On isotopies of homeomorphisms of Riemann surfaces*

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1. Introduction

Let X, $\mathbf{X}$ be orientable surfaces. Let $(p, X, \mathbf{X})$ be a regular covering space, possibly branched. A homeomorphism $g: X \to X$ is said to be "fiber-preserving" with respect to the triplet $(p, X, \mathbf{X})$ if for every pair of points x, $x' \in X$ the condition $p(x) = p(x')$ implies $pg(x) = pg(x')$. If g is fiber-preserving and isotopic to the identity map via an isotopy g_s, then g is said to be "fiber-isotopic to 1" if for every $s \in [0, 1]$ the homeomorphism g_s is fiber-preserving. This paper studies the relationship between isotopies and fiber-isotopies of g. Our main results are:

THEOREM 1. *Let $(p, X, \mathbf{X})$ be a regular covering space, either branched or unbranched, with a finite group of covering transformations and a finite number of branch points. Let the covering transformations leave each branch point fixed. In the case of a branched covering, assume that X is not homeomorphic to the closed sphere or closed torus. Let $g: X \to X$ be a fiber-preserving homeomorphism of X which is isotopic to the identity map. Then g is fiber-isotopic to the identity.*

THEOREM 2. *Let $(p, X, \mathbf{X})$ be a regular covering space, either branched or unbranched, with at most finitely many branch points. Let the group of covering transformations be finite and solvable. Let $g: X \to X$ be a fiber-preserving homeomorphism of X which is isotopic to the identity map. Let $\mathbf{g}$ be the projection of g to $\mathbf{X}$. Then $\mathbf{g}$ is also isotopic to the identity map.*

Section 2 contains the proofs of Theorems 1 and 2. A special case of Theorem 1 was established by the authors in an earlier paper [5], for the particular case where $\mathbf{X}$ is a 2-sphere, and X is a 2-sheeted covering of $\mathbf{X}$ with $2g + 2$ branch points. The proof given here is considerably simpler than the version in [5], and at the same time it holds in a much more general situation.

In Section 3 we consider applications of Theorem 1 to mapping class

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groups of surfaces, that is the group $\mathfrak{M}(X)$ of all orientation-preserving homeomorphisms of $X \to X$ modulo the subgroup of those homeomorphisms which are isotopic to the identity map. Let $T_{g,n}$ denote a Riemann surface of genus g with n points removed. The groups $\mathfrak{M}(T_{g,0})$ can be expected to play an important role in understanding the topology of 3-manifolds, in Teichmuller theory, and again in the theory of automorphism groups of infinite groups. However for the cases $g \geqq 3$ very little is known about these groups. As a step in this direction, we investigate the subgroups $\mathfrak{M}_p(T_{g,0})$ of those elements in $\mathfrak{M}(T_{g,0})$ which can be represented by fiber-preserving maps with respect to a particular covering $(p, T_{g,0}, \mathbf{X})$. It is shown in Theorems 3 and 4 that the subgroups $\mathfrak{M}_p(T_{g,0})$ may be described algebraically as the normalizers of all elements of finite order in $\mathfrak{M}(T_{g,0})$. To apply this result, we restrict ourselves to k-sheeted cyclic coverings of the sphere by the closed surface $T_{g,0}$. In this situation the covering will have n branch points, where k, n, and g are related by the formula $2g = (k-1)(n-2)$. It is proved in Theorem 5 that for these coverings the group $\mathfrak{M}_p(T_{g,0})$ is an extension of a cyclic group of order k by the group $\mathfrak{M}(T_{0,n})$. Since generators and defining relations are known for the group $\mathfrak{M}(T_{0,n})$ for every integer n, we can use this result (which is constructive) to determine explicit presentations for all of the groups $\mathfrak{M}_p(T_{g,0})$. (This latter calculation is outlined in Section 5.)

For the special case $g = 2$, $k = 2$, $n = 6$ it was shown in [5] that the group $\mathfrak{M}_p(T_{2,0})$ coincides with the full mapping class group $\mathfrak{M}(T_{2,0})$. Theorem 6 shows that this situation was special indeed, and in fact if $g \geqq 3$ all the subgroups $\mathfrak{M}_p(T_{g,0})$ are proper subgroups of $\mathfrak{M}(T_{g,0})$.

In Section 4 we discuss and settle a conjecture (due to W. Magnus) about Artin's braid group B_n. The braid group can be defined as that group of automorphisms of a free group $F_n = \langle x_1, \cdots, x_n \rangle$ of rank n which maps every generator x_i into a conjugate of itself, and preserves the product $x_1 x_2 \cdots x_n$ [1]. Let k be any integer $\geqq 2$, and let N_k be the normal closure in F_n of the n elements $x_1^k, \cdots, x_n^k$. Then the elements in B_n induce a group of automorphisms of F_n/N_k, which we denote by $B_{n,k}$. Theorem 7 shows that B_n is isomorphic to $B_{n,k}$. In other words, for every integer $k \geqq 2$ the standard representation of the braid group B_n acting on F_n goes over to a faithful representation of B_n as a group of automorphism of the free product $Z_k * \cdots * Z_k$ of n cyclic groups Z_k of order k.

2. Isotopies and fiber-isotopies

In this section we will prove Theorems 1 and 2. The proofs will be via a sequence of Lemmas. These Lemmas are numbered to correspond to the associated Theorem (e.g. Lemmas 1.1–1.6 are successive steps in the proof of Theorem 1).

We may without loss of generality assume that X and $\mathbf{X}$ are Riemann surfaces, that p is an analytic map, and that the covering transformations are analytic homeomorphisms of X.

Theorem 1 relates to both branched and unbranched coverings. The branched case will be treated first. Our object in Lemmas 1.1–1.5 will be to reduce the branched case to the unbranched case. Hence in Lemmas 1.1–1.5 we assume that $(p, X, \mathbf{X})$ is a regular covering, with finitely many branch points and a finite group of covering transformations, and that each covering transformation leaves the branch points individually fixed. (This last assumption is equivalent to the assumption that the preimage of a branch point under p is a single point.) We will moreover assume, in Lemmas 1.1–1.5, that X is not homeomorphic to $T_{0,0}$, $T_{0,1}$, $T_{0,2}$, or $T_{1,0}$. This allows us to use two properties of X which are essential to the arguments which follow:

(i) The universal covering surface U of X is hyperbolic (see [13], p. 230) and

(ii) The center of $\pi_1 X$ is trivial (see [10], Cor. 4.5). The proof of Theorem 1 for the case of branched coverings with $X = T_{0,1}$ or $T_{0,2}$ will be treated later, separately. The theorem is false if $X = T_{0,0}$ or $T_{1,0}$.

LEMMA 1.1* *Let f be a non-trivial analytic homeomorphism of X. Suppose that f has a fixed point, P. Let f_* be the induced automorphism of $\pi_1(X, P)$. Then f_* leaves no element of $\pi_1(X, P)$ fixed except the identity.*

Proof. Suppose there exists an element $[\gamma] \in \pi_1 X$, $[\gamma] \neq 1$, $f(\gamma) \cong \gamma$. Lift f to an analytic homeomorphism $\tilde{f}\colon U \to U$. Since U is by hypothesis hyperbolic, it follows that $\tilde{f}$ is a Moebius transformation. We may assume (by composing with a covering transformation if necessary) that there is a point $\tilde{P} \in U$ and lying over P such that $\tilde{f}(\tilde{P}) = \tilde{P}$. Since γ is a P-based loop, and $f(\gamma) \cong \gamma$, it follows that $\tilde{f}(\tilde{Q}) = \tilde{Q}$ where $\tilde{Q}$ is the endpoint of the lift of γ beginning at $\tilde{P}$. But then $\tilde{f}$ is a Moebius transformation with two fixed points, which is impossible unless $\tilde{f} = \widetilde{\mathrm{id}}$. This implies that $f = \mathrm{id}$. ||

LEMMA 1.2. *Let g be a fiber-preserving homeomorphism of $X \to \mathbf{X}$ which is isotopic to the identity. Then g commutes with covering transformations.*

Proof. Let t be a covering transformation, and consider $r = gtg^{-1}t^{-1}$. Since g preserves fibers, it follows that gtg^{-1} is a covering transformation,

* Lemma 1.1 was originally proved by J. Nielsen, Acta Math. **75**, p. 39. However Nielsen's proof is very different from the proof given here.

therefore r is also a covering transformation, and hence an analytic homeomorphism of X. Since p is 1–1 on the set of branch points, it follows that r must leave each branch point fixed. Also g is isotopic to the identity, therefore $tg^{-1}t^{-1}$ is isotopic to the identity, therefore r is isotopic to the identity. Thus r induces an inner automorphism on $\pi_1(X)$. Since an inner automorphism always leaves non-trivial elements fixed, it follows from Lemma 1.1 that r must be the identity. ||

Using Lemmas 1.1 and 1.2 we are now able to establish the first main step in the proof of Theorem 1. Let $\mathbf{P}_1, \cdots, \mathbf{P}_n \in \mathbf{X}$ be the branch points, and let P_i be the preimage of $\mathbf{P}_i$ under p.*

LEMMA 1.3. *Let g be a fiber-preserving homeomorphism of X which is isotopic to the identity map via an isotopy g_s. Then:*

(i) *$g(P_i) = P_i$ for every $i = 1, \cdots, n$.*

(ii) *The orbit $g_s(P_i)$ is homotopic to the constant curve in the group $\pi_1(X, P_i)$.*

Proof. Suppose $g(P_i) = P_j$ for some $i \neq j$. Let γ be any P_i-based loop and let t be any covering transformation. By Lemma 1.2:

$$g(t(\gamma)) = t(g(\gamma)) \; . \tag{1}$$

Let β denote the path $\beta(s) = g_s(P_i)$ joining P_i to P_j. Then

$$\gamma \cong \beta g(\gamma)\beta^{-1} \tag{2}$$

where the homotopy taking γ to $\beta g(\gamma)\beta^{-1}$ is defined by the isotopy $g_s(\gamma(t))$. Applying t to the homotopy in (2) and using (1) we obtain

$$t(\gamma) \cong t(\beta)g(t(\gamma))t(\beta)^{-1} \; . \tag{3}$$

Now consider the P_i-based loop $t(\gamma)$. Just as in (2) we have:

$$t(\gamma) \cong \beta g(t(\gamma))\beta^{-1} \; . \tag{4}$$

Combining (3) and (4) we see that $\beta^{-1}t(\beta)$, which is a closed loop based at P_j, commutes with $g(t(\gamma))$. Since γ was arbitrary, and g and t are homeomorphisms, $\beta^{-1}t(\beta)$ commutes with every element of $\pi_1(X, P_j)$. Since the center of $\pi_1(X, P_j)$ is trivial $\beta^{-1}t(\beta) \cong 0$. Thus:

$$\beta \cong t(\beta)$$

where the homotopy is a homotopy of curves from P_i to P_j keeping endpoints fixed.

Now lift t to a Moebius transformation $\tilde{t}$ of U. By composing with a

* By hypothesis the covering transformations leave the branch points individually fixed. This implies that the preimage of each point $\mathbf{P}_i$ is a single point.

covering transformation if necessary we may assume $\tilde{t}(\tilde{P}_j) = \tilde{P}_j$ for some $\tilde{P}_j$ lying over P_j. Since $\beta \cong t(\beta)$ we also have $\tilde{t}(\tilde{P}_i) = \tilde{P}_i$ where $\tilde{P}_i$ is the unique endpoint of the lift of β^{-1} which begins at $\tilde{P}_j$. Since $P_i \neq P_j$, it follows that $\tilde{P}_i \neq \tilde{P}_j$. But then $\tilde{t}$ has two fixed points, which is impossible. Hence the assumption that $P_i \neq P_j$ must be false.

To prove Statement (ii), we now take t to be any non-trivial covering transformation which leaves P_i fixed. The homotopy in (3) is still satisfied. Since $g(\gamma)$ is now a P_i-based loop, the homotopy in (2) also gives:

$$t(\gamma) \cong \beta g(t(\gamma))\beta^{-1} \tag{5}$$

where β is now a closed loop based at P_i. Combining (3) and (5), we see that $\beta^{-1}t(\beta)$ commutes with $gt(\gamma)$, therefore as above $\beta^{-1}t(\beta) \in$ center $\pi_1(X, P_i)$, therefore $\beta \cong t(\beta)$. But t is a *non-trivial* covering transformation, hence by Lemma 1.1 it leaves no element of $\pi_1(X, P_i)$ fixed except the identity. Hence $\beta \cong 1$. ||

LEMMA 1.4. *Let P be a point in a p.l. manifold X without boundary. Let $\beta(s)$ be a curve in X homotopic to 0 in $\pi_1(X, P)$. There is an isotopy k_s of X such that $k_0 = k_1 =$ id, where k_s has compact support and $k_s(P) = \beta(s)$.*

Proof. The proof of this lemma follows from the simplicial approximation theorem and the 2-isotopy extension theorem (see page 154 of [6]).

Using Lemma 1.4, we can now improve Lemma 1.3 to:

LEMMA 1.5. *Let g be a fiber-preserving homeomorphism of X which is isotopic to the identity map via an isotopy g_s. Then there is another isotopy $\bar{g}_s$ of g with the identity such that $\bar{g}_s(P_i) = P_i$ for every $i = 1, \cdots, n$ and $0 \leqq s \leqq 1$.*

Proof. By Lemma 1.3, $g(P_1) = P_1$ and $\beta_1(s) = g_s(P_1) \cong 1$ in $\pi_1(X, P_1)$. By Lemma 1.4 there is an isotopy k_s of X with $k_0 = k_1 =$ id and $k_s(P_1) = \beta_1(s)$. Let $h_s = k_s^{-1}g_s$. Then $h_s(P_1) = P_1$ for all $s \in [0, 1]$ and h_s is an isotopy of g to id. Now consider the covering $(p, X - P_1, \mathbf{X} - \mathbf{P}_1)$, and the homeomorphism $g \mid X - P_1$. By enumerating the Riemann surfaces that do not have hyperbolic universal cover or fundamental group with trivial center we see that $X - P_1$ satisfies the hypotheses about the cover. Since $h_s(P_1) = P_1$ for all $s \in [0, 1]$, we can restrict h_s to $X - P_1$.

Now repeat the argument for $P_2, P_3, \cdots, P_n$. We finally achieve an isotopy $\bar{g}_s$ which has the desired properties. ||

We have now shown that if $(P, X, \mathbf{X})$ is a *branched* covering, then every homeomorphism $g: X \rightarrow X$ which is fiber-preserving and isotopic to the identity map may without loss of generality be assumed to be fiber-preserving and isotopic to the identity with respect to the associated *unbranched* covering space $(\tilde{P}, X - P_1 \cup \cdots \cup P_n, \mathbf{X} - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$. Our next step in the proof of Theorem 1 applies to a larger class of spaces which, by virtue of Lemma 1.5, include the branched coverings we have been treating.

LEMMA 1.6. *Let* $(q, Y, \mathbf{Y})$ *be a regular, unbranched covering space, where* Y, $\mathbf{Y}$ *are connected oriented 2-manifolds. Let* $g: Y \to Y$ *be fiber-preserving and isotopic to the identity. Let the centralizer of* $q_*\pi_1 Y$ *in* $\pi_1\mathbf{Y}$ *be trivial. Then* g *is fiber-isotopic to the identity.*

Proof. Since g is fiber-preserving it projects to $\boldsymbol{g}: \mathbf{Y} \to \mathbf{Y}$. Pick points P and $\mathbf{P}$ such that $q(P) = \mathbf{P}$. Let $\beta(s)$ be the curve $g_s(P)$, where g_s is an isotopy of g with the identity, and let $\boldsymbol{\beta}$ be the projection of β. Let γ be a P-based loop, and let $\boldsymbol{\gamma} = q(\gamma)$. We may define $\boldsymbol{g}_*$, an automorphism of $\pi_1(\mathbf{Y}, \mathbf{P})$, by $\boldsymbol{g}_*[\boldsymbol{\gamma}] = \boldsymbol{\beta g}(\boldsymbol{\gamma})\boldsymbol{\beta}^{-1}$. If $[\boldsymbol{\gamma}] \in q_*\pi_1(Y, P)$, we have $\boldsymbol{q}_*[\boldsymbol{\gamma}] = [\boldsymbol{\gamma}]$, since $\gamma \cong \beta g(\gamma)\beta^{-1}$. Thus we may assume that $\boldsymbol{g}_*$, restricted to $q_*\pi_1(Y, P)$, is the identity.

Now choose any $\boldsymbol{\alpha} \in q_*\pi_1(Y, P)$ and any $\boldsymbol{\beta} \in \pi_1(\mathbf{Y}, \mathbf{P})$. Since the covering is regular we have $\boldsymbol{\beta\alpha\beta}^{-1} \in q_*\pi_1(Y, P)$. Thus $\boldsymbol{\beta\alpha\beta}^{-1} = \boldsymbol{g}_*(\boldsymbol{\beta\alpha\beta}^{-1}) = \boldsymbol{g}_*(\boldsymbol{\beta})\boldsymbol{\alpha g}_*(\boldsymbol{\beta}^{-1})$. It follows that $\boldsymbol{\beta}^{-1}\boldsymbol{g}_*(\boldsymbol{\beta})$ is in the centralizer of $q_*\pi_1(Y, P)$ and is therefore trivial. Thus $\boldsymbol{g}_*$ is the identity. Since $\mathbf{Y}$ is a surface, it follows that $\boldsymbol{g}$ must be isotopic to the identity [11], and this isotopy lifts to a fiber isotopy taking g to the identity. ||

We are now ready to prove Theorem 1.

Proof. We first treat the case where the covering is unbranched, and the base space $\mathbf{X}$ is not homeomorphic to $T_{0,0}$, $T_{0,1}$, $T_{0,2}$ or $T_{1,0}$. In this case we need only verify that the conditions of Lemma 1.6 are satisfied, i.e. that $H = q_*\pi_1 X$ has a trivial centralizer in $G = \pi_1\mathbf{X}$. Since $\mathbf{X}$ is a surface, the group G is either a free group or a 1-relator group. If free, G must have rank $\geqq 2$ (because $\mathbf{X}$ is not $T_{0,0}$, $T_{0,1}$ or $T_{0,2}$), hence any subgroup has trivial centralizer. If G is a 1-relator group, then G admits a presentation of the following type:

$$G = \langle a_1, \cdots, a_m, b_1, \cdots, b_m; \textstyle\prod_{i=1}^m a_i b_i a_i^{-1} b_i^{-1} = 1\rangle$$

where $m \geqq 2$ (because $\mathbf{X}$ is not $T_{1,0}$). Hence, by Corollary 4.5 of [10], we know that G is centerless. Now, the subgroup H is of finite index in G, hence H is also a surface group, hence H has a trivial center. Suppose $\alpha \in$ centralizer of H in G. Since H is centerless, either $\alpha = 1$ or else we can choose α to be a non-trivial coset representative of H in G. Since H has finite index in G, this implies that a finite power of α is in H. But every power of α will commute with all elements in H, hence this is impossible, hence $\alpha^{\lambda} = 1$ for some integer λ. But by Theorem 4.12 of [10] the groups G under consideration are torsion free, hence we can only have $\alpha = 1$. Thus Lemma 1.6 applies.

If $\mathbf{X}$ is homeomorphic to $T_{0,0}$ or $T_{0,1}$ the theorem is trivial since every orientation preserving homeomorphism $g: \mathbf{X} \to \mathbf{X}$ is isotopic to 1. If $\mathbf{X}$ is homeomorphic to $\mathrm{T}_{0,2}$, the only orientation preserving homeomorphism of $\mathbf{X}$, not isotopic to 1, is the one that exchanges the missing points. Its' lift must also exchange the missing points, and so cannot be isotopic to the identity. That leaves the case $\mathbf{X} = T_{1,0}$.

As in the proof of Lemma 1.6, we may assume that g_* restricted to $p_*\pi_1 X$ is the identity. But $\pi_1\mathbf{X} = Z \oplus Z$, and any automorphism of $Z \oplus Z$ whose restriction to a subgroup of finite index is the identity, is itself the identity. Thus by [10] g is isotopic to 1.

We turn now to the branched case. Lemma 1.5 shows that if $X = T_{0,0}$, $T_{0,1}$, $T_{0,2}$, or $T_{1,0}$ we may replace the branched covering by the associated unbranched covering. But then the argument given above, for unbranched coverings, applies, and again Theorem 1 is true.

Since the cases $X = T_{0,0}$ and $T_{0,1}$ are excluded by hypothesis, all that remains to complete the proof of Theorem 1 is to establish it for branched coverings with $X = T_{0,2}$ or $T_{0,1}$. A covering transformation of $T_{0,2}$ is a rotation. If it leaves a branch point fixed, it must be the identity, hence this case is trivial. If there is a non-trivial covering transformation of $T_{0,1}$ it can have at most one fixed point, which we take to be the branch point at the origin. The group of covering transformations is a finite group of rotations and is therefore cyclic. The proof of the theorem follows from the fact that any homeomorphism of $T_{0,1}$ is isotopic to the identity and the isotopy can be lifted. ||

Theorem 1 cannot be extended to the sphere or torus. To see this consider the two sheeted covering of the sphere by the sphere with two branch points or by the torus with four branch points. In either case there are fibre preserving homeomorphisms of X isotopic to the identity exchanging branch points. Since the isotopies exchange branch points, they cannot be isotopic to the identity via an isotopy fixing the branch points.

We would like to extend Theorem 1 to a more general result, which holds for an arbitrary solvable group of covering transformations. In principle, solvable groups could be handled by factoring the covering into a sequence of cyclic coverings, but the difficulty which arises is that at some intermediate stage the base space (which will become the covering space at the next stage) might be $T_{0,0}$ or $T_{1,0}$. To overcome this difficulty, we consider a somewhat weaker result than the statement of Theorem 1, which is true for the exceptional cases $T_{0,0}$ and $T_{1,0}$:

LEMMA 2.1. *Let* $p\colon X \to \mathbf{X}$ *be a regular branched covering of Riemann surfaces with finitely many sheets, with at least one branch point, and with* X *either the torus or sphere. Assume the group of covering transformations leaves the branch points fixed. Let* $g\colon X \to X$ *be a fibre preserving homeomorphism of* X *and let* $\boldsymbol{g}\colon X \to X$ *be the projection of* g*. Then, if* g *is isotopic to the identity, so is* $\boldsymbol{g}$*.*

Proof. If $X = T_{0,0}$, then X can only cover S^2 or P^2, hence $\mathbf{X}$ must also be S^2, hence the lemma is trivial.

Now let $X = T_{1,0}$. We shall think of X as C modulo the group of translations generated by 1 and i. We can choose [0] as one of the branch points. The covering transformations of X can be lifted to Moebius transformations of C fixing 0 and leaving the lattice points of C invariant. The only Moebius transformations that do this are multiples of 90° rotation about 0. Thus $\mathbf{X} = C$ modulo the group of transformations generated by translations by 1 and i and 90° rotation about 0, or the group of transformations generated by translations by 1 and i and 180° rotation about 0. In either case $\mathbf{X}$ is the sphere and the Lemma is trivial. ||

Now we can prove Theorem 2 (stated in Section 1).

Proof. Suppose first that the group G of covering transformations is cyclic of prime order. Then p must be 1-1 on the branch points, because the number of elements in an orbit divides the number of elements in the group of covering transformations. Thus every covering transformation leaves the branch points fixed. Hence, if $X \neq T_{0,0}$ or $T_{1,0}$ the statement follows from Theorem 1, while if $X = T_{0,0}$ or $T_{1,0}$ the statement follows from Theorem 1, while if $X = T_{0,0}$ or $T_{1,0}$ it follows from Lemma 2.1.

The proof in the more general case where G is solvable follows immediately via an inductive argument from the case where G is cyclic of prime order, using a factorization of the map p. ||

3. Mapping class groups of Riemann surfaces

For the remainder of this paper we will be concerned with applications of Theorem 1 to the investigation of the mapping class group $\mathfrak{M}(X)$ of a Riemann surface X.

Let $(p, X, \mathbf{X})$ be a covering space, either branched or unbranched. The "subgroup of the covering" is defined to be the subgroup $p_*\pi_1 X$ of $\pi_1 \mathbf{X}$ (if $(p, X, \mathbf{X})$ is unbranched), or the corresponding subgroup for the associated unbranched covering (if $(p, X, \mathbf{X})$ is branched). A covering $(p, X, \mathbf{X})$ is said to be *cyclic* if the subgroup of the covering is normal in $\pi_1\mathbf{X}$, and if the quotient group is cyclic of finite order. If the covering is cyclic, the group of covering transformations, T, will be cyclic of finite order, k, and will determine a subgroup $\mathcal{T}$ of the mapping class group $\mathfrak{M}(X)$ which is again a cyclic group of order k.

Conversely, let X be a closed, orientable surface; let $[t]$ be any element of finite order k in $\mathfrak{M}(X)$; and let $\mathcal{T}$ be the cyclic subgroup of $\mathfrak{M}(X)$ generated by $[t]$. By a theorem of J. Nielsen [8], the mapping class $[t]$ can be represented by a surface homeomorphism t which has order k. Let $\mathbf{X}$ be the quotient space of X defined by identifying points which are mapped into each other by powers of t. Let p be the natural mapping $p: X \to \mathbf{X}$. Then $(p, X, \mathbf{X})$ is a cyclic covering. The covering will be branched if and only if t has fixed points.

We restrict ourselves in this section to cyclic coverings for which the covering space X is a closed orientable surface of genus g, denoted $T_{g,0}$. Let $\mathfrak{M}_p(T_{g,0})$ denote the subgroup of $\mathfrak{M}(T_{g,0})$ which is generated by isotopy classes of fiber-preserving homeomorphisms of $T_{g,0}$. Since every covering transformation is fiber-preserving, the group $\mathcal{T}$ is included (normally) in $\mathfrak{M}_p(T_{g,0})$. Our first result in this section is to show that the subgroup $\mathfrak{M}_p(T_{g,0})$ is precisely the normalizer of $\mathcal{T}$ in $\mathfrak{M}(T_{g,0})$. We conjecture that this result generalizes to the case where $\mathcal{T}$ is any finite subgroup of $\mathfrak{M}(X)$, however the stronger result depends on a strong form of a theorem due to Kravetz [7], which is, to the authors' knowledge, an open question.

We begin with a proof of an extension of a theorem of J. Nielsen [8] which is perhaps of interest in its own right.

THEOREM 3. *Let* $[t], [h] \in \mathfrak{M}(T_{g,0})$ *where* $[t]$ *has finite order* k. *Suppose that* $[h]$ *belongs to the normalizer of* $[t]$, *i.e.*

$$[h][t][h]^{-1} = [t]^s \qquad 1 \leqq s \leqq k-1 .$$

Then $[h]$ *and* $[t]$ *can be represented by topological mappings* h, t *which have the properties**

* The referee has pointed out that a similar result was established by E. Vogt in Lemma 3.1.2 of his PhD thesis, "Vierdimensionale Siefertsche Faserraume", Univ. of Bochum, 1970, for the case $s = 1$. The proof given there is not the same as our proof. A similar result, using a third method of proof, was also established more recently by D. Kaufman in her PhD thesis, Princeton Univ., Sept. 1972.

$$hth^{-1} = t^s \textit{ and } t^k = 1 .$$

Proof. Since $[t]$ has finite order in $\mathfrak{M}(T_{g,0})$, by Nielsen's theorem [8] we know that $[t]$ can be represented by a homeomorphism $t: T_{g,0} \to T_{g,0}$ which has the property that $t^k =$ identity.

By Kravetz [7], t has a fixed point as a mapping of Teichmuller space, and therefore t can be chosen to be conformal, for some analytic structure on $T_{g,0}$.

Let h' be any homeomorphism of $T_{g,0}$ which represents the element $[h]$. We can without loss of generality assume that h' is quasiconformal. Among all quasiconformal mappings which are isotopic to h', choose h to be the unique quasiconformal mapping which has the smallest dilatation (see, for example, [2]).

Since $[hth^{-1}] = [t^s]$, we know that ht is isotopic to $t^s h$. Since multiplication of a quasiconformal mapping by a conformal mapping does not alter the dilatation, it follows that the dilatations of ht and $t^s h$ are both equal to the dilatation of h. But then ht and $t^s h$ are also extremal quasiconformal mappings, and since by Teichmuller's uniqueness theorem [2] there is precisely one extremal quasiconformal mapping in an isotopy class, it follows that $ht = t^s h$.

THEOREM 4. *The symmetric subgroups $\mathfrak{M}_p(T_{g,0})$ are the normalizers of the cyclic subgroups $\mathcal{T}$ generated by any element $[t]$ of finite order in $\mathfrak{M}(T_{g,0})$.*

Proof. Let $[t] \in \mathfrak{M}(T_{g,0})$ be any element of finite order k. By Nielsen's theorem, $[t]$ can be represented by $t: T_{g,0} \to T_{g,0}$ where $t^k = \mathrm{id}$. The homeomorphism t can be used to define a covering space $(p, T_{g,0}, \mathbf{X})$: define $\mathbf{X}$ to be the quotient space obtained by identifying points of $T_{g,0}$ which are mapped into one another by powers of t. The cyclic group T of order k generated by t will be the group of covering transformations.

Suppose that $[h] \in \mathfrak{M}(T_{g,0})$ is in the normalizer of the cyclic group $\mathcal{T}$ of order k generated by $[t]$. By Theorem 3, we know that $[h]$ has a representative h which is in the normalizer of T. It follows that h is fiber-preserving with respect to the covering $(p, T_{g,0}, \mathbf{X})$, hence $[h] \in \mathfrak{M}_p(T_{g,0})$. Conversely, if $[h] \in \mathfrak{M}_p(T_{g,0})$, then $[h]$ can be represented by a fiber-preserving homeomorphism $h: T_{g,0} \to T_{g,0}$. Let t^s be any covering transformation. Then ht^sh^{-1} is also a covering transformation, therefore $ht^sh^{-1} \in T$, therefore h is in the normalizer of T, therefore $[h]$ is in the normalizer of $\mathcal{T}$. ||

Our object now is to apply Theorem 4 to the study of the mapping class groups $\mathfrak{M}(T_{g,0})$. For our next result we restrict our attention to the case where the base space X is the *sphere*. This restriction is not essential, however it simplifies matters because we have:

LEMMA 5.1. *Let* $(p, T_{g,0}, T_{0,0})$ *be a cyclic branched covering. Let* $(\tilde{p}, T_{g,n}, T_{0,n})$ *be the associated unbranched covering. Then every homeomorphism of* $T_{0,n}$ *lifts to a homeomorphism of* $T_{g,n}$. (Remark: the lift is only unique up to covering transformations.)

Proof. A homeomorphism lifts if and only if it maps every closed curve which lifts to a closed curve into a closed curve which lifts to a closed curve. Since the covering is a k-sheeted cyclic covering, a closed curve lifts to a closed curve if and only if it encircles a multiple of k branch points. The property of encircling k branch points is preserved by every homeomorphism of the punctured sphere. ||

(In the more general situation where **X** is an arbitrary surface, one must modify Theorem 4 so that it relates to the subgroup of those elements in $(\mathbf{X} - \mathbf{P}_1, \cdots, \mathbf{P}_n)$ which can be represented by homeomorphisms which lift to X).

THEOREM 5. *Projecting fiber-preserving homeomorphisms induces an isomorphism i between the groups* $\mathfrak{M}_p(T_{g,0})/\mathcal{T}$ *and* $\mathfrak{M}(T_{0,n})$.

Proof. Choose any element $[h] \in \mathfrak{M}_p(T_{g,0})$. Let h be a representative of $[h]$, which we may choose to be fiber-preserving. Then h projects to $\boldsymbol{h} = php^{-1}$, where $\boldsymbol{h}$ necessarily maps the set of branch points into itself, hence $\boldsymbol{h}$ represents a well-defined element $[h] \in \mathfrak{M}(T_{0,n})$. Now suppose that h' is a second fiber-preserving representative of $[h]$, so that h and h' are isotopic. By Theorem 1 we may choose the isotopy h_s to be fiber-preserving, hence the projections $\boldsymbol{h}$ and $\boldsymbol{h}'$ of h and h' respectively are isotopic via the isotopy $\boldsymbol{h}_s = ph_sp^{-1}$, hence $[\boldsymbol{h}] = [\boldsymbol{h}']$.

It is immediate that the projection of fiber-preserving homeomorphisms from $T_{g,0}$ to $T_{0,n}$ induces a homomorphism from $\mathfrak{M}_p(T_{g,0})/\mathcal{T}$ to $\mathfrak{M}(T_{0,n})$. We denote this homomorphism by the symbol "i". To see that the homomorphism i is onto, we note that by Lemma 5.1 every homeomorphism of $T_{0,n} \rightarrow T_{0,n}$ lifts. (However, if we are given $\boldsymbol{h}$, its lift h is only defined up to an arbitrary covering transformation.) Moreover, the homomorphism i is invertible because if $\boldsymbol{h}$ represents an element $[\boldsymbol{h}] \in \mathfrak{M}(T_{0,n})$, and if $\boldsymbol{h}$ is altered by an isotopic deformation, then by the homotopy lifting property for covering spaces this isotopy will lift to an isotopic deformation of the lift h of $\boldsymbol{h}$. Hence i is an isomorphism onto.

Since defining relations for the groups $\mathfrak{M}(T_{g,0})$ are not known if $g \geqq 3$, a natural question to ask is whether by a clever choice of a covering we might not find a symmetric subgroup $\mathfrak{M}_p(T_{g,0})$ which coincides with all of $\mathfrak{M}(T_{g,0})$? It was shown in [5] that if $g = 2$, the symmetric subgroup for $k = 2$, $n = 6$ coincides with all of $\mathfrak{M}(T_{2,0})$. Our next result says that for cyclic coverings this won't happen again:

THEOREM 6. *If* $g \geqq 3$, *there does not exist any finite cyclic covering with the property that* $\mathfrak{M}_p(T_{g,0})$ *coincides with* $\mathfrak{M}(T_{g,0})$.

Proof. Suppose we could find such a covering. By Theorem 5, the projection of fiber-preserving homeomorphisms would then induce a homomorphism from $G_1 = \mathfrak{M}_p(T_{g,0})$ onto $G_2 = \mathfrak{M}(T_{0,n})$, the kernel being the finite cyclic group $\mathcal{T}$. This homomorphism induces a homomorphism from the abelianized group $G_1/[G_1, G_1]$ onto $G_2/[G_2, G_2]$.

Now, it is shown in [4] that if $g \geqq 3$ the group $G_1/[G_1, G_1]$ has order 2 or 1. From the presentation for G_2 in Th. N9 [10], we find that $G_2/[G_2, G_2]$ has order $2(n - 1)$ if n is even, or $n - 1$ if n is odd. Since $n \geqq 3$, this tells us that $n = 3$ is the only possible case where a homomorphism might exist. However if $n = 3$ the group $G_2 = \mathfrak{M}(T_{0,3})$ is a finite group. But by Theorem 5: G_1 modulo the finite cyclic group $\mathcal{T}$ is isomorphic to G_2. Since G_1 is infinite for every $g \geqq 1$, this is clearly impossible. ||

The possibility remains that if we relax the requirements on $(p, X, \mathbf{X})$ to admit coverings of other Riemann surfaces, or to admit all *regular* coverings, or to admit *non*-regular coverings that we will have better luck. (However we conjecture that all such efforts will fail.)

4. A theorem about Artin's braid group

Our object now is to apply Theorem 1 to establish an interesting new property of Artin's braid group, B_n. The reader is referred to Section 1 for definitions of F_n and N_k. The braid group B_n is defined to be that subgroup of Aut F_n which is generated by automorphisms $\sigma_1, \cdots, \sigma_{n-1}$, where

$$(8) \qquad \begin{aligned} \sigma_i\colon x_i &\longrightarrow x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\longrightarrow x_i \\ x_k &\longrightarrow x_k \qquad\qquad k \neq i,\ i+1;\ k = 1, \cdots, n\,. \end{aligned}$$

For a review of the basic properties of B_n see Section 3.7 of [10]. We observe that every element in B_n maps $N_k \to N_k$. We will prove:

THEOREM 7. *Let* $B_{n,k} \subset \text{Aut}\,(F_n/N_k)$ *be the group of automorphisms of* F_n/N_k *which is induced by the action of* B_n *on* F_n. *Let* $\Psi_k\colon B_n \to B_{n,k}$ *be* *the natural homomorphism. Then* Ψ_k *is an isomorphism for every integer* $k \geqq 2$.

Proof. In order for an automorphism of F_n to be a braid automorphism, it must satisfy two characteristic properties (see [1]): it must map each generator x_i of F_n into a conjugate of itself or some other x_j, and it must map the product $x_1x_2 \cdots x_n$ into itself. Suppose first that $\beta \in B_n \cap \operatorname{Inn} F_n$. Since the action of β is inner, it follows that β must map the product $(x_1x_2 \cdots x_n) \rightarrow T(x_1x_2 \cdots x_n)T^{-1}$ for some $T \in F_n$, and since $\beta(x_1x_2 \cdots x_n) = x_1x_2 \cdots x_n$, it follows that T must commute with $x_1x_2 \cdots x_n$. Since F_n is a free group, this is possible only if $T = (x_1x_2 \cdots x_n)^\lambda$ for some integer λ. One verifies by calculating, using the action given in equation (8), that the automorphism $\sigma = (\sigma_1\sigma_2 \cdots \sigma_{n-1})$ and its powers have precisely this effect:

$$\sigma^\lambda\colon x_i \longrightarrow (x_1x_2 \cdots x_n)^\lambda x_i (x_1x_2 \cdots x_n)^{-\lambda} \qquad i = 1, \cdots, n\,. \tag{9}$$

Hence $B_n \cap \operatorname{Inn} F_n$ is the infinite cyclic group I generated by σ. ||

We ask what happens to I under the homomorphism $\Psi_k\colon B_n \rightarrow B_{n,k}$? Clearly $\Psi_k(I)$ is cyclic, so the only question is whether $\Psi_k(\sigma)$ might have finite order? Now, the group F_n/N_k is a free product of n cyclic groups of order k, hence by Theorem 4.1 of [10] each element in F_n/N_k has a unique representation as a product of elements in the factors. Therefore $\Psi_k(\sigma^\lambda)$ cannot possibly be 1 unless $\sigma^\lambda(x_i)$, when freely reduced, contains symbols of the form x_i^k. But from equation (9) we see that $\sigma^\lambda(x_i)$ contains no k^{th} powers, hence it follows that $\Psi_k(\sigma^\lambda) \neq 1$. Thus we have proved:

LEMMA 7.1. $\ker \Psi_k$ *contains no non-trivial inner automorphism of* F_n.

To proceed further, we place a geometrical interpretation on the groups F_n and N_k. Let $\mathbf{X}$ be the complex plane E^2, and let $(p, X, \mathbf{X})$ be a k-sheeted cyclic covering, with n (interchangeable) branch points $\mathbf{P}_1, \cdots, \mathbf{P}_n$. Let $\mathbf{P}_0 \in (\mathbf{X} - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$ be a base point for $\pi_1(X - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$. Then the group F_n can be interpreted as the fundamental group $\pi_1(\mathbf{X} - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$; each x_i is understood to be the homotopy class of a simple $\mathbf{P}_0$-based loop enclosing precisely one branch point $\mathbf{P}_i$.

Let H be the subgroup of index k in F_n consisting of all words of exponent sum k in $x_1, \cdots, x_n$. We can interpret H geometrically as the group $\tilde{p}_*\pi_1(X - P_1 \cup \cdots \cup P_n)$, where $\tilde{p}$ is the covering space projection of the unbranched covering

$$(\tilde{p}, X - P_1 \cup \cdots \cup P_n, \mathbf{X} - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$$

associated with the branched covering $(p, X, \mathbf{X})$. The group N_k is the normal closure in F_n of the elements $\{x_j^k; j = 1, \cdots, n\}$. We can identify $H \cap N_k$ as the subgroup of $\pi_1(\mathbf{X} - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$ consisting of those elements which are represented by curves which lift to closed curves which are homotopically trivial on X, but homotopically non-trivial on $(X - P_1 \cup \cdots \cup P_n)$. The quotient group $H/H \cap N_k$ can then be identified geometrically as the fundamental group $\pi_1 X$ of the covering space in the *branched* covering $(p, X, \mathbf{X})$.

To complete the proof of Theorem 7, we now note that every braid automorphism can be induced by a topological mapping. This is easily established by noting that each generator σ_i of B_n can be induced by a homeomorphism of E^2 which interchanges the branch points $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$, and is the identity outside a small disc which includes $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ but no other $\mathbf{P}_j$. Suppose $\beta \in \ker \Psi_k$, $\beta \neq 1$. Let $\boldsymbol{b}$ be a topological mapping of $(E^2 - \mathbf{P}_1 \cup \cdots \cup \mathbf{P}_n)$ which induces the automorphism β. Lift $\boldsymbol{b}$ to a topological mapping $b: (X - P_1, \cdots, P_n) \to (X - P_1, \cdots, P_n)$. This mapping extends in an obvious way to a topological mapping $\mathfrak{b}$ of the surface X, and the mapping $\mathfrak{b}$ induces an automorphism $\mathfrak{b}_*$, which we recognize is precisely $\Psi_k(\beta)$, of $\pi_1 X$. By hypothesis $\beta \in \ker \Psi_k$, therefore it follows that $\mathfrak{b}_* = 1$. By a classical result [11], this means that $\mathfrak{b}$ is isotopic to the identity map. Now, $\mathfrak{b}$ is fiber-preserving (because it was the lift of the topological mapping $\boldsymbol{b}$) and isotopic to the identity, hence by Theorem 2 it must be fiber-isotopic to the identity. This fiber-isotopy projects to an isotopy taking $\boldsymbol{b}$ to the identity. But then the automorphism β induced by $\boldsymbol{b}$ on $\pi_1(\mathbf{X} - \mathbf{P}_1, \cdots, \mathbf{P}_n)$ must be inner, hence by Lemma 7.1 it follows that $\beta = 1$. Thus Ψ_k is an isomorphism.

5. Generators and relations for $\mathfrak{M}_p(T_{g,0})$

In concluding, we note that Theorem 5 is a constructive result, in that it enables us to determine explicit presentations for the subgroups $\mathfrak{M}_p(T_{g,0})$ for the special case of cyclic branched coverings of the sphere. The method used is exactly the same as that described in [5] for the special case of 2-sheeted coverings, and therefore we omit details and simply summarize the results.

Choose generators $x_1, \cdots, x_n$ for $\pi_1 T_{0,n}$, similar to those as described in the proof of Lemma 7.1, but replacing the n-punctured complex plane by its one point compactification, so that these generators now satisfy the single relation $x_1 x_2 \cdots x_n = 1$. The group $\pi_1 T_{0,n}$ is thus a free group of rank $n - 1$. The covering space $T_{g,0}$ will be a closed surface of genus g, where by a well-known formula (see e.g. page 275 of [13]) the integers g, k, and n are related by:

(10) $$2g = (k-1)(n-2)\ .$$

Let s_i ($i = 1, \cdots, n-1$) denote the isotopy class in $\mathfrak{M}(T_{0,n})$ of a simple twist map which interchanges the branch points $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$, but is the identity map outside of a small disc neighborhood which encloses $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ and avoids all P_j with $j \neq i,\ i+1$. The group $\mathfrak{M}(T_{0,n})$ was studied by W. Magnus in [9], and admits a presentation in terms of the generators $s_1, \cdots, s_{n-1}$, with defining relations (2.1)–(2.4) of [5]. Let t_i be the lift of s_i to $\mathfrak{M}(T_{g,0})$. (The meaning of t_i as a product of standard twist generators of $\mathfrak{M}(T_{g,0})$, as given for example in [5], will of course depend upon k, g and the explicit choice of coset representatives for the subgroup of the covering.) Then, using the method described in § 4 of [5], it can be shown that $\mathfrak{M}_p(T_{g,0})$ admits a presentation with generators $t_1, \cdots, t_{n-1}$ and defining relations:

(11) $$t_it_j = t_jt_i \qquad i, j = 1, \cdots, n-1;\ |i-j| \geqq 2$$

(12) $$t_it_{i+1}t_1 = t_{i+1}t_it_{i+1} \qquad i = 1, \cdots, n-2$$

(13) $$(t_1t_2 \cdots t_{n-1})^n = 1$$

(14) $$(t_1t_2 \cdots t_{n-1}t_{n-1} \cdots t_2t_1)^k = 1$$

(15) $$[t_1t_2 \cdots t_{n-1}t_{n-1} \cdots t_2t_1,\ t_1] = 1\ .$$

We note that the presentation above depends only on the integers k and n, while the interpretation of t_i as a surface homeomorphism depends on the explicit definition of the covering projection.

Added in proof. It has recently come to our attention that a generalized version of Theorem 1 has been obtained (by different methods) by W. J. Harvey and C. MacLachlan. It will be reported in their forthcoming paper "On mapping class groups and Teichmuller spaces".

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