

# Finite rigid sets and homologically non-trivial spheres in the curve complex of a surface

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## Abstract

Aramayona and Leininger have provided a “finite rigid subset”  $\mathfrak{X}(\Sigma)$  of the curve complex  $\mathcal{C}(\Sigma)$  of a surface  $\Sigma = \Sigma_g^n$ , characterized by the fact that any simplicial injection  $\mathfrak{X}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$  is induced by a unique element of the mapping class group  $\text{Mod}(\Sigma)$ . In this paper we prove that, in the case of the sphere with  $n \geq 5$  marked points, the reduced homology class of the finite rigid set of Aramayona and Leininger is a  $\text{Mod}(\Sigma)$ -module generator for the reduced homology of the curve complex  $\mathcal{C}(\Sigma)$ , answering in the affirmative a question posed in [1]. For the surface  $\Sigma = \Sigma_g^n$  with  $g \geq 3$  and  $n \in \{0, 1\}$  we find that the finite rigid set  $\mathfrak{X}(\Sigma)$  of Aramayona and Leininger contains a proper subcomplex  $X(\Sigma)$  whose reduced homology class is a  $\text{Mod}(\Sigma)$ -module generator for the reduced homology of  $\mathcal{C}(\Sigma)$  but which is not itself rigid.

## 1 Introduction and Statement of Results

Let  $\Sigma_g^n$  be the surface with genus  $g \geq 0$  and  $n \geq 0$  marked points. The *mapping class group*  $\text{Mod}(\Sigma_g^n)$  is the group

$$\text{Mod}(\Sigma_g^n) = \text{Homeo}^+(\Sigma_g^n) / \text{Homeo}^0(\Sigma_g^n)$$

where  $\text{Homeo}^+(\Sigma_g^n)$  is the group of orientation preserving self-homeomorphisms of the surface  $\Sigma_g^n$  which permute the marked points and  $\text{Homeo}^0(\Sigma_g^n)$  is the path component of  $\text{Homeo}^+(\Sigma_g^n)$  containing the identity map.

Let  $V \subset \Sigma_g^n$  be the set of marked points. An *essential curve*  $\gamma$  in  $\Sigma_g^n$  is a simple closed curve in  $\Sigma_g^n - V$  such that any disk in  $\Sigma_g^n$  bounded by  $\gamma$  must contain at least two marked points. The isotopy class of  $\gamma$  is its orbit under the action of  $\text{Homeo}^0(\Sigma_g^n)$ . A *curve system* is a non-empty set of isotopy classes of essential curves which have pairwise disjoint representatives and a *sub curve system* of a curve system is any non-empty subset of a curve system. The *curve complex*  $\mathcal{C} = \mathcal{C}(\Sigma_g^n)$  is the simplicial complex whose vertices correspond to isotopy classes of essential curves in  $\Sigma_g^n$  and whose  $k$ -simplices correspond to curve systems with  $k + 1$  curves. Henceforth we will abuse notation by not distinguishing between essential curves and their isotopy classes. In the cases of the sporadic surfaces of low complexity,  $\Sigma_0^4$ ,  $\Sigma_1^0$  and  $\Sigma_1^1$ , it is customary to include simplices for sets of essential curves with pairwise minimal intersection, but we will not do this here. Hence for our purposes  $\mathcal{C}(\Sigma_0^4)$ ,  $\mathcal{C}(\Sigma_1^0)$  and  $\mathcal{C}(\Sigma_1^1)$  will be disconnected complexes of dimension 0.

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The curve complex plays dual geometric and homological roles for the structure of the mapping class group. On the one hand we have results in the vein of Ivanov [8], which retrieve the mapping class group as the set of simplicial isomorphisms of the curve complex and other related simplicial complexes. On the other hand we have the work of Harer [5] which shows that the single non-trivial reduced homology group of the curve complex is the dualizing module for the mapping class group, thereby linking its group homology and cohomology. One should not be too surprised at these dual functions for the curve complex, as the curve complex was originally proposed by Harvey [6] as an analog for the mapping class group of the spherical building at infinity for nonuniform lattices in semi-simple Lie groups.

Recent work of Aramayona and Leininger [1] advanced the Ivanov side of the picture by giving a finite rigid subset  $\mathfrak{X}(\Sigma)$  in the curve complex with the property that any simplicial injection of their set into the curve complex is induced by a unique mapping class. In this paper we relate the work in [1] to the Harer side of that picture by relating the finite rigid set  $\mathfrak{X}(\Sigma)$  of Aramayona and Leininger to a homologically non-trivial sphere  $X(\Sigma)$  in the curve complex whose orbit under the action of the mapping class group generates the reduced homology of the curve complex. First, we focus on the sphere with  $n$  marked points. In Proposition 25 we find, for each  $n \geq 5$ , an explicit set of curves on  $\Sigma_0^n$  whose associated vertices in  $\mathcal{C}(\Sigma_0^n)$  determine an essential simplicial sphere in  $\mathcal{C}(\Sigma_0^n)$ . The information in Proposition 25 will allow us to prove our first main result in §3.2:

**Theorem 1.** *Assume that  $n \geq 5$ . Then the finite rigid set  $\mathfrak{X}(\Sigma_0^n) \subset \mathcal{C}(\Sigma_0^n)$  given in [1] is precisely the essential  $(n - 4)$ -sphere  $X(\Sigma_0^n) \subset \mathcal{C}(\Sigma_0^n)$  whose reduced homology class is determined in Proposition 25 of this paper to be a  $\text{Mod}(\Sigma_0^n)$ -module generator of the reduced homology of the curve complex. This answers Question 2 of [1] in the affirmative for  $n \geq 5$ .*

Given the surprising coincidence of our essential sphere and the finite rigid set of Aramayona and Leininger for the surface  $\Sigma_0^n$ , and the fact that  $X(\Sigma_0^6) = X(\Sigma_2^0) = \mathfrak{X}(\Sigma_2^0)$ , one might be tempted to conjecture that homologically non-trivial spheres in the curve complex are always finite rigid sets. Attempting to test our conjecture in the cases  $\Sigma_g^0$ ,  $g \geq 3$  we learned that it is not true. In Proposition 31 we construct essential spheres  $X(\Sigma_g^0)$  and  $X(\Sigma_g^1)$  in  $\mathcal{C}(\Sigma_g^0)$  and  $\mathcal{C}(\Sigma_g^1)$  respectively. Our second main result, proved in §4.2 is:

**Theorem 2.** *Let  $g \geq 3$  and  $n \in \{0, 1\}$  or  $g = 2$  and  $n = 1$ . Then the essential  $(2g - 2)$ -sphere  $X(\Sigma_g^n) \subset \mathcal{C}(\Sigma_g^n)$  that is determined in Proposition 31 (i) represents a  $\text{Mod}(\Sigma_g^n)$ -module generator for the reduced homology of  $\mathcal{C}(\Sigma_g^n)$ , (ii) is a proper subset of  $\mathfrak{X}(\Sigma_g^n)$ , but (iii) is not rigid.*

Modified versions of Theorems 1 and 2 hold for the sporadic surfaces  $\Sigma_0^4$ ,  $\Sigma_1^0$  and  $\Sigma_1^1$ . See §4.3 for the precise statements.

While Theorem 2 rules out the possibility that homologically non-trivial spheres in the curve complex must be rigid, we have the following suggestive corollary:

**Corollary 3.** *Suppose  $g = 0$  and  $n \geq 5$  or  $g \geq 2$  and  $n \in \{0, 1\}$ . Let  $\mathfrak{X}(\Sigma_g^n) \subset \mathcal{C}(\Sigma_g^n)$  be the finite rigid set given in [1]. Then the inclusion map  $i : \mathfrak{X}(\Sigma_g^n) \rightarrow \mathcal{C}(\Sigma_g^n)$  induces a homomorphism  $i_* : \tilde{H}_*(\mathfrak{X}(\Sigma_g^n)) \rightarrow \tilde{H}_*(\mathcal{C}(\Sigma_g^n))$  with non-trivial image.*

*Proof.* In both cases the inclusion map  $j : X \rightarrow \mathcal{C}$  factors through the inclusion map  $i : \mathfrak{X} \rightarrow \mathcal{C}$  and induces a homomorphism  $j_* : \tilde{H}_*(X) \rightarrow \tilde{H}_*(\mathcal{C})$  with non-trivial image.  $\square$

**Question 4.** *Suppose  $3g + n \geq 5$ . Does the inclusion map  $i : \mathfrak{X}(\Sigma_g^n) \rightarrow \mathcal{C}(\Sigma_g^n)$  of the finite rigid set given in [1] induce a homomorphism  $i_* : \tilde{H}_*(\mathfrak{X}(\Sigma_g^n)) \rightarrow \tilde{H}_*(\mathcal{C}(\Sigma_g^n))$  with non-trivial image?*

Here is a guide to the paper. We begin, in §2, with a review of the necessary background. The associahedron will enter into this review because it concisely describes the simplicial structure of the spheres in the curve complex which we demonstrate to be homologically non-trivial. We introduce it in §2.1.

Our new work begins in §3, where we restrict our discussion to surfaces of genus 0 with  $n \geq 4$  marked points and give a finite Mod-module resolution of the Steinberg module (the reduced homology of the curve complex) using the cellular chain complex of the arc complex relative to the arc complex at infinity. The last two terms of this resolution give a Mod-module presentation of this homology group which we use to give a single class whose orbit under the mapping class group generates the entire relative homology group. In §3.1 we use Harer's homotopy equivalence from §2.2 to convert our non-trivial class in the homology of the arc complex relative to the arc complex at infinity to a class in the homology of the curve complex. We then simplify our representative of this class. In Proposition 25 we give, explicitly, a finite subset  $X$  of the curve complex  $\mathcal{C}$  the orbit of whose homology class under the action of the mapping class group generates the reduced homology of the curve complex. In §3.2 we find that the simplified representative is precisely the finite rigid set of Aramayona and Leininger, proving Theorem 1.

In §4 we consider the same situation when  $g \geq 1$  and  $n \in \{0, 1\}$ . This situation was already treated in [2], however we are now able to simplify the results given there drastically. We give, in Proposition 31, an explicit simplified homologically non-trivial sphere in the curve complex the orbit of whose homology class under the action of the mapping class group generates the reduced homology of the curve complex. We establish that it is a proper subset of the finite rigid set of Aramayona and Leininger. In §4.2 we prove Theorem 2, by showing that it is not a finite rigid set.

Modified versions of Theorems 1 and 2, for sporadic surfaces, are discussed and proved in §4.3.

## 2 Review of the Background

Our results in §3 and §4 will concern the surfaces  $\Sigma_0^n$  where  $n \geq 4$  and  $\Sigma_g^n$  where  $g \geq 1$  and  $n \in \{0, 1\}$ . However, many of the results that we review in this section apply in greater generality so we provide the stronger statements and more general definitions when possible.

### 2.1 The curve complex, arc complex and associahedron

Let  $n \geq 4$  or  $g \geq 1$  and set

$$\tau = \begin{cases} n - 4, & g = 0 \\ 2g - 2, & g \geq 1 \text{ and } n = 0 \\ 2g + n - 3, & g \geq 1 \text{ and } n \geq 1 \end{cases} . \quad (1)$$

By the work of Harer we have the following theorem:

**Known Result 5** ([5, Theorem 3.5]). *Assume  $n \geq 4$  or  $g \geq 1$ . The curve complex  $\mathcal{C}(\Sigma_g^n)$  has the homotopy type of a countably infinite wedge sum of spheres of dimension  $\tau$  thus*

$$\mathcal{C}(\Sigma_g^n) \simeq \bigvee^{\infty} S^{\tau}.$$

where  $\tau$  is as in (1).

In particular the reduced homology  $\tilde{H}_*(\mathcal{C}(\Sigma_g^n); \mathbf{Z})$  of the curve complex is non-trivial only in dimension  $\tau$  where we get

$$\tilde{H}_{\tau}(\mathcal{C}(\Sigma_g^n); \mathbf{Z}) \cong \bigoplus^{\infty} \mathbf{Z}.$$

The *dualizing module* of a duality group  $\Gamma$  of cohomological dimension  $d$  is a  $\Gamma$ -module  $D$  such that for any  $\Gamma$ -module  $A$  and any  $k \in \mathbf{Z}$  we have

$$H^k(\Gamma; A) \cong H_{d-k}(\Gamma; A \otimes_{\mathbf{Z}} D)$$

where  $A \otimes_{\mathbf{Z}} D$  has the diagonal module structure. (See [3, VIII.10] for a general introduction to duality groups.)

In particular if  $\Gamma$  is any torsion free finite index subgroup of the mapping class group  $\text{Mod}(\Sigma_g^n)$  where  $n \geq 4$  or  $g \geq 1$  then by the work of Harer [5, Theorem 4.1] the dualizing module of  $\Gamma$  is the *Steinberg module*

$$\text{St} = \text{St}(\Sigma_g^n) := \tilde{H}_\tau(\mathcal{C}(\Sigma_g^n); \mathbf{Z}).$$

We view this as motivation for a careful study of the homotopy type of the curve complex.

To help us in this study, it will be convenient to introduce a closely related complex, the “arc complex” of a surface with marked points. Let  $\Sigma_g^n$  be the surface with genus  $g \geq 0$  and  $n \geq 1$  marked points and let  $V \subset \Sigma_g^n$  be the set of marked points. An *arc* in  $\Sigma_g^n$  is either the unoriented image of a properly embedded path in  $\Sigma_g^n$  joining two points of  $V$  which is disjoint from  $V$  except at its endpoints or else the image of an unoriented simple loop in  $\Sigma_g^n$  based at a point in  $V$  disjoint from  $V$  except at its basepoint. An *essential arc* in  $\Sigma_g^n$  is an arc which does not bound an embedded disk in  $\Sigma_g^n$  whose interior is disjoint from  $V$ . We will say that two arcs are *disjoint* if they are disjoint except possibly at their endpoints. The isotopy class of an essential arc is its orbit under the elements of  $\text{Homeo}^0(\Sigma_g^n)$  which fix the marked points pointwise. An *arc system* is a set of isotopy classes of arcs with pairwise disjoint representatives and a *sub arc system* of an arc system is any non-empty subset of the arc system. The *arc complex*  $\mathcal{A} = \mathcal{A}(\Sigma_g^n)$  is the simplicial complex whose vertices are isotopy classes of essential arcs and whose  $k$ -simplices correspond to arc systems with  $k + 1$  isotopy classes of arcs. Observe that a maximal arc system gives a triangulation of the surface  $\Sigma_g^n$  with  $n$  vertices so by euler characteristic it will have  $6g + 3n - 6$  arcs. Therefore we have

$$\dim(\mathcal{A}(\Sigma_g^n)) = 6g + 3n - 7.$$

The arc complex has a very nice property:

**Known Result 6** ([7]). *If  $n \geq 2$  or if both  $g \geq 1$  and  $n \geq 1$  then  $\mathcal{A}(\Sigma_g^n)$  is contractible.*

An arc system *fills* the surface  $\Sigma_g^n$  if the arcs in the arc system cut the surface into disks with at most one marked point in their interior. The *arc complex at infinity*  $\mathcal{A}_\infty = \mathcal{A}_\infty(\Sigma_g^n)$  is the union of all simplices of  $\mathcal{A}(\Sigma_g^n)$  whose corresponding arc systems do *not* fill the surface. One may also alternatively describe  $\mathcal{A}_\infty(\Sigma_g^n)$  as the subset of points in  $\mathcal{A}(\Sigma_g^n)$  with infinite stabilizers under the action of  $\text{Mod}(\Sigma_g^n)$ .

Now suppose that  $g = 0$  and  $n \geq 4$  or  $g \geq 1$  and  $n = 1$  and let  $\tau$  be as in equation (1). Then at least  $\tau + 2$  arcs are needed to fill the surface  $\Sigma_g^n$  thus  $\mathcal{A}_\infty(\Sigma_g^n)$  contains the entire  $\tau$ -skeleton of  $\mathcal{A}(\Sigma_g^n)$ . If on the other hand we have  $g \geq 1$  and  $n \geq 2$  then at least  $\tau + 1$  arcs are needed to fill the surface which implies that  $\mathcal{A}_\infty(\Sigma_g^n)$  contains the  $(\tau - 1)$ -skeleton of  $\mathcal{A}(\Sigma_g^n)$  but not the entire  $\tau$ -skeleton of  $\mathcal{A}(\Sigma_g^n)$ . This fact renders an essential part of our argument given in §2.3 applicable only to the cases where  $g = 0$  and  $n \geq 4$  or  $g \geq 1$  and  $n = 1$ .

We now introduce a simplicial complex which we will later embed in the curve complex as a homologically non-trivial sphere. Let  $m \geq 3$ . The associahedron  $K_m$  is a convex polytope homeomorphic to the closed  $(m - 2)$ -ball. Its boundary  $\partial K_m$  is a cell complex homeomorphic to the  $(m - 3)$ -sphere. The dual of the cell complex  $\partial K_m$  is a *simplicial* complex  $D_m$  which is again homeomorphic to the  $(m - 3)$ -sphere. Here we provide a convenient description of  $D_m$  taken from [4, Corollary 2.7].

Let  $\Gamma$  be a graph. A *tube* is any proper non-empty subset of the vertices of  $\Gamma$  whose induced subgraph in  $\Gamma$  is connected. Let  $t_1$  and  $t_2$  be two tubes of  $\Gamma$ . We say that  $t_1$  and  $t_2$  are *nested* if  $t_1 \subset t_2$  or  $t_2 \subset t_1$ . The tubes  $t_1$  and  $t_2$  *overlap* if  $t_1 \cap t_2 \neq \emptyset$  and  $t_1$  and  $t_2$  are not nested. Tubes  $t_1$  and  $t_2$  are *adjacent* if  $t_1 \cap t_2 = \emptyset$  and  $t_1 \cup t_2$  is a tube. Two tubes in  $\Gamma$  are *compatible* if they do not overlap and are not adjacent.

**Definition 7.** Let  $\mathcal{D}(\Gamma)$  be the simplicial complex with a vertex for each tube of  $\Gamma$  and a  $k$ -simplex for each set of  $k + 1$  pairwise compatible tubes in  $\Gamma$ .

**Definition 8.** The dual  $D_m$  of the boundary  $(m - 3)$ -sphere of the associahedron  $K_m$  is the simplicial complex  $\mathcal{D}(\Lambda_{m-1})$  where  $\Lambda_{m-1}$  is the path graph with  $m - 1$  vertices given in Figure 1.

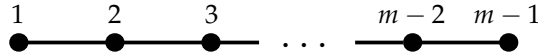


Figure 1: A path graph  $\Lambda_{m-1}$  with vertex set  $\{1, 2, \dots, m-1\}$ .

**Example 9.**  $D_5$  is the simplicial complex homeomorphic to  $S^2$  depicted in Figure 2.

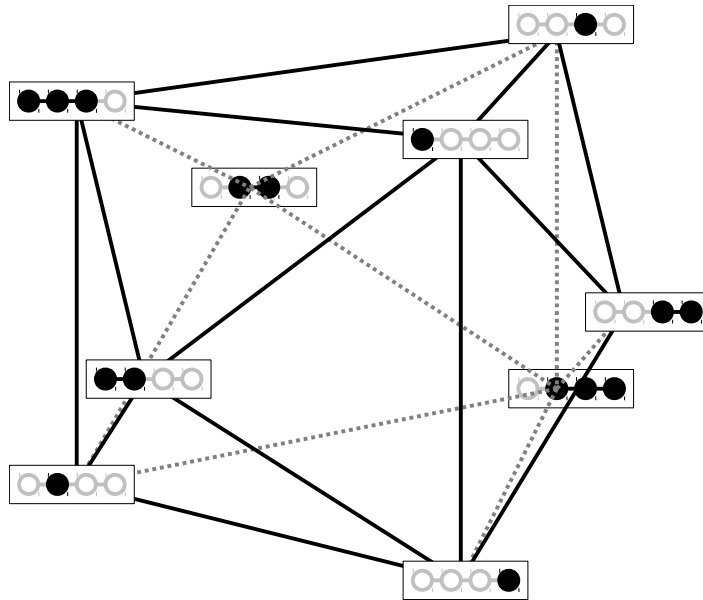


Figure 2: The simplicial complex  $D_5$  is the dual of the boundary of the associahedron  $K_5$

The curious reader will discover that the literature on the combinatorics of simplicial complexes is filled with references to the associahedron. In particular, the reader may wish to compare Figure 2 below with Figure 10 of [2], which illustrates the role that the associahedron will play in this paper. Indeed, it is used to describe both the finite rigid subsets sets of  $\mathcal{C}(\Sigma_0^6)$  and of  $\mathcal{C}(\Sigma_2^0)$ , illustrating the well-known isomorphism between these two complexes [10].

## 2.2 Maps between simplicial complexes

In the discussion that follows we will encounter a number of maps between simplicial complexes which are not simplicial maps, but are nonetheless very well-behaved piecewise linear maps. For example we

will describe below a continuous map from the arc complex at infinity  $\mathcal{A}_\infty(\Sigma_g^n)$  to the curve complex  $\mathcal{C}(\Sigma_g^n)$  which induces a homotopy equivalence (see Known Result 17). Here we develop some convenient language and observations for working with these maps.

If  $X$  is a simplicial complex with vertex set  $X^0$  we will specify a general point  $p \in X$  as a linear combination

$$p = \sum_{v \in X^0} p_v v$$

where for all  $v \in X^0$  we have  $0 \leq p_v \leq 1$  and  $\sum_{v \in X^0} p_v = 1$ . For a point  $p \in X$  the *support* of  $p$  is the set

$$\text{supp } p = \{v \in X^0 \mid p_v > 0\}$$

and for a subset  $U \subset X$  we set  $\text{supp } U = \bigcup_{p \in U} \text{supp } p$ . Our simplicial complexes will all be finite dimensional so we will have  $\dim X = \max_{p \in X} |\text{supp } p|$ . Conversely, given a set  $S \subset X^0$  we will define the *span* of  $S$  to be the set

$$\text{span } S = \{p \in X \mid \text{supp } p \subset S\}.$$

**Definition 10** (Interpolability). Let  $X$  and  $Y$  be simplicial complexes with vertex sets  $X^0$  and  $Y^0$  respectively. Let  $f^0 : X^0 \rightarrow Y$  be a function. We say that  $f^0$  is *interpolable* if for any simplex  $\sigma$  of  $X$  there is some simplex  $\eta$  of  $Y$  such that for every vertex  $v$  of  $\sigma$  we have  $f^0(v) \in \eta$ .

Note that it suffices to verify the condition of Definition 10 for the maximal simplices of a finite dimensional simplicial complex  $X$ .

**Definition 11** (Linear interpolation). Let  $X$  and  $Y$  be simplicial complexes with vertex sets  $X^0$  and  $Y^0$  respectively. We say that  $f : X \rightarrow Y$  is a *linear interpolation* if the restriction  $f|_{X^0} : X^0 \rightarrow Y$  is interpolable and for any point  $p = \sum_{v \in X^0} p_v v \in X$  we have

$$f \left( \sum_{v \in X^0} p_v v \right) = \sum_{v \in X^0} p_v f(v).$$

**Remark 12.** Note that any interpolable function  $f^0 : X^0 \rightarrow Y$  from the 0-skeleton  $X^0$  of a simplicial complex  $X$  to a simplicial complex  $Y$  extends to a unique linear interpolation  $f : X \rightarrow Y$  via the equation

$$f \left( \sum_{v \in X^0} p_v v \right) = \sum_{v \in X^0} p_v f^0(v).$$

**Lemma 13** (Linear interpolations are continuous). *Let  $f : X \rightarrow Y$  be a linear interpolation. Then  $f$  is continuous.*

*Proof.* This follows from the gluing lemma from basic topology and the observation that linear interpolations agree on intersections of simplices.  $\square$

**Lemma 14.** *Let  $f, g : X \rightarrow Y$  be linear interpolations and suppose that for each simplex  $\sigma$  of  $X$  there is a simplex  $\eta$  of  $Y$  such that for each vertex  $v$  of  $\sigma$  we have  $f(v), g(v) \in \eta$ . Then  $f$  and  $g$  are homotopic functions.*

*Proof.* For  $t \in [0, 1]$  define  $F_t^0 : X^0 \rightarrow Y$  by setting

$$F_t^0(v) = (1-t)f(v) + tg(v)$$

for each vertex  $v \in X^0$ . Note that  $F_t^0$  is interpolable so we may extend it to a linear interpolation  $F_t : X \rightarrow Y$ . Now define  $F : X \times I \rightarrow Y$  by setting  $F(x, t) = F_t(x)$ . One observes that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$  for all  $x \in X$ . The continuity of  $F$  follows from the continuity of the restriction  $F|_{\sigma \times I}$  for each simplex  $\sigma$  of  $X$  and the gluing lemma.  $\square$

**Corollary 15.** *Let  $f : X \rightarrow Y$  be a linear interpolation. For each vertex  $v \in X^0$  choose a vertex  $w_v \in \text{supp } f(v)$  and define  $h^0 : X^0 \rightarrow Y$  by setting  $h^0(v) = w_v$ . Then  $h^0$  is interpolable and  $f$  is homotopic to the linear interpolation  $h : X \rightarrow Y$  of  $h^0$ .*

*Proof.* Let  $\sigma$  be a simplex of  $X$ . Then since  $f$  is a linear interpolation there is a simplex  $\eta$  of  $Y$  such that  $f(\sigma) \subset \eta$  and hence  $\text{supp } f(\sigma) \subset \eta$ . By construction if  $v$  is a vertex of  $\sigma$  then  $\text{supp } h^0(v) \subset \text{supp } f(\sigma)$  so  $\text{supp } h^0(v) \subset \eta$ . It follows that  $h^0$  is interpolable with linear interpolation  $h : X \rightarrow Y$ . Moreover,  $h(v), f(v) \in \eta$  so we may apply Lemma 14 to conclude that  $h$  and  $f$  are homotopic.  $\square$

Finally we note that if  $X$  is a simplicial complex then its *barycentric subdivision*  $\text{bs } X$  is the simplicial complex with 0-skeleton given by the set

$$(\text{bs } X)^0 = \{\sigma \mid \sigma \text{ is a simplex of } X\}$$

and with  $n$ -simplices given by flags of simplices of  $X$  of length  $n + 1$ . One gets a homeomorphism between  $X$  and  $\text{bs } X$  by letting  $j^0 : (\text{bs } X)^0 \rightarrow X$  be the function sending the vertex  $\sigma \in (\text{bs } X)^0$  to the point  $\frac{1}{|\text{supp } \sigma|} \sum_{v \in \text{supp } \sigma} v$  and letting

$$j : \text{bs } X \rightarrow X$$

be the linear interpolation of  $j^0$ . The *barycentric subdivision map*

$$\text{bs} : X \rightarrow \text{bs } X \tag{2}$$

is the inverse of  $j$ .

We end our general discussion of linear interpolations with the following observation:

**Remark 16.** A linear interpolation  $f : X \rightarrow Y$  is a simplicial map if and only if for all  $v \in X^0$  we have  $|\text{supp } f(v)| = 1$ .

With the language of linear interpolations we now review Harer's homotopy equivalence between the arc complex at infinity and the curve complex. Suppose that  $g \geq 1$  and  $n \geq 1$  or  $g = 0$  and  $n \geq 4$ . Let  $\mathcal{A}_\infty = \mathcal{A}_\infty(\Sigma_g^n)$  and  $\mathcal{C} = \mathcal{C}(\Sigma_g^n)$ . Let  $\text{bs } \mathcal{A}_\infty$  be the barycentric subdivision of  $\mathcal{A}_\infty$ . The vertices of  $\text{bs } \mathcal{A}_\infty$  will correspond to non-filling arc systems with  $k$ -simplices corresponding to increasing flags of non-filling arc systems of length  $k + 1$ .

When an arc system  $\alpha$  does not fill the surface  $\Sigma_g^n$  we can associate a curve system  $\gamma(\alpha)$  to it by letting  $N(\alpha) \subset \Sigma_g^n$  be a closed regular neighborhood of the union of the arcs in the arc system  $\alpha$  and letting the curve system  $\gamma(\alpha)$  be the set of essential (in  $\Sigma_g^n$ ) boundary curves of the subsurface  $\Sigma_g^n - N(\alpha)$  with duplicate curves removed. Let

$$\Phi^0 : (\text{bs } \mathcal{A}_\infty)^0 \rightarrow \mathcal{C}$$

be the function sending the non-filling arc system  $\alpha$  to the barycenter  $\frac{1}{|\gamma(\alpha)|} \sum_{c \in \gamma(\alpha)} c$  of the simplex in  $\mathcal{C}$  corresponding to the the curve system  $\gamma(\alpha)$ .

One sees that  $\Phi^0$  is interpolable as follows. If  $\sigma$  is a simplex of  $\text{bs } \mathcal{A}_\infty$  then it corresponds to an increasing flag of non-filling arc systems  $\alpha_1 \subsetneq \alpha_2 \subsetneq \cdots \subsetneq \alpha_k$ . We may arrange that the corresponding regular neighborhoods satisfy  $N(\alpha_i) \subsetneq \text{int } N(\alpha_{i+1})$  when  $1 \leq i < k$ . Thus all boundary curves can be made to be simultaneously disjoint. Let  $\eta$  be the simplex of  $\mathcal{C}$  corresponding to the curve system which is the union of all such sets of boundary curves. If  $v$  is a vertex of  $\sigma$  then  $\Phi^0(v) \in \eta$ . Thus  $\Phi^0$  is interpolable. Let

$$\Phi : \text{bs } \mathcal{A}_\infty \rightarrow \mathcal{C} \tag{3}$$

be the linear interpolation of  $\Phi^0$ .

**Known Result 17** ([5, Theorem 3.4]). *Suppose that  $g \geq 1$  and  $n \geq 1$  or  $g = 0$  and  $n \geq 4$ . Then the map  $\Phi$  in (3) induces a homotopy equivalence between  $\mathcal{A}_\infty(\Sigma_g^n)$  and  $\mathcal{C}(\Sigma_g^n)$ .*

### 2.3 A free resolution of the homology of the Steinberg module

We now restrict to the cases where either  $g = 0$  and  $n \geq 4$  or  $g \geq 1$  and  $n = 1$  and utilize techniques similar to those in [2] to give a  $\text{Mod}(\Sigma_g^n)$ -module resolution of  $\text{St}(\Sigma_g^n)$ . It is the failure of (5) below when  $g \geq 1$  and  $n \geq 2$  which renders our argument invalid for that case.

By Known Result 6 the arc complex  $\mathcal{A} = \mathcal{A}(\Sigma_g^n)$  is contractible and by Known Result 17 the arc complex at infinity  $\mathcal{A}_\infty = \mathcal{A}_\infty(\Sigma_g^n)$  is homotopy equivalent to the curve complex  $\mathcal{C} = \mathcal{C}(\Sigma_g^n)$ . Known Result 5 says that curve complex has the homotopy type of a wedge of  $\tau$ -dimensional spheres. Therefore for  $k \geq 1$

$$\begin{aligned} H_k(\mathcal{A}, \mathcal{A}_\infty; \mathbf{Z}) &\cong \tilde{H}_{k-1}(\mathcal{A}_\infty; \mathbf{Z}) \\ &\cong \tilde{H}_{k-1}(\mathcal{C}; \mathbf{Z}) \\ &= \begin{cases} 0, & k \neq \tau + 1 \\ \text{St}(\Sigma_g^n), & k = \tau + 1. \end{cases} \end{aligned} \quad (4)$$

Now consider the cellular chain complex  $C_*(\mathcal{A}, \mathcal{A}_\infty)$  for the pair of spaces  $(\mathcal{A}, \mathcal{A}_\infty)$ .

$$\cdots \rightarrow 0 \rightarrow C_{6g+3n-7}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \cdots \rightarrow C_1(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_0(\mathcal{A}, \mathcal{A}_\infty) \rightarrow 0$$

where as usual we define  $C_k(\mathcal{A}, \mathcal{A}_\infty) := C_k(\mathcal{A}) / C_k(\mathcal{A}_\infty)$ . A chain complex is an exact sequence when all of its homology groups are 0. Note that every homology group of the pair of spaces  $(\mathcal{A}, \mathcal{A}_\infty)$  is zero except for  $H_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty; \mathbf{Z})$  so the chain complex  $C_*(\mathcal{A}, \mathcal{A}_\infty)$  is very close to being an exact sequence.

No arc system with  $2g - 1$  or fewer arcs can fill the surface  $\Sigma_g^1$  and no arc system with  $n - 3$  or fewer arcs can fill the surface  $\Sigma_0^n$ . Hence in both cases no arc system with  $\tau + 1$  or fewer arcs can fill the surface so the entire  $\tau$ -skeleton of  $\mathcal{A}$  is contained in  $\mathcal{A}_\infty$ . Thus

$$C_\tau(\mathcal{A}, \mathcal{A}_\infty) := C_\tau(\mathcal{A}) / C_\tau(\mathcal{A}_\infty) = C_\tau(\mathcal{A}_\infty) / C_\tau(\mathcal{A}_\infty) = 0. \quad (5)$$

From equations (4) and (5) we have that

$$\begin{aligned} \text{St}(\Sigma_g^n) &\cong H_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \\ &= \frac{\ker \left( C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_\tau(\mathcal{A}, \mathcal{A}_\infty) \right)}{\text{Image} \left( C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \right)} \\ &= \frac{\ker \left( C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow 0 \right)}{\text{Image} \left( C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \right)} \\ &= \frac{C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty)}{\partial C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty)} \\ &\cong \frac{C_{\tau+1}(\mathcal{A})}{C_{\tau+1}(\mathcal{A}_\infty) \oplus \partial C_{\tau+2}(\mathcal{A})} \end{aligned} \quad (6)$$

where  $\partial$  is the boundary operator for  $C_*(\mathcal{A}, \mathcal{A}_\infty)$  or  $C_*(\mathcal{A})$  depending on the context. We thus have the following very useful description of the homology of the curve complex:



**Proposition 18.** *Let  $g = 0$  and  $n \geq 4$  or  $g \geq 1$  and  $n = 1$ . Let  $\tau$  be as in (1) and  $C_*(\mathcal{A})$  and  $C_*(\mathcal{A}_\infty)$  be the simplicial chain complexes of the arc complex  $\mathcal{A} = \mathcal{A}(\Sigma_g^n)$  and arc complex at infinity  $\mathcal{A}_\infty = \mathcal{A}_\infty(\Sigma_g^n)$  respectively. Then the Steinberg module  $\text{St}(\Sigma_g^n) := \tilde{H}_\tau(\mathcal{C}(\Sigma_g^n); \mathbf{Z})$  has the following presentation:*

$$\text{St}(\Sigma_g^n) \cong \frac{C_{\tau+1}(\mathcal{A})}{C_{\tau+1}(\mathcal{A}_\infty) \oplus \partial C_{\tau+2}(\mathcal{A})}.$$

Proposition 18 says that the reduced homology of the curve complex is the quotient of the free abelian group on all arc systems in  $\Sigma_g^n$  with  $\tau + 2$  arcs modulo all *non-filling* arc systems with  $\tau + 2$  arcs and boundaries of arc systems with  $\tau + 3$  arcs.

Notice that if  $k \geq \tau + 2$  then by equation (4) we have  $H_k(\mathcal{A}, \mathcal{A}_\infty) = 0$  so

$$\cdots \rightarrow 0 \rightarrow C_{6g+3n-7}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \cdots \rightarrow C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \quad (7)$$

is an *exact* sequence. As with any terminating exact sequence we can append two more terms to get a slightly longer exact sequence

$$\cdots \rightarrow 0 \rightarrow C_{6g+3n-7}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \cdots \rightarrow C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \frac{C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty)}{\partial C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty)} \rightarrow 0.$$

From (6) we know that the penultimate term of this sequence is isomorphic to  $\text{St}(\Sigma_g^n)$ . Since the sequence in (7) is exact we have a  $\text{Mod}(\Sigma_g^n)$ -module resolution of  $\text{St}(\Sigma_g^n)$ . Note that a filling arc system in  $\Sigma_g^n$  always has a finite but possibly non-trivial stabilizer in  $\text{Mod}(\Sigma_g^n)$ . Hence for  $k \geq \tau + 1$  the relative chain group  $C_k(\mathcal{A}, \mathcal{A}_\infty)$  is not quite a free  $\text{Mod}(\Sigma_g^n)$ -module. However, if one restricts to a torsion free subgroup  $\Gamma < \text{Mod}(\Sigma_g^n)$  then  $C_k(\mathcal{A}, \mathcal{A}_\infty)$  will be a free  $\Gamma$ -module.

**Proposition 19.** *Let  $g = 0$  and  $n \geq 4$  or  $g \geq 1$  and  $n = 1$ . Let  $\tau$  be as in (1). The Steinberg module  $\text{St}(\Sigma_g^n) := \tilde{H}_\tau(\mathcal{C}(\Sigma_g^n); \mathbf{Z})$  has the following  $\text{Mod}(\Sigma_g^n)$ -module resolution of finite length:*

$$\cdots \rightarrow 0 \rightarrow C_{6g+3n-7}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \cdots \rightarrow C_{\tau+2}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow C_{\tau+1}(\mathcal{A}, \mathcal{A}_\infty) \rightarrow \text{St}(\Sigma_g^n).$$

*This resolution is not free but will be free for any torsion free subgroup  $\Gamma < \text{Mod}(\Sigma_g^n)$ . The terms of this resolution will be finitely generated as  $\Gamma$ -modules if the index of  $\Gamma$  in  $\text{Mod}(\Sigma_g^n)$  is finite.*

### 3 Essential spheres and finite rigid sets for the surfaces $\Sigma_0^n$

We begin the new work in this paper. In this section we assume that  $n \geq 4$ , and let  $\Sigma_0^n$  be the surface of genus 0 with marked point set  $V \subset \Sigma_0^n$  of order  $n$ . We turn our attention to the goal of finding a single class in  $\text{St}(\Sigma_0^n)$  whose orbit under  $\text{Mod}(\Sigma_0^n)$  generates  $\text{St}(\Sigma_0^n)$  as a  $\mathbf{Z}$ -module.

From Proposition 18 we have that

$$\text{St}(\Sigma_0^n) \cong \frac{C_{n-3}(\mathcal{A})}{C_{n-3}(\mathcal{A}_\infty) \oplus \partial C_{n-2}(\mathcal{A})}.$$

$C_{n-3}(\mathcal{A})$  is the free abelian group on all arc systems with  $n - 2$  arcs. The mapping class group is not transitive on the set of these arc systems, but there are a finite number of orbits. We will show that after quotienting by the relations  $C_{n-3}(\mathcal{A}_\infty)$  and  $\partial C_{n-2}(\mathcal{A})$  that every element of  $\text{St}(\Sigma_0^n)$  will be a linear combination of elements of the  $\text{Mod}(\Sigma_0^n)$ -orbit of the class of a single arc system.

The set of non-filling arc systems with  $n - 2$  arcs forms a basis for  $C_{n-3}(\mathcal{A}_\infty)$ . This is a subset of the set of all arc systems with  $n - 2$  arcs which forms a basis for  $C_{n-3}(\mathcal{A})$ . Thus we see that non-filling arc systems give the trivial class in  $\text{St}(\Sigma_0^n)$ .

The union of the arcs in an arc system  $\alpha$  is a graph  $G(\alpha)$ . Note that in general the vertices of  $G(\alpha)$  will be only those marked points in  $V$  which are endpoints of at least one arc in the arc system  $\alpha$ . The arc system  $\alpha$  fills the surface if  $\Sigma_0^n - G(\alpha)$  is a disjoint union of open disks containing at most one marked point. If the filling arc system has exactly  $n - 2$  arcs then each of these disks must contain exactly one marked point. We note that if  $\alpha$  is a filling arc system then  $G(\alpha)$  must be connected since its complement is a disjoint union of open disks.

**Lemma 20.** *If  $\alpha$  is a filling arc system in  $\Sigma_0^n$  with  $n - 2$  arcs then its class in the Steinberg module  $\text{St}(\Sigma_0^n)$  is a  $\mathbf{Z}$ -linear combination of the classes of filling arc systems  $\beta$  with  $n - 2$  arcs such that the graph  $G(\beta)$  is a path graph.*

*Proof.* Let  $\alpha$  be an arc system in  $\Sigma_0^n$  with  $n - 2$  arcs. If  $\alpha$  does not fill then its class in  $\text{St}(\Sigma_0^n)$  is trivial. Define a *tine* of the graph  $G(\alpha)$  to be a path subgraph  $P$  of  $G(\alpha)$  which is either any single vertex of  $G(\alpha)$  or else has at least one vertex which has valence one in  $G(\alpha)$  and at most one vertex which has valence greater than 2 in  $G(\alpha)$  (see Figure 3). Note that every vertex of  $G(\alpha)$  is a tine of length 0. Define the complexity  $\mu(\alpha)$  of the arc system  $\alpha$  to be the length of the longest tine of  $G(\alpha)$ .

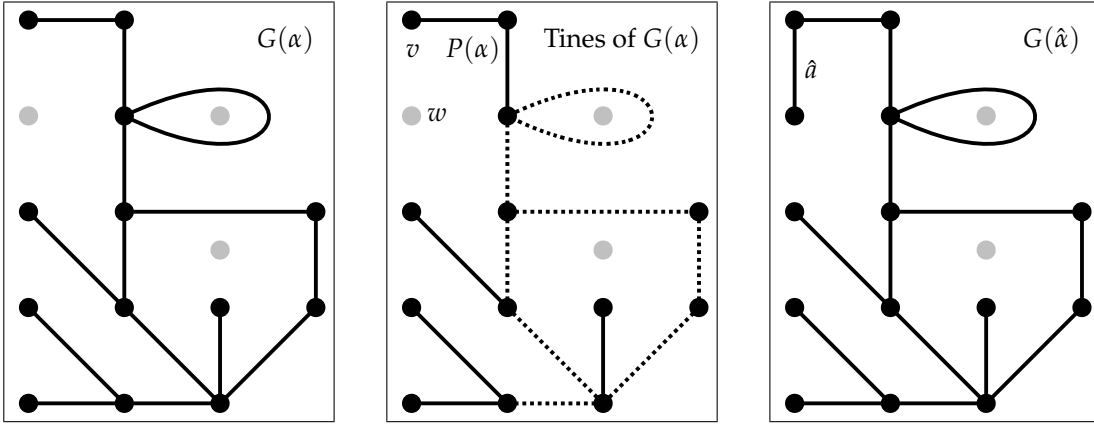


Figure 3: On the left we have a filling arc system  $\alpha$  in the surface  $\Sigma_0^{16}$  with 14 arcs with complexity  $\mu(\alpha) = 2$ . In the center the solid part of the graph is the union of the tines of  $G(\alpha)$ . The graph  $G(\alpha)$  has 13 tines of length zero, four tines of length one and one tine  $P(\alpha)$  of length two. On the right we have a filling arc system  $\hat{\alpha}$  such that the relation in  $\text{St}(\Sigma_0^{16})$  induced by  $\partial\hat{\alpha}$  expresses the class of  $\alpha$  as a linear combination of classes of filling arc systems all with with complexity at least 3.

Let  $P(\alpha)$  be a tine of  $\alpha$  with maximal length. If  $\mu(\alpha) = n - 2$  then  $G(\alpha)$  is a path graph and we are done. Assume that  $\mu(\alpha) < n - 2$ . If  $\mu(\alpha) = 0$  let  $v$  be the unique vertex of  $P(\alpha)$ . Otherwise let  $v$  be the unique vertex of  $P(\alpha)$  which has valence 1 in  $G(\alpha)$ . The vertex  $v$  must be in the closure of at least one component of  $\Sigma_0^n - \bigcup \alpha$ . Let  $w$  be the marked point contained in the interior of such a component (see the center of Figure 3). Let  $\hat{\alpha}$  be an arc disjoint from the arcs of  $\alpha$  connecting  $v$  to  $w$ . Let  $\hat{\alpha} = \alpha \cup \{\hat{\alpha}\}$  which is an arc system with  $n - 1$  arcs (see the right of Figure 3).

Given  $b \in \hat{\alpha}$  set  $\alpha_b = \hat{\alpha} - \{b\}$ . We claim exactly one of the following holds:

1.  $\alpha_b = \alpha$ .
2.  $\alpha_b$  is a non-filling arc system in  $\Sigma_0^n$ .
3.  $\alpha_b$  is a filling arc system with  $\mu(\alpha_b) > \mu(\alpha)$ .

Note that  $\alpha_b = \alpha$  if and only if  $b = \hat{a}$ . Suppose  $b \neq \hat{a}$ . Then  $b \in \alpha$ . If  $b \subset P(\alpha)$  then  $G(\alpha_b)$  is disconnected so  $\alpha_b$  is non-filling. If  $b \not\subset P(\alpha)$  then  $\alpha_b$  may or may not be a filling system but in either case  $\mu(\alpha_b) \geq \mu(\alpha) + 1$ .

We may use the relation  $\partial \hat{a}$  in  $\text{St}(\Sigma_0^n)$  to express the class of the arc system  $\alpha$  as a  $\mathbf{Z}$ -linear combination of classes of arc systems with higher complexity. By induction we may write any such class as a  $\mathbf{Z}$ -linear combination of classes of filling arcs systems with the maximal complexity  $n - 2$ . The graphs for each of these arc systems must be a path graph.  $\square$

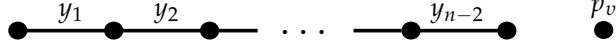


Figure 4: A filling arc system  $v = \{y_1, y_2, \dots, y_{n-2}\}$  with  $n - 2$  arcs in the surface  $\Sigma_0^n$  whose class generates  $\text{St}(\Sigma_0^n)$  as a  $\text{Mod}(\Sigma_0^n)$ -module.

**Proposition 21.** *Suppose  $n \geq 4$ . Let  $\theta_v \subset \mathcal{A}(\Sigma_0^n)$  be the  $(n - 3)$ -simplex corresponding to the arc system  $v$  in Figure 4. Then the class*

$$[\theta_v] \in \text{St}(\Sigma_0^n) = \frac{C_{n-3}(\mathcal{A})}{C_{n-3}(\mathcal{A}_\infty) \oplus \partial C_{n-2}(\mathcal{A})}$$

*generates  $\text{St}(\Sigma_0^n)$  as a  $\text{Mod}(\Sigma_0^n)$ -module.*

*Proof.* By Lemma 20 the class of any arc system with  $n - 2$  arcs is a  $\mathbf{Z}$ -linear combination of classes of arc systems whose graphs are path graphs. All path graphs with  $n - 2$  edges in  $\Sigma_0^n$  are in the  $\text{Mod}(\Sigma_0^n)$ -orbit of the arc system  $v$  pictured in Figure 4. Hence the  $\text{Mod}(\Sigma_0^n)$ -orbit of the class of  $\theta_v$  generates  $\text{St}(\Sigma_0^n)$  as a  $\mathbf{Z}$ -module. In other words  $[\theta_v]$  generates  $\text{St}(\Sigma_0^n)$  as a  $\text{Mod}(\Sigma_0^n)$ -module.  $\square$

**Corollary 22.** *Suppose  $n \geq 4$ . Let  $\theta_v \subset \mathcal{A}(\Sigma_0^n)$  be the  $(n - 3)$ -simplex corresponding to the arc system  $v$  in Figure 4. Then the class*

$$[\theta_v] \in \text{St}(\Sigma_0^n) = \frac{C_{n-3}(\mathcal{A})}{C_{n-3}(\mathcal{A}_\infty) \oplus \partial C_{n-2}(\mathcal{A})}$$

*is non-trivial.*

*Proof.* If the  $[\theta_v]$  is the trivial class in  $\text{St}(\Sigma_0^n)$  then by Proposition 21 we must have that  $\text{St}(\Sigma_0^n)$  is the zero module. Let  $\Gamma < \text{Mod}(\Sigma_0^n)$  be a torsion free finite index subgroup. Then by [5, Theorem 4.1] the cohomological dimension of  $\Gamma$  is  $n - 3$ . We may use duality to show that for all  $k$  and all  $\Gamma$ -modules  $A$

$$H^k(\Gamma; A) \cong H_{n-3-k}(\Gamma; A \otimes \text{St}(\Sigma_0^n)) = H_{n-3-k}(\Gamma; A \otimes 0) = H_{n-3-k}(\Gamma; 0) = 0.$$

This implies that the cohomological dimension of  $\Gamma$  is 0. Hence we have a contradiction.  $\square$

### 3.1 An essential sphere in the curve complex of the surface $\Sigma_0^n$

Again we assume that  $g = 0$  and  $n \geq 4$ . In Corollary 22 we have a non-trivial class in

$$\text{St}(\Sigma_0^n) = \frac{C_{n-3}(\mathcal{A})}{C_{n-3}(\mathcal{A}_\infty) \oplus \partial C_{n-2}(\mathcal{A})}.$$

We will now use the homotopy equivalence between  $\mathcal{A}_\infty$  and  $\mathcal{C}$  and the techniques of §2.2 to get a  $(n - 4)$ -sphere in  $\mathcal{C}$  representing this non-trivial class. After simplification our  $(n - 4)$ -sphere will be the dual  $D_{n-1}$  of the boundary of the associahedron  $K_{n-1}$  (See Definition 8).

The class of the filling arc system in Figure 4 is represented by a single  $(n-3)$ -simplex  $\theta_v \subset \mathcal{A}(\Sigma_g^n)$  all of whose proper faces are contained in the arc complex at infinity. The connecting homomorphism

$$\partial : H_{n-3}(\mathcal{A}, \mathcal{A}_\infty; \mathbf{Z}) \rightarrow \tilde{H}_{n-4}(\mathcal{A}_\infty; \mathbf{Z})$$

sends the class  $[\theta_v] \in H_{n-3}(\mathcal{A}, \mathcal{A}_\infty; \mathbf{Z})$  to the class  $[\theta_v^{n-4}] \in \tilde{H}_{n-4}(\mathcal{A}_\infty; \mathbf{Z})$  where  $\theta_v^{n-4}$  denotes the  $(n-4)$ -skeleton of the  $(n-3)$ -simplex  $\theta_v$ . Let  $\text{bs} : \mathcal{A}_\infty \rightarrow \text{bs } \mathcal{A}_\infty$  be the barycentric subdivision map from (2). Recall that the homotopy equivalence  $\Phi : \text{bs } \mathcal{A}_\infty(\Sigma_0^n) \rightarrow \mathcal{C}(\Sigma_0^n)$  from (3) sends a non-filling arc system to the collection of boundary curves of a regular neighborhood of the arc system. It follows that the homology class of the  $(n-4)$ -sphere  $\Phi(\text{bs } \theta_v^{n-4})$  is a  $\text{Mod}(\Sigma_0^n)$ -module generator for  $\tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$  and is non-trivial.

There are a number of properties of the  $(n-4)$ -sphere  $\Phi(\text{bs } \theta_v^{n-4})$  which are undesirable. Firstly, since  $\Phi$  is defined on the barycentric subdivision of  $\mathcal{A}_\infty$  the sphere  $\Phi(\text{bs } \theta_v^{n-4})$  has  $2^{(n-2)} - 2$  vertices which is exponential in the number of marked points. Secondly, it is a subcomplex of the barycentric subdivision of the curve complex, whereas one would like to describe the sphere as a subcomplex of the curve complex itself. In order to address these issues we will replace the sphere  $\Phi(\text{bs } \theta_v^{n-4})$  with a simpler one representing the same free homotopy class. In fact our simplified sphere  $X \subset \mathcal{C}$  described below will agree with  $\Phi(\text{bs } \theta_v^{n-4})$  setwise but will have a much simpler structure as a simplicial complex.

Let  $Y \subset \text{bs } \mathcal{A}_\infty$  be the flag complex whose vertices correspond to non-empty proper sub arc systems of the filling arc system  $v$  in Figure 4 and whose  $k$ -simplices correspond to increasing flags of such sub arc systems of length  $k+1$ . Notice that  $Y = \text{bs } \theta_v^{n-4}$  is the homeomorphic image of  $\theta_v^{n-4}$  under the barycentric subdivision map  $\text{bs} : \mathcal{A}_\infty \rightarrow \text{bs } \mathcal{A}_\infty$  from (2) and hence topologically a sphere of dimension  $n-4$ . Let  $i : Y \rightarrow \text{bs } \mathcal{A}_\infty$  be the inclusion map and let  $\varphi = \Phi \circ i$  where  $\Phi : \text{bs } \mathcal{A}_\infty \rightarrow \mathcal{C}$  is as in (3). Now we will apply the techniques of §2.2 to simplify  $\varphi$ .

Let  $Y^0$  be the 0-skeleton of  $Y$ . Define the function  $\varphi_1^0 : Y^0 \rightarrow \mathcal{C}$  as follows. Let  $\alpha \subsetneq v$  be a vertex of  $Y$ . Set  $\alpha_* \subset \alpha$  to be the left-most connected component of the arc system  $\alpha$  and  $\varphi_1^0(\alpha)$  to be the unique boundary curve of  $\alpha_*$ . Notice that  $\varphi_1^0(\alpha) \in \text{supp } \varphi(\alpha)$ . Hence by Corollary 15 the function  $\varphi_1^0$  is interpolable. Let

$$\varphi_1 : Y \rightarrow \mathcal{C}$$

be its linear interpolation which by Corollary 15 is homotopic to  $\varphi$ .

Define the subcomplex  $X = X(\Sigma_0^n) \subset \mathcal{C}(\Sigma_0^n)$  as follows. For each non-empty interval  $J = \{j, j+1, \dots, m\} \subsetneq \{1, 2, \dots, n-2\}$  let  $x_J$  be the boundary curve of the arc system  $\{y_j, y_{j+1}, \dots, y_m\} \subset v$ . Let

$$X^0 = \{x_J \mid J \subsetneq \{1, 2, \dots, n-2\} \text{ is a non-empty interval}\}$$

Let  $X(\Sigma_0^n) = \text{span } X^0 \subset \mathcal{C}(\Sigma_0^n)$ .

**Lemma 23.** *The complex  $X$  and the simplicial complex  $D_{n-1}$  (see Definition 8) are isomorphic as simplicial complexes.*

*Proof.* Let  $\Lambda_{n-2}$  be the graph in Figure 1. If  $J, J' \subsetneq \{1, 2, \dots, n-2\}$  are two non-empty intervals then the boundary curves  $x_J$  and  $x_{J'}$  form a curve system exactly when  $J$  and  $J'$  are compatible tubings of the graph  $\Lambda_{n-2}$ . We caution the reader not to confuse the graph  $\Lambda_{n-2}$  which has a *vertex* for each arc in the arc system  $v$  and the underlying graph  $G(v)$  for the arc system  $v$  from §3 in which each arc of  $v$  is an *edge*.

The linear interpolation  $f : D_{n-1} \rightarrow X$  of the map sending the tubing  $J \in D_{n-1}$  of  $\Lambda_{n-2}$  to the curve  $x_J \in X$  is a simplicial map by Remark 16 and its inverse is the linear interpolation of the function sending the curve  $x_J$  to the maximal subinterval of  $J' \subset \{1, 2, \dots, n-2\}$  such that the arc system  $\{y_j \mid j \in J'\}$  is entirely contained in the component of  $\Sigma_0^n - x_J$  which does not contain the marked point  $p_v$ .  $\square$

**Lemma 24.** *The map  $\varphi_1$  is a simplicial map,  $\varphi_1(Y) = X$  and  $\varphi_1 : Y \rightarrow X$  is a homotopy equivalence.*

*Proof.* The map  $\varphi_1$  is a simplicial map onto its image by Remark 16 and the observation that for each vertex  $\alpha$  of  $Y$  we have

$$|\text{supp } \varphi_1(\alpha)| = |\{\varphi_1^0(\alpha)\}| = 1.$$

If  $\alpha$  is a vertex of  $Y$  then  $\varphi_1(\alpha)$  is a vertex of  $X$ . The map  $\varphi_1$  is simplicial so  $\varphi_1(Y) \subset \text{span } X^0 = X$ .

Observe that  $Y$  is topologically an  $(n-4)$ -sphere and by Lemma 23 that  $X$  is topologically an  $(n-4)$ -sphere. Hence the map  $\varphi_1 : Y \rightarrow X$  will be a homotopy equivalence if and only if it is a degree  $\pm 1$  map. Let  $J = \{1, 2, \dots, n-3\}$  then  $\varphi_1^{-1}(x_J) = \{\alpha\}$  where  $\alpha = \{y_1, y_2, \dots, y_{n-3}\}$ . Thus the degree of  $\varphi_1$  is equal to the degree of the induced map  $\varphi_1^* : \text{Lk}(\alpha, Y) \rightarrow \text{Lk}(x_J, X)$  where  $\text{Lk}(w, W)$  denotes the link of the vertex  $w$  in the complex  $W$ . The map  $\varphi_1^*$  is exactly the map  $\varphi_1$  for the surface with one less puncture. For the base case surface  $\Sigma_0^4$  the map  $\varphi_1 : Y \rightarrow X$  is a homeomorphism of 0-spheres which has degree  $\pm 1$ . Hence, inductively, for all  $n \geq 4$  the map  $\varphi_1 : Y \rightarrow X$  has degree  $\pm 1$  and is therefore a homotopy equivalence. It follows that  $\varphi_1 : Y \rightarrow X$  is surjective.  $\square$

We have arrived at the following simplification of our  $\text{Mod}(\Sigma_0^n)$ -module generator for  $\tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$ .

**Proposition 25.** *Assume  $n \geq 4$ . Let  $X = X(\Sigma_0^n) \subset \mathcal{C}(\Sigma_0^n)$  be the subcomplex of all simplices of  $\mathcal{C}$  whose vertices are boundary curves of connected sub arc systems of the arc system  $v$  in Figure 4. Then as a simplicial complex  $X$  is the dual  $D_{n-1}$  of the boundary of the associahedron  $K_{n-1}$  (see Definition 8) and with a proper choice of orientation the class  $[X] \in \tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$  is the same as that of  $\theta_v$  from Proposition 21 and hence is a non-trivial generator for  $\tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$  as a  $\text{Mod}(\Sigma_0^n)$ -module.*

*Proof.* Lemma 23 shows that  $X$  and  $D_{n-1}$  are isomorphic simplicial complexes. The map  $\varphi : Y \rightarrow \mathcal{C}$  above represents a  $\text{Mod}(\Sigma_0^n)$ -module generator for  $\tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$ . As we noted  $\varphi_1 : Y \rightarrow \mathcal{C}$  is homotopic to  $\varphi$  and thus represents the same class. Let  $i_X : X \rightarrow \mathcal{C}$  be the inclusion map. By Lemma 24 the map  $\varphi_1 : Y \rightarrow X$  has a homotopy inverse  $h : X \rightarrow Y$ . Thus  $i_X$  is homotopic to  $\varphi_1 \circ h$  which represents the same class as  $\varphi$  in  $\tilde{H}_{n-4}(\mathcal{C}; \mathbf{Z})$ .  $\square$

### 3.2 The proof of Theorem 1

In [1] Aramayona and Leininger show that certain finite subcomplexes  $\mathfrak{W} \subset \mathcal{C}$  of the curve complex are *rigid sets*. That is, any injective simplicial map  $f : \mathfrak{W} \rightarrow \mathcal{C}$  extends to a simplicial injection  $F : \mathcal{C} \rightarrow \mathcal{C}$  and hence is induced by a mapping class.

In the case that  $\Sigma = \Sigma_0^n$  their finite rigid set  $\mathfrak{X} = \mathfrak{X}(\Sigma_0^n)$  is a topological sphere of dimension  $n-4$  and they ask [1, Question 2] if the class  $[\mathfrak{X}] \in \tilde{H}_{n-4}(\mathcal{C}(\Sigma_0^n))$  is non-trivial. The construction of the set  $\mathfrak{X} \subset \mathcal{C}(\Sigma_0^n)$  is as follows: Let  $G_n$  be a regular  $n$ -gon and let  $\Sigma_0^n$  be the double of  $G_n$  along its boundary with each vertex of  $G_n$  giving a marked point. For each line segment  $w$  joining two nonadjacent sides of  $G_n$  the double  $c_w$  of  $w$  will be an essential curve in  $\Sigma_0^n$ . Let  $\mathfrak{X}^0$  be the set of all  $\binom{n}{2} - n$  essential curves in  $\Sigma_0^n$  which may be constructed in this manner. Define  $\mathfrak{X} = \text{span } \mathfrak{X}^0$  to be the union of all simplices of  $\mathcal{C}(\Sigma_0^n)$  whose vertices all lie in the set  $\mathfrak{X}^0$ .

The reader is referred to the Introduction to this paper for the statement of Theorem 1. Using Proposition 25, we are now ready to give the proof.

*Proof of Theorem 1.* Assume that  $n \geq 5$ . Arrange the arc system in Figure 4 so that it is entirely contained in the image of the boundary of the  $n$ -gon  $G_n$  in  $\Sigma_0^n$ . Arrange the arcs in Figure 4 so that they coincide with all

but two adjacent edges of  $G_n$ . Let  $p_v$  be the marked point of  $\Sigma_0^n$  which is not incident with any arcs. If  $w$  is a line segment joining two nonadjacent sides of  $G_n$  then its double  $c_w$  will be the boundary curve for the maximal connected sub arc system of the arc system in Figure 4 which is disjoint from  $c_w$  and which is in the component of  $\Sigma_0^n - c_w$  which does not contain the marked point  $p_v$ . This pairing identifies the vertices of  $\mathfrak{X}$  and  $X$ . Since both subcomplexes are the full subcomplexes of  $\mathcal{C}(\Sigma_0^n)$  spanned by their vertices the subcomplexes must coincide.  $\square$

We address the case of the surface  $\Sigma_0^4$  in §4.3 below.

## 4 Essential spheres and finite rigid sets for the surfaces $\Sigma_g^0$ and $\Sigma_g^1$

We now turn our attention to the surface  $\Sigma_g^n$  where  $g \geq 1$  and  $n \in \{0, 1\}$ . Our main interest is in the case  $n = 0$ , but in the process of investigating that we will need to study the case  $n = 1$ . To get started, we review what was done in [2] to identify the homologically non-trivial sphere in  $\mathcal{C}(\Sigma_g^n)$  described there. After that, we use the techniques of §2.2 to radically simplify this sphere. Finally in §4.2 we compare this simplified sphere to the finite rigid sets for  $\Sigma_g^n$  given in [1].

### 4.1 An essential sphere in the curve complex of a surface of genus $g > 0$

Consider the surface of genus  $g$  with 1 marked point depicted in Figure 5. Let  $\theta_\zeta \subset \mathcal{A}(\Sigma_g^1)$  be the  $(2g - 1)$ -simplex corresponding to the filling arc system  $\zeta$  given in Figure 5. In [2] it is shown that the class  $[\theta_\zeta]$  generates

$$\text{St}(\Sigma_g^1) \cong \frac{C_{2g-1}(\mathcal{A})}{C_{2g-1}(\mathcal{A}_\infty) \oplus \partial C_{2g}(\mathcal{A})}$$

as a  $\text{Mod}(\Sigma_g^1)$ -module.

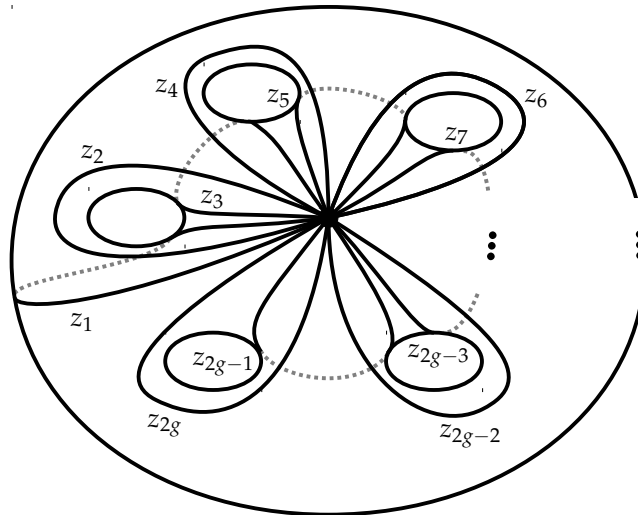


Figure 5: A filling arc system  $\zeta = \{z_1, z_2, \dots, z_{2g}\}$  with  $2g$  arcs in the surface  $\Sigma_g^1$  whose class generates  $\text{St}(\Sigma_g^1)$  as a  $\text{Mod}(\Sigma_g^1)$ -module.

As was the case in Corollary 22 this class in  $\text{St}(\Sigma_g^1)$  must be non-trivial.

Now consider the surface  $\Sigma_g^0$  without marked points. Here we have no arc complex, but by a result of Harer following from more general work by Kent, Leininger and Schleimer [9, Corollary 1.1] the map

$$\Psi : \mathcal{C}(\Sigma_g^1) \rightarrow \mathcal{C}(\Sigma_g^0) \quad (8)$$

which “forgets” the marked point is a homotopy equivalence. Hence we may identify  $\text{St}(\Sigma_g^1)$  and  $\text{St}(\Sigma_g^0)$  via the induced map

$$\Psi_* : \tilde{\text{H}}_{2g-2}(\mathcal{C}(\Sigma_g^1); \mathbf{Z}) \rightarrow \tilde{\text{H}}_{2g-2}(\mathcal{C}(\Sigma_g^0); \mathbf{Z})$$

for all  $g \geq 1$ .

To continue, suppose that  $g \geq 1$  and  $n = 1$ . Let  $Z \subset \text{bs } \mathcal{A}_\infty(\Sigma_g^1)$  be the flag complex whose vertices correspond to non-empty proper sub arc systems of the filling arc system  $\zeta$  in Figure 5 and whose  $k$ -simplices correspond to increasing flags of such sub arc systems of length  $k + 1$ . The simplicial complex  $Z$  is the barycentric subdivision of the boundary of a  $(2g - 1)$ -simplex and hence topologically a sphere of dimension  $2g - 2$ . Again we let  $i : Z \rightarrow \text{bs } \mathcal{A}_\infty$  be the inclusion map and set  $\varphi = \Phi \circ i$  where  $\Phi : \text{bs } \mathcal{A}_\infty \rightarrow \mathcal{C}$  is as in (3).

The *sequential components* of a sub arc system  $\alpha$  of the arc system  $\zeta$  in Figure 5 will be the maximal subsets of  $\alpha$  of the form  $\{z_j, z_{j+1}, \dots, z_k\}$  where  $1 \leq j \leq k \leq 2g$ . Note that  $\alpha$  is the disjoint union of its sequential components. If  $\alpha_1, \alpha_2 \subset \alpha$  are sequential components of  $\alpha$  then we say that  $\alpha_1$  *precedes*  $\alpha_2$  if for all  $z_j \in \alpha_1$  and all  $z_k \in \alpha_2$  we have  $j < k$ . Under this order we will refer to the *first, second, last, etc.* sequential component of  $\alpha$ . The order on the sequential components of a sub arc system  $\alpha \subsetneq \zeta$  should not be confused with the containment partial order on all arc systems under which the sequential components of  $\alpha$  are incomparable. A sequential component will be called *odd* if it has an odd number of elements and *even* if it has an even number of elements. A key observation is that if the arc system  $\alpha \subsetneq \zeta$  has  $m$  odd sequential components then it will have  $m + 1$  boundary curves.

This first step in the simplification of  $\varphi : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  will be a straightforward application of Corollary 15. In order to apply Corollary 15 we would like to choose a particular boundary component for a regular neighborhood of each sub arc system of the arc system in Figure 5. Define the function  $\varphi_1^0 : Z^0 \rightarrow \mathcal{C}(\Sigma_g^1)$  as follows. Let  $\alpha \in Z^0$  be a proper sub arc system of  $\zeta$ . If every sequential component of  $\alpha$  is even then there is only one boundary curve. Let  $\varphi_1^0(\alpha)$  be this curve. If  $\alpha$  has an odd sequential component then let  $\alpha^* \subset \alpha$  be the first odd sequential component of  $\alpha$ . Let  $j$  be the minimum number such that  $a_j \in \alpha^*$ . Let  $N(\alpha^*)$  be a closed regular neighborhood of  $\alpha^*$  in  $\Sigma_g^1$  and let  $\varphi_1^0(\alpha)$  be the boundary component of  $N(\alpha^*)$  which is to the right when following  $a_j$  in the clockwise direction (see Figure 6). By Corollary 15 the function  $\varphi_1^0$  extends to a linear interpolation  $\varphi_1 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  which is homotopic to  $\varphi$ .

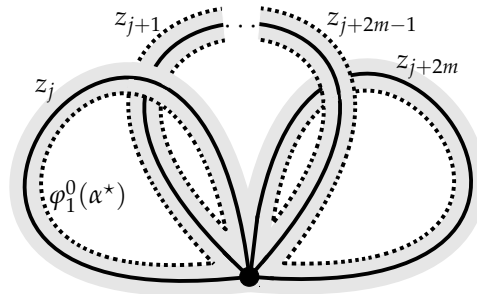


Figure 6: An odd sequential component  $\alpha^* = \{z_j, z_{j+1}, \dots, z_{j+2m}\}$ , its closed regular neighborhood  $N(\alpha^*)$  (the grey region) and one of its boundary curves  $\varphi_1^0(\alpha^*)$  (the dotted line).

We now further simplify  $\varphi_1 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  to get a map  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  whose image in  $\mathcal{C}(\Sigma_g^1)$  will involve

fewer curves. First we will describe the new map  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  and then construct an explicit homotopy between  $\varphi_1$  and  $\varphi_2$ .

For each arc system  $\alpha \in Z^0$  all of whose sequential components has an even number of arcs let  $\alpha_* \subset \alpha$  the first sequential component. Let  $c$  be the unique boundary curve of a closed regular neighborhood  $N(\alpha)$  of  $\alpha$  and  $c_*$  be the unique boundary component of a closed regular neighborhood  $N(\alpha_*)$  for  $\alpha_*$  chosen so that  $N(\alpha_*) \subset \text{int } N(\alpha)$ . Lastly, modify the boundary curve  $c_*$  in  $\Sigma_g^1$  by pulling it past the marked point to get  $c'_*$  (see Figures 7 and 8).

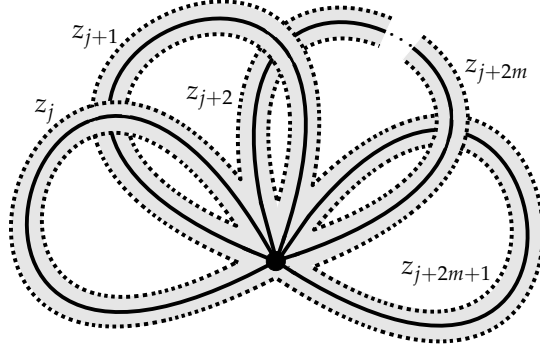


Figure 7: An even sequential component  $\alpha_* = \{z_j, z_{j+1}, \dots, z_{j+2m+1}\}$ , its regular neighborhood  $N(\alpha_*)$  (the grey region) and its single boundary curve  $c_*$  (the dotted line).

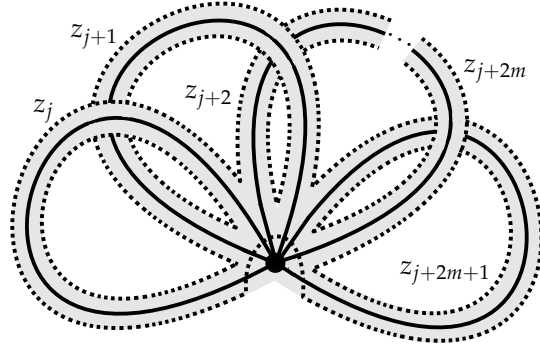


Figure 8: An even sequential component  $\alpha_* = \{z_j, z_{j+1}, \dots, z_{j+2m+1}\}$  and its modified boundary curve  $c'_*$  (the dotted line).

Let  $\varphi_2^0 : Z^0 \rightarrow \mathcal{C}(\Sigma_g^1)$  be the function

$$\varphi_2^0(\alpha) = \begin{cases} \varphi_1(\alpha), & \alpha \text{ has an odd sequential component} \\ c'_*, & \alpha \text{ has no odd sequential components} \end{cases}.$$

**Lemma 26.** *If  $\alpha \subset \beta$  are two proper sub arc systems of the arc system  $\zeta$  in Figure 5 then  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\}$  is a curve system.*

*Proof.* Let  $\alpha \subset \beta$  are two proper sub arc systems of the arc system  $\zeta$  in Figure 5.

Suppose  $\beta$  has an odd sequential component. Choose closed regular neighborhoods  $N(\alpha)$  and  $N(\beta)$  of the arc systems  $\alpha$  and  $\beta$  respectively such that  $N(\alpha) \subset \text{int } N(\beta)$ . We may arrange that the curve  $\varphi_2^0(\alpha)$  is contained in  $N(\alpha)$  and  $\varphi_2^0(\beta) = \varphi_1(\beta)$  is a boundary curve of  $N(\beta)$ . Hence they must be disjoint curves.



Suppose  $\beta$  has no odd sequential components and  $\alpha$  has an odd sequential component. Let  $\beta_* \subset \beta$  be the first (even) sequential component of  $\beta$  and  $\alpha^*$  be the first odd sequential component of  $\alpha$ . Either  $\alpha^* \subset \beta_*$  or the arc system  $\alpha^* \cup \beta_*$  has two distinct sequential components. If  $\alpha^* \subset \beta_*$  then choose closed regular neighborhoods  $N(\alpha^*)$  and  $N(\beta_*)$  such that  $N(\alpha^*) \subset \text{int } N(\beta_*)$ . Let  $c'_*$  be the modified boundary curve of  $N(\beta_*)$  as in Figure 8 chosen so that its intersection with  $N(\alpha^*)$  is in a sufficiently small neighborhood of the marked point that  $c'_*$  is disjoint from the boundary curve  $\varphi_1(\alpha^*)$  of  $N(\alpha^*)$ . Then  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\} = \{c'_*, \varphi_1(\alpha^*)\}$  is a curve system. If  $\alpha^* \cup \beta_*$  has two distinct sequential components then we can choose closed regular neighborhoods  $N(\alpha^*)$  and  $N(\beta_*)$  and a closed subset  $N' \subset N(\beta_*)$  such that  $N(\alpha^*)$  and  $N'$  are disjoint,  $\varphi_2^0(\alpha)$  is a boundary curve of  $N(\alpha^*)$  and  $\varphi_2^0(\beta)$  is the boundary curve of  $N'$ . Thus  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\}$  is a curve system.

Suppose neither  $\beta$  nor  $\alpha$  has an odd sequential component. Let  $\beta_* \subset \beta$  and  $\alpha_* \subset \alpha$  be the first (even) sequential components of  $\beta$  and  $\alpha$  respectively. Either  $\alpha_* \subset \beta_*$  or the arc system  $\alpha_* \cup \beta_*$  has two distinct sequential components. If  $\alpha_* \subset \beta_*$  then choose closed regular neighborhoods  $N(\alpha_*)$  and  $N(\beta_*)$  such that  $N(\alpha_*) \subset \text{int } N(\beta_*)$ . Let  $\varphi_2^0(\beta)$  be the modified boundary curve of  $N(\beta_*)$  as in Figure 8 chosen so that it intersects with the boundary curve of  $N(\alpha_*)$  exactly twice and choose  $\varphi_2^0(\alpha)$  to be the boundary curve of  $N(\alpha_*)$  surgered to follow along the segment  $\varphi_2^0(\beta) \cap N(\alpha_*)$ . Then after an arbitrarily small perturbation we may arrange that  $\varphi_2^0(\alpha)$  and  $\varphi_2^0(\beta)$  are disjoint. Hence  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\}$  is a curve system. If  $\alpha_* \cup \beta_*$  has two distinct sequential components then we can choose closed regular neighborhoods  $N(\alpha_*)$  and  $N(\beta_*)$  and closed subsets  $N'(\alpha_*) \subset N(\alpha_*)$  and  $N'(\beta_*) \subset N(\beta_*)$  such that  $N'(\alpha_*)$  and  $N'(\beta_*)$  are disjoint,  $\varphi_2^0(\alpha)$  is the boundary curve of  $N'(\alpha_*)$  and  $\varphi_2^0(\beta)$  is the boundary curve of  $N'(\beta_*)$ . Thus  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\}$  is a curve system.  $\square$

Recall that a set of curves forms a curve system if and only if every subset of order two is a curve system. Let  $\sigma$  be a simplex of  $Z$  with vertex set  $\sigma^0$ . Then every subset of  $\varphi_2^0(\sigma^0)$  of order two is of the form  $\{\varphi_2^0(\alpha), \varphi_2^0(\beta)\}$  for some arc systems  $\alpha, \beta$  with  $\alpha \subsetneq \beta \subsetneq \zeta$ . By Lemma 26 all such sets of order two are curve systems. It follows that  $\varphi_2^0(\sigma^0)$  is a curve system and that  $\varphi_2^0$  is interpolable. Let

$$\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$$

be its linear interpolation.

In Lemma 28 below we will construct an explicit homotopy between  $\varphi_1 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  and  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$ . In order to do this we will need Lemma 27 below.

**Lemma 27.** *If  $\alpha \subset \beta$  are two proper sub arc systems of the arc system  $\zeta$  in Figure 5 then  $\{\varphi_2(\alpha), \varphi_1(\beta)\}$  is a curve system.*

*Proof.* Suppose  $\alpha$  has an odd sequential component. Then  $\varphi_2(\alpha) = \varphi_1(\alpha)$  and  $\varphi_1(\beta)$  are both boundary curves for the arc systems  $\alpha$  and  $\beta$  and therefore form a curve system.

Suppose  $\alpha$  has no odd sequential components. Let  $\alpha_* \subset \alpha$  be its first sequential component and choose closed regular neighborhoods  $N(\alpha_*)$  and  $N(\beta)$  of the arc systems  $\alpha_*$  and  $\beta$  so that  $N(\alpha_*) \subset N(\beta)$ . If  $c'_*$  is the modified boundary curve of  $N(\alpha_*)$  as in Figure 8 then it is clearly disjoint from all boundary curves of  $N(\beta)$ . The curve  $\varphi_1(\beta)$  is one such boundary curve of  $N(\beta)$ . Hence  $\varphi_2(\alpha) = c'_*$  and  $\varphi_1(\beta)$  are disjoint curves.  $\square$

Now we have the tools to construct an explicit homotopy equivalence between  $\varphi_1$  and  $\varphi_2$ .

**Lemma 28.** *The maps  $\varphi_1 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  and  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  are homotopic.*

*Proof.* Let  $\sigma$  be a simplex of  $Z$  with vertex set  $\sigma^0 = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$  satisfying  $\alpha_0 \subsetneq \alpha_1 \subsetneq \dots \subsetneq \alpha_k$ . Consider the simplicial complex  $P = \sigma \times I$  which is a prism triangulated by  $k + 1$  simplices of dimension  $k + 1$  with vertex sets:

$$\begin{aligned} P_0 &= \{(\alpha_0, 0), (\alpha_0, 1), (\alpha_1, 1), \dots, (\alpha_k, 1)\} \\ P_1 &= \{(\alpha_0, 0), (\alpha_1, 0), (\alpha_1, 1), \dots, (\alpha_k, 1)\} \\ &\vdots \\ P_k &= \{(\alpha_0, 0), (\alpha_1, 0), \dots, (\alpha_k, 0), (\alpha_k, 1)\} \end{aligned}$$

The 0-skeleton of  $P$  is the set  $\sigma^0 \times \{0, 1\}$ . Let  $F_\sigma^0 : \sigma^0 \times \{0, 1\} \rightarrow \mathcal{C}(\Sigma_g^1)$  be the function

$$F_\sigma^0(\alpha_i, \varepsilon) = \begin{cases} \varphi_2(\alpha_i), & \varepsilon = 0 \\ \varphi_1(\alpha_i), & \varepsilon = 1 \end{cases} .$$

We wish to show that  $F_\sigma^0$  is interpolable. Each maximal simplex of  $P = \sigma \times I$  has vertex set  $P_i$  for some  $i$ . Thus we must show that  $F_\sigma^0(P_i)$  is a curve system for each  $i$ . Recall that a set of curves is a curve system if and only if each subset of order two is a curve system. Subsets of order two of the set  $F_\sigma^0(P_i)$  are all of the form  $\{F_\sigma^0(\alpha, 0), F_\sigma^0(\beta, 0)\}$ ,  $\{F_\sigma^0(\alpha, 0), F_\sigma^0(\beta, 1)\}$ , or  $\{F_\sigma^0(\alpha, 1), F_\sigma^0(\beta, 1)\}$  where  $\alpha \subset \beta \subsetneq \zeta$ . The set  $\{F_\sigma^0(\alpha, 0), F_\sigma^0(\beta, 0)\} = \{\varphi_2(\alpha), \varphi_2(\beta)\}$  is a curve system by Lemma 26. The set  $\{F_\sigma^0(\alpha, 0), F_\sigma^0(\beta, 1)\} = \{\varphi_2(\alpha), \varphi_1(\beta)\}$  is a curve system by Lemma 27. Finally the set  $\{F_\sigma^0(\alpha, 1), F_\sigma^0(\beta, 1)\} = \{\varphi_1(\alpha), \varphi_1(\beta)\}$  is a curve system since it contains boundary curves of two closed subsurfaces  $N(\alpha)$  and  $N(\beta)$  with  $N(\alpha) \subset \text{int } N(\beta)$ . Hence  $F_\sigma^0$  is interpolable. Let  $F_\sigma : \sigma \times I \rightarrow \mathcal{C}(\Sigma_g^1)$  be its linear interpolation. By the gluing lemma these  $F_\sigma$ 's assemble to a homotopy  $F : Z \times I \rightarrow \mathcal{C}(\Sigma_g^1)$  from  $\varphi_2$  to  $\varphi_1$ .  $\square$

We now describe a subcomplex  $X(\Sigma_g^1) \subset \mathcal{C}(\Sigma_g^1)$  which will prove to be the image of the map  $\varphi_2$ . For  $1 \leq i \leq 2g$  let  $z'_i = \varphi_2(z_i)$  (see Figure 9). For each interval  $J = \{j, j + 1, \dots, m\} \subsetneq \{1, 2, \dots, 2g\}$  let  $N_J$

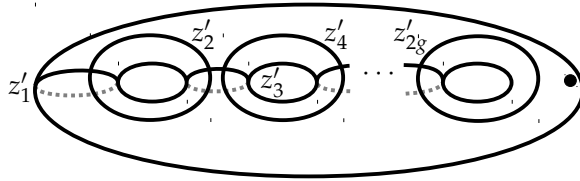


Figure 9

be a closed regular neighborhood of the set  $z'_j \cup z'_{j+1} \cup \dots \cup z'_m$ . If  $|J|$  is even then  $N_J$  has one boundary component. Let  $x_j$  be that boundary component. If  $|J|$  is odd then  $N_J$  has two boundary components. In this case let  $x_j$  be the boundary component of  $N_J$  in the “back half” (if  $j$  is even) or “top half” (if  $j$  is odd) of the surface. Let  $X^0$  be the set

$$X^0 = \{x_J \mid J \subsetneq \{1, 2, \dots, 2g\} \text{ is an interval}\} \quad (9)$$

and define  $X(\Sigma_g^1) = \text{span } X^0$  to be the subcomplex of all simplices of  $\mathcal{C}(\Sigma_g^1)$  whose vertices are in  $X^0$ . Let  $X(\Sigma_g^0) = \Psi(X(\Sigma_g^1))$  be the image of  $X(\Sigma_g^1)$  under the map  $\Psi : \mathcal{C}(\Sigma_g^1) \rightarrow \mathcal{C}(\Sigma_g^0)$  from (8) above which “forgets” the marked point.

**Lemma 29.** *Assume  $g \geq 1$  and  $n \in \{0, 1\}$ . The simplicial complex  $X(\Sigma_g^n)$  is isomorphic as a simplicial complex to  $D_{2g+1}$  (see Definition 8).*

*Proof.* For intervals  $J_1, J_2 \subsetneq \{1, 2, \dots, 2g\}$  the curves  $x_{J_1}$  and  $x_{J_2}$  are disjoint precisely when the tubes  $J_1$  and  $J_2$  for the graph  $\Lambda_{2g}$  in Figure 1 are compatible. Let  $D_{2g+1}^0$  be the 0-skeleton of the dual  $D_{2g+1}$  of the boundary of the associahedron  $K_{2g+1}$ . The bijection  $f^0 : X^0 \rightarrow D_{2g+1}^0$  given by  $f^0(x_J) = J$  and its inverse are interpolable maps showing that  $X(\Sigma_g^1)$  and  $D_{2g+1}$  are isomorphic as simplicial complexes. The map  $\Psi|_{X(\Sigma_g^1)} : X(\Sigma_g^1) \rightarrow X(\Sigma_g^0)$  is an isomorphism of simplicial complexes.  $\square$

**Lemma 30.** *Assume  $g \geq 1$  and  $n \in \{0, 1\}$ . The maps  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  and  $\Psi \circ \varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^0)$  are simplicial maps. The images of  $\varphi_2$  and  $\Psi \circ \varphi_2$  are  $X(\Sigma_g^1)$  and  $X(\Sigma_g^0)$  respectively. Furthermore  $\varphi_2 : Z \rightarrow X(\Sigma_g^1)$  and  $\Psi \circ \varphi_2 : Z \rightarrow X(\Sigma_g^0)$  are homotopy equivalences.*

*Proof.* The map  $\varphi_2 : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  is a simplicial map onto its image by Remark 16 and the observation that for each vertex  $\alpha$  of  $Z$  we have

$$|\text{supp } \varphi_2(\alpha)| = 1$$

The map  $\Psi$  a simplicial map so the composition  $\Psi \circ \varphi_2$  is simplicial.

The curves in the set  $X^0$  given (9) are precisely the curves in the image of  $\varphi_2$ . Thus if  $\alpha$  is a vertex of  $Z$  then  $\varphi_2(\alpha)$  is a vertex of  $X(\Sigma_g^1)$  and  $\Psi \circ \varphi_2(\alpha)$  is a vertex of  $X(\Sigma_g^0)$ . The maps  $\varphi_2$  and  $\Psi \circ \varphi_2$  are simplicial so  $\varphi_2(Z) \subset X(\Sigma_g^1)$  and  $\Psi \circ \varphi_2(Z) \subset X(\Sigma_g^0)$ .

Observe that  $Z$  is topologically a  $(2g - 2)$ -sphere and by Lemma 29 that  $X(\Sigma_g^1)$  is topologically an  $(2g - 2)$ -sphere. Hence the map  $\varphi_2 : Z \rightarrow X(\Sigma_g^1)$  will be a homotopy equivalence if and only if it is a degree  $\pm 1$  map. Let  $J = \{1, 2, \dots, 2g - 1\}$  then  $\varphi_2^{-1}(x_J) = \{\alpha\}$  where  $\alpha = \{z_1, z_2, \dots, z_{2g-1}\}$ . Thus the degree of  $\varphi_2$  is equal to the degree of the induced map  $\varphi_2^* : \text{Lk}(\alpha, Z) \rightarrow \text{Lk}(x_J, X(\Sigma_g^1))$  where  $\text{Lk}(w, W)$  denotes the link of the vertex  $w$  in the complex  $W$ . Applying the link argument once more we see that the degree of  $\varphi_2^*$  is equal to the degree of the map  $\varphi_2$  for the surface with genus  $g - 1$ . For the base case surface  $\Sigma_1^1$  the map  $\varphi_2 : Z \rightarrow X(\Sigma_1^1)$  is a homeomorphism of 0-spheres which has degree  $\pm 1$ . Hence, inductively, for all  $g \geq 1$  the map  $\varphi_2 : Z \rightarrow X(\Sigma_g^1)$  has degree  $\pm 1$  and is therefore a homotopy equivalence. It follows that  $\varphi_2 : Z \rightarrow X(\Sigma_g^1)$  is surjective.

The restriction  $\Psi|_{X(\Sigma_g^1)} : X(\Sigma_g^1) \rightarrow X(\Sigma_g^0)$  is an isomorphism of simplicial complexes and hence a degree  $\pm 1$  map. It follows that  $\Psi \circ \varphi_2 : Z \rightarrow X(\Sigma_g^0)$  has degree  $\pm 1$ . It is therefore a surjective simplicial homotopy equivalence.  $\square$

**Proposition 31.** *Let  $g \geq 1$  and  $n \in \{0, 1\}$ . Let  $X^0$  be the set of essential curves in  $\Sigma_g^n$  given in (9) and  $X(\Sigma_g^n) \subset \mathcal{C}(\Sigma_g^n)$  be the full subcomplex of  $\mathcal{C}(\Sigma_g^n)$  generated by  $X^0$ . Then as a simplicial complex  $X(\Sigma_g^n)$  is isomorphic to the dual  $D_{2g+1}$  of the boundary of the associahedron  $K_{2g+1}$  (see Definition 8). Furthermore  $X(\Sigma_g^n)$  is topologically a sphere of dimension  $2g - 2$  and, choosing an orientation, the class  $[X(\Sigma_g^n)] \in \tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g^n); \mathbf{Z})$  is non-trivial and generates  $\tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g^n); \mathbf{Z})$  as a  $\text{Mod}(\Sigma_g^n)$ -module.*

*Proof.* Lemma 29 shows that  $X(\Sigma_g^1)$ ,  $X(\Sigma_g^0)$  and  $D_{2g+1}$  are isomorphic simplicial complexes. The map  $\varphi : Z \rightarrow \mathcal{C}(\Sigma_g^1)$  (respectively  $\Psi \circ \varphi : Z \rightarrow \mathcal{C}(\Sigma_g^0)$ ) above represents a  $\text{Mod}(\Sigma_g^1)$ -module (respectively  $\text{Mod}(\Sigma_g^0)$ -module) generator for  $\tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g^1); \mathbf{Z})$  (respectively  $\tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g^0); \mathbf{Z})$ ). As we noted  $\varphi_2$  is homotopic to  $\varphi$  and  $\Psi \circ \varphi_2$  is homotopic to  $\Psi \circ \varphi$ . Let  $i_{X(\Sigma_g^1)} : X(\Sigma_g^1) \rightarrow \mathcal{C}(\Sigma_g^1)$  and  $i_{X(\Sigma_g^0)} : X(\Sigma_g^0) \rightarrow \mathcal{C}(\Sigma_g^0)$  be the inclusion maps. By Lemma 30 the maps  $\varphi_2 : Z \rightarrow X(\Sigma_g^1)$  and  $\Psi \circ \varphi_2 : Z \rightarrow X(\Sigma_g^0)$  have a homotopy inverses  $h_1 : X(\Sigma_g^1) \rightarrow Z$  and  $h_0 : X(\Sigma_g^0) \rightarrow Z$  respectively. Thus  $i_{X(\Sigma_g^1)}$  and  $i_{X(\Sigma_g^0)}$  are homotopic to  $\varphi_2 \circ h_1$  and  $\Psi \circ \varphi_2 \circ h_0$  respectively which represent the same classes as  $\varphi$  in  $\tilde{H}_{n-4}(\mathcal{C}(\Sigma_g^1); \mathbf{Z})$  and  $\Psi \circ \varphi$  in  $\tilde{H}_{n-4}(\mathcal{C}(\Sigma_g^0); \mathbf{Z})$  respectively.  $\square$

## 4.2 The proof of Theorem 2

Again let  $g \geq 1$  and  $n \in \{0, 1\}$ . In [1] a finite rigid set  $\mathfrak{X}(\Sigma_g^n) \subset \mathcal{C}(\Sigma_g^n)$  is given. Our essential sphere  $X(\Sigma_g^n)$  is a subset of  $\mathfrak{X}(\Sigma_g^n)$ . Given the surprising coincidence of our essential sphere and the finite rigid set of Aramayona and Leininger for the surface  $\Sigma_0^n$  one might be tempted to conjecture that a subcomplex of the curve complex is essential if and only if it is finitely rigid. As suggested by Theorem 2 this is not the case. See §1 of this paper for the statement of Theorem 2.

*Proof of Theorem 2.* Suppose that  $g \geq 3$  and  $n \in \{0, 1\}$ . Let  $X^0$  be the set of curves described in (9). Let  $N_{\{1,2,3\}}$  be a closed regular neighborhood of  $z'_1 \cup z'_2 \cup z'_3$  where the curves  $z'_1, z'_2, z'_3$  are as in Figure 9 above. Then  $N_{\{1,2,3\}}$  has two boundary components. One boundary component  $x_{\{1,2,3\}}$  of  $N_{\{1,2,3\}}$  is in  $X^0$  the other  $x'_{\{1,2,3\}}$  is not. Modify  $x'_{\{1,2,3\}}$  as in Figure 8 by pulling it passed the marked point to get  $x''_{\{1,2,3\}}$ . Let  $f^0 : X^0 \rightarrow \mathcal{C}(\Sigma_g^n)$  be the function

$$f^0(y) = \begin{cases} y, & y \neq x_{\{1,2,3\}} \\ x''_{\{1,2,3\}}, & y = x_{\{1,2,3\}} \end{cases}.$$

Then  $f^0$  is interpolable with continuous linear interpolation  $f : X \rightarrow \mathcal{C}(\Sigma_g^n)$ . The map  $f$  is simplicial since  $|\text{supp } f(x)| = 1$  for all  $x \in X^0$ . The map  $f$  is an injection since if  $x, y \in X^0$  are disjoint curves then  $f(x)$  and  $f(y)$  are disjoint curves.

We claim that  $f$  is not induced by an extended mapping class  $h \in \text{Mod}^\pm(\Sigma_g^n)$ . Note that  $f$  is the identity on the chain curves  $z'_1, z'_2, \dots, z'_{2g}$ . Suppose that the mapping class  $h \in \text{Mod}^\pm(\Sigma_g^n)$  extends  $f$ . Then without loss of generality we may represent the class of  $h$  by a map which induces an isometry on the regular  $(8g-4)$ -gon  $G_{8g-4}$  one gets by cutting along all of the chain curves. There are 4 possibilities for  $h$  generated by the orientation reversing reflection  $\varepsilon$  about the plane through the even curves  $z'_2, z'_4, \dots, z'_{2g}$  and the orientation reversing reflection  $\rho$  of the surface about the plane through the odd curves  $z'_1, z'_3, \dots, z'_{2g-1}$ . Since  $f$  takes the curve  $x_{\{2,3,4\}}$  to itself,  $h$  must be in the subgroup generated by  $\rho$ . Since  $f(x_{\{1,2,3\}}) = x''_{\{1,2,3\}}$  we must have  $f = \rho$ . However  $f(x_{\{3,4,5\}}) = x_{\{3,4,5\}}$  so  $f \neq \rho$ . Thus we have a contradiction.

If  $g = 2$  and  $n = 1$  then  $X \subsetneq \mathfrak{X}$ . Again, we claim that  $X$  is not a finite rigid set. Let  $X^0$  be the set of curves described in (9). Let  $N_{\{1,2\}}$  be a closed regular neighborhood of  $z'_1 \cup z'_2$  where the curves  $z'_1$  and  $z'_2$  are as in Figure 9 above. Then  $N_{\{1,2\}}$  has a single boundary component  $x_{\{1,2\}}$ . Modify  $x_{\{1,2\}}$  reversing the procedure in Figure 8 by pulling it passed the marked point to get  $x''_{\{1,2\}}$ . Let  $f^0 : X^0 \rightarrow \mathcal{C}(\Sigma_g^n)$  be the function

$$f^0(y) = \begin{cases} y, & y \neq x_{\{1,2\}} \\ x''_{\{1,2\}}, & y = x_{\{1,2\}} \end{cases}.$$

Then  $f^0$  is interpolable with continuous linear interpolation  $f : X \rightarrow \mathcal{C}(\Sigma_g^n)$ . The map  $f$  is simplicial since  $|\text{supp } f(x)| = 1$  for all  $x \in X^0$ . The map  $f$  is an injection since if  $x, y \in X^0$  are disjoint curves then  $f(x)$  and  $f(y)$  are disjoint curves.

We claim that  $f$  is not induced by an extended mapping class  $h \in \text{Mod}^\pm(\Sigma_g^n)$ . Note that  $f$  is the identity on the chain curves  $z'_1, z'_2, z'_3, z'_4$ . Suppose that the mapping class  $h \in \text{Mod}^\pm(\Sigma_g^n)$  extends  $f$ . Then without loss of generality we may represent the class of  $h$  by a map which induces an isometry on the regular 12-gon  $G_{12}$  one gets by cutting along all of the chain curves. There are 4 possibilities for  $h$  generated by the orientation reversing reflection  $\varepsilon$  about the plane through the even curves  $z'_2, z'_4$  and the orientation reversing reflection  $\rho$  of the surface about the plane through the odd curves  $z'_1, z'_3$ . All of these 4 maps fix  $X$  pointwise and therefore cannot induce  $f$ .  $\square$

### 4.3 Theorems 1 and 2 for sporadic surfaces of low complexity

We address the “missing cases” in Theorems 1 and 2. The case of  $\Sigma_2^1$  was included in the statement of Theorem 2, but  $\Sigma_2^0$  was omitted. Indeed, by direct computation, we see that  $X(\Sigma_2^0) = \mathfrak{X}(\Sigma_2^0)$  and hence  $X(\Sigma_2^0)$  is a finite rigid set.

Recall that with our nonstandard definition the curve complexes  $\mathcal{C}(\Sigma_0^4)$ ,  $\mathcal{C}(\Sigma_1^0)$  and  $\mathcal{C}(\Sigma_1^1)$  are disconnected countable discrete sets. Clearly these complexes have no finite rigid subsets. In particular our 0-spheres  $X(\Sigma_0^4)$ ,  $X(\Sigma_1^0)$  and  $X(\Sigma_1^1)$  cannot be finite rigid subsets of their respective curve complexes.

However, in these three sporadic cases if  $\mathcal{C}'$  is the standard 2-dimensional curve complex with 0-skeleton given by the set of essential simple closed curves and an  $n$ -simplex for each set of  $n + 1$  curves with minimal pairwise intersection number, then the 1-dimensional subcomplex  $X'$  of  $\mathcal{C}'$  spanned by  $X$  is a finite rigid set. The finite rigid set  $\mathfrak{X}$  of Aramayona and Leininger [1, §4] is a maximal 2-simplex and we may take our finite rigid set  $X'$  to be a proper subcomplex of  $\mathfrak{X}$ . In this case  $X'$  is not a sphere, but a cell of dimension one.

## References

- [1] J. Aramayona and C. Leininger, Finite rigid sets in curve complexes, *J. Topol. Anal.* **5** (2013), no. 2, 183–203.
- [2] N. Broaddus, Homology of the curve complex and the Steinberg module of the mapping class group, *Duke Math. J.* **161** (2012), no. 10, 1943–1969. MR2954621
- [3] K. S. Brown, *Cohomology of groups*, Springer, New York, 1982.
- [4] M. P. Carr and S. L. Devadoss, Coxeter complexes and graph-associahedra, *Topology Appl.* **153** (2006), no. 12, 2155–2168.
- [5] J. L. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, *Invent. Math.* **84** (1986), no. 1, 157–176.
- [6] W. J. Harvey, Boundary structure of the modular group, in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, 245–251, *Ann. of Math. Stud.*, 97, Princeton Univ. Press, Princeton, N.J.
- [7] A. Hatcher, On triangulations of surfaces, *Topology Appl.* **40** (1991), no. 2, 189–194. Updated version available at <http://www.math.cornell.edu/~hatcher/Papers/TriangSurf.pdf>
- [8] N. V. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices* **1997**, no. 14, 651–666. MR1460387
- [9] R. P. Kent, IV, C. J. Leininger and S. Schleimer, Trees and mapping class groups, *J. Reine Angew. Math.* **637** (2009), 1–21. MR2599078 (2011a:57002)
- [10] F. Luo, Automorphisms of the complex of curves, *Topology.* **39** (2000), 283–298.

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