \\ 0 & 1 & 0 & 0 & m & 0 \\ 0 & 0 & 1 & 0 & 0 & m \end{bmatrix},$$

$$\sigma_2 \to \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & l^{-1} & ml^{-1} & m & 0 & 0 \\ 1 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{-1} \end{bmatrix},$$

$$\sigma_3 \to \begin{bmatrix} l^{-1} & m & 0 & 0 & \alpha^{-1}m & 0 \\ 0 & m & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

We ask the question: Is this representation faithful? By the methods described in [Bi] it would probably suffice to prove that the matrix group generated by $\sigma_1\sigma_3^{-1}$ and $\sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ is free of rank 2.

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