ON MINIMAL HEEGAARD SPLITTINGS

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1. INTRODUCTION

In 1936 J. H. C. Whitehead developed an algorithm for transforming a set of words in a free group to a new set which has minimum length among all systems which are equivalent to the given system under automorphisms of the free group (see [7] and also [2]). Whitehead's stated motivation for studying this question was a related (but different) topological question: if a Heegaard diagram for a 3-manifold M does not have minimal genus, is there an algorithm for systematically reducing the genus and then further modifying the diagram to some "simplest" possible diagram for that M? (See section 4 of [7].) Such an algorithm would in certain cases solve the homeomorphism problem for 3-manifolds. The results of Whitehead were later sharpened by Zieschang [8], with further positive contributions by Waldhausen [5], [6].

The question which was asked by Whitehead is still open, however interest continues in this particular approach because of its clear similarity to many solved problems. The current status of the problem of simplifying Heegaard diagrams for 3-manifolds was reviewed recently in an article by Waldhausen [6], which contained in particular various suggestions for approaches to the original question. The purpose of this note is to give two examples which illuminate the difficulties in Waldhausen's suggestions.

As in [6] a Heegaard-diagram for a connected, closed, orientable 3-manifold is denoted by (M, F; v, w), where F is a connected, closed, orientable 2-manifold embedded in M, such that the closure of the two components of $M - F$ consists of two handlebodies V and W, and where v, w are complete systems of meridian discs for V and W, respectively. The *complexity,* c(M, F; v, w), of the Heegaard-diagram is the cardinality of the set $\partial v \cap \partial w$, (we assume here, and during the rest of the paper, that ∂v and ∂w are in general position).

We call (M, F; v, w) *pseudominimal* if

$$c(M, F; v', w) \geq c(M, F; v, w) \leq c(M, F; v, w'),$$

for any arbitrary systems v′, w′ of meridian discs for V and W, respectively. If the stronger condition, $c(M, F; v, w) \leq c(M, F; v', w')$ holds, the diagram will be called *minimal* (here we do not follow the nomenclature of [6, section 4]). The *underlying splitting,* (M, F), is *minimal* if (M, F) cannot be obtained from

Received April 11, 1978. Revision received July 15, 1978.

The work of the first author was partially supported by National Science Foundation Grant No. 76-0823

Michigan Math. J. 27 (1980).

a Heegaard splitting of lower genus by trivial handle addition. Note, however, that this does not necessarily imply that (M, F) has minimal genus for all Heegaard splittings of M.

An algorithm exists to change a given Heegaard-diagram into one which is pseudominimal (see [1, pp. 149–150]; also the proof of a weak version of a theorem of Zieschang which is given in [6, section 4], gives a different algorithm). No algorithm is known to change a Heegaard diagram into one which is minimal. If one existed for S^3, the homeomorphism problem for S^3 would be solved [5]. (*Note added in proof:* T. Homma, M. Ochiai and M. Takahashi have recently developed an algorithm for recognizing Heegaard diagrams of genus 2 for S^3. Their algorithm systematically reduces the complexity until a cancelling pair of handles is obtained. The algorithm fails for diagrams of genus greater than or equal to 4 for S^3.)

While, certainly, minimal implies pseudominimal, our first example shows that the converse is in general false. Figures 1, 2 exhibit Heegaard-diagrams for the lens space L(7, 2) (in Poincaré's notation [4]); both diagrams are pseudominimal, as can be checked by applying the algorithm referred to above, but they have different complexity. Hence the one in Figure 2 is not minimal. Note that both diagrams have a canceling pair of handles.

Question 1. (Waldhausen [6]). *Suppose that* (M, F) *is not minimal, and that* (M, F; v, w) *is pseudominimal. Does* (M, F; v, w) *have a canceling pair of handles?*

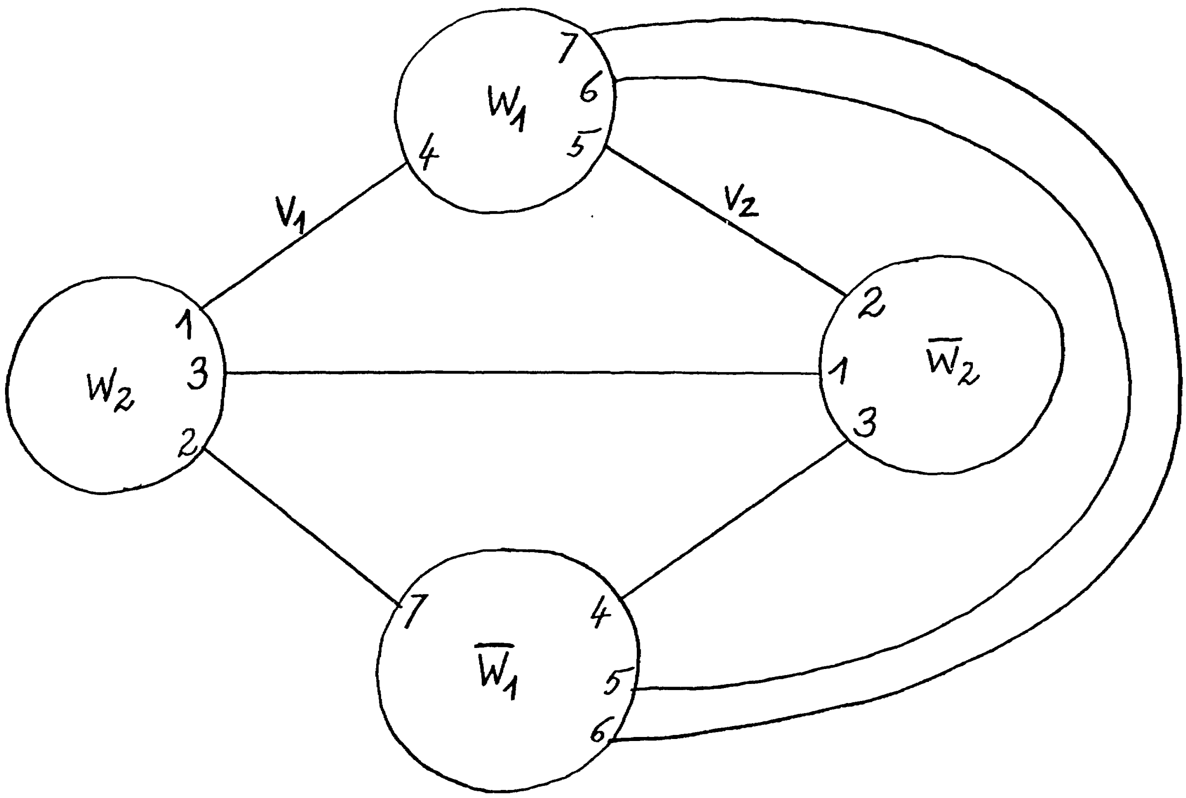

Figure 1

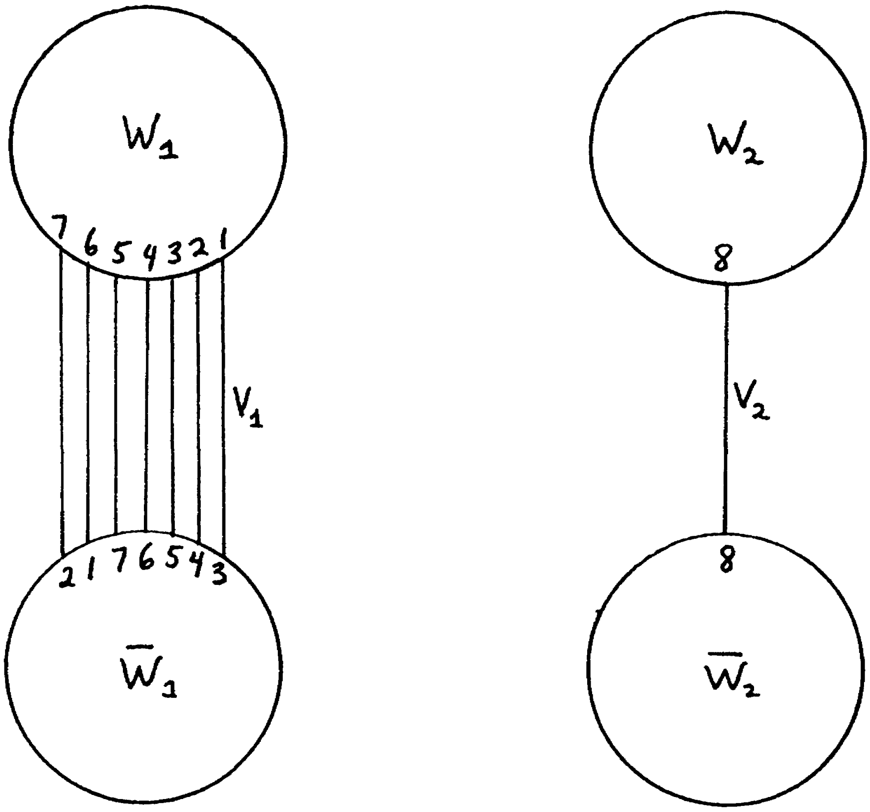

Figure 2

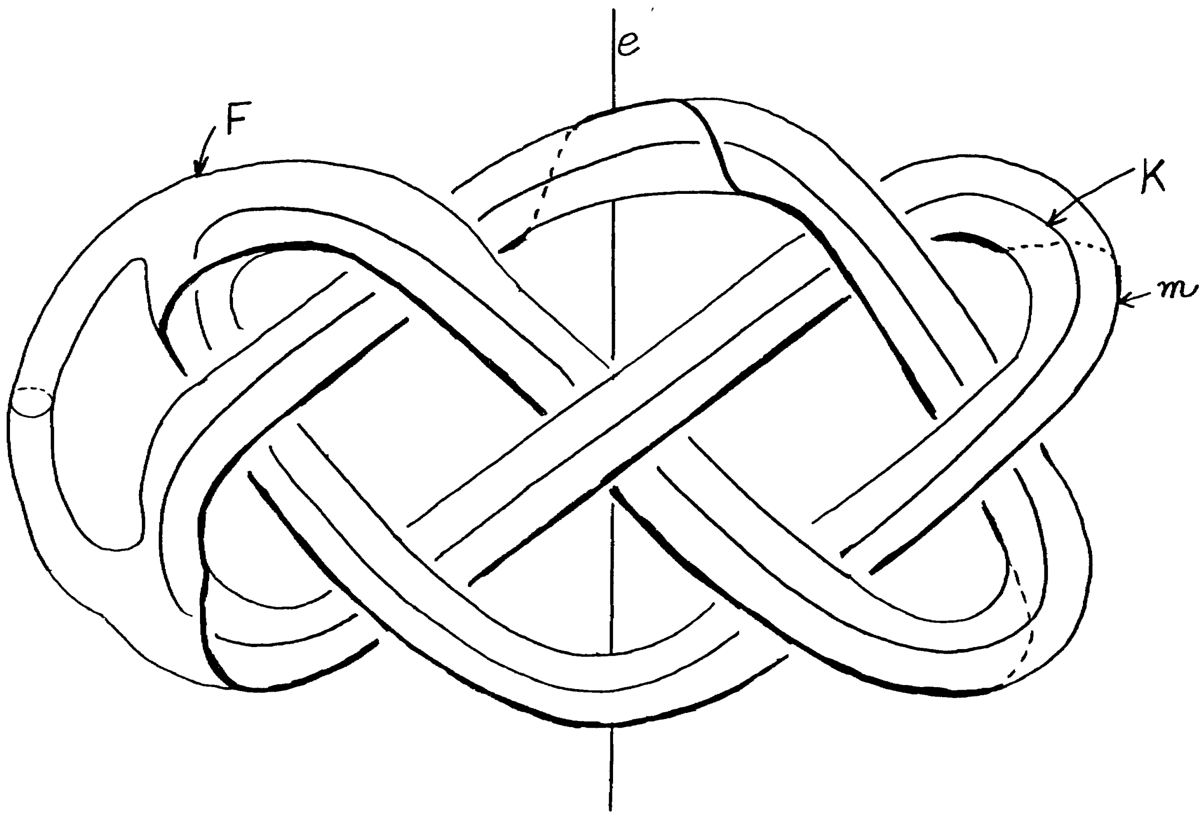

Figure 3

We supplement this with the following

Question 2. *Suppose that* (M, F) *is not minimal, and that* (M, F; v, w) *is minimal. Does* (M, F; v, w) *have a canceling pair of handles?*

We remark that the answer to Question 2 is yes for $M = S^3$ and connected sum of copies of $S^1 \times S^2$ by Waldhausen [5].

Both questions are related to the following

Question 3. (Waldhausen [5]). *Is there any manifold with two minimal Heegaard splittings of different genus?*

In the next section we present a Heegaard-diagram (M, F; v, w) such that either (M, F) answers Question 3 affirmatively, or else (M, F; v, w) answers Question 1 negatively. We conjecture that the former is the case.

2. DESCRIPTION OF THE EXAMPLE

Let M be the manifold which is obtained by Dehn surgery in the knot K of Figure 3, in such a way that the new meridian is the curve m.

LEMMA 1. *The Heegaard genus of* M *is at most* 2.

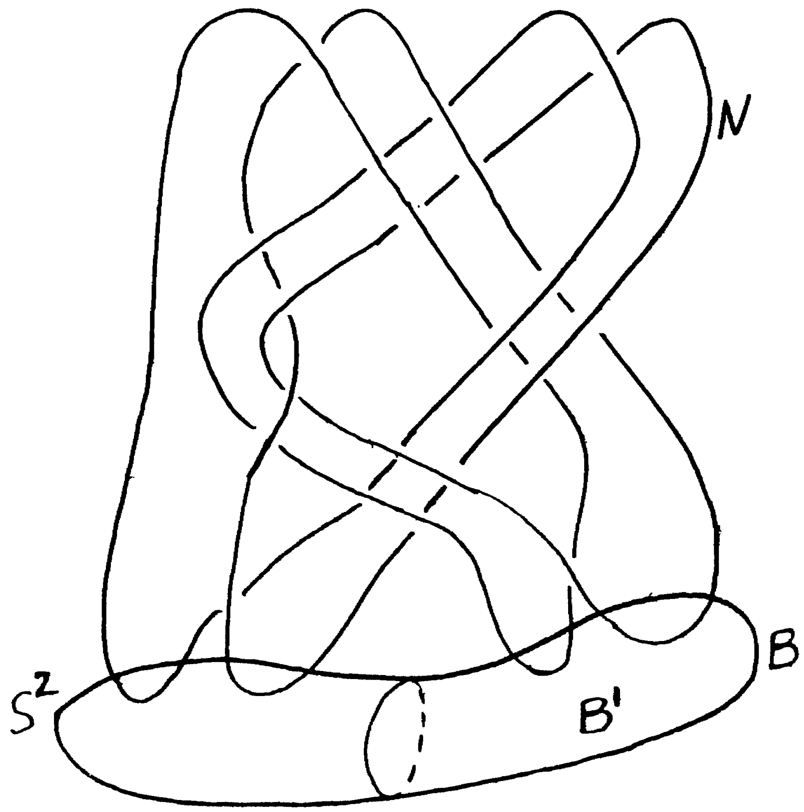

Figure 4

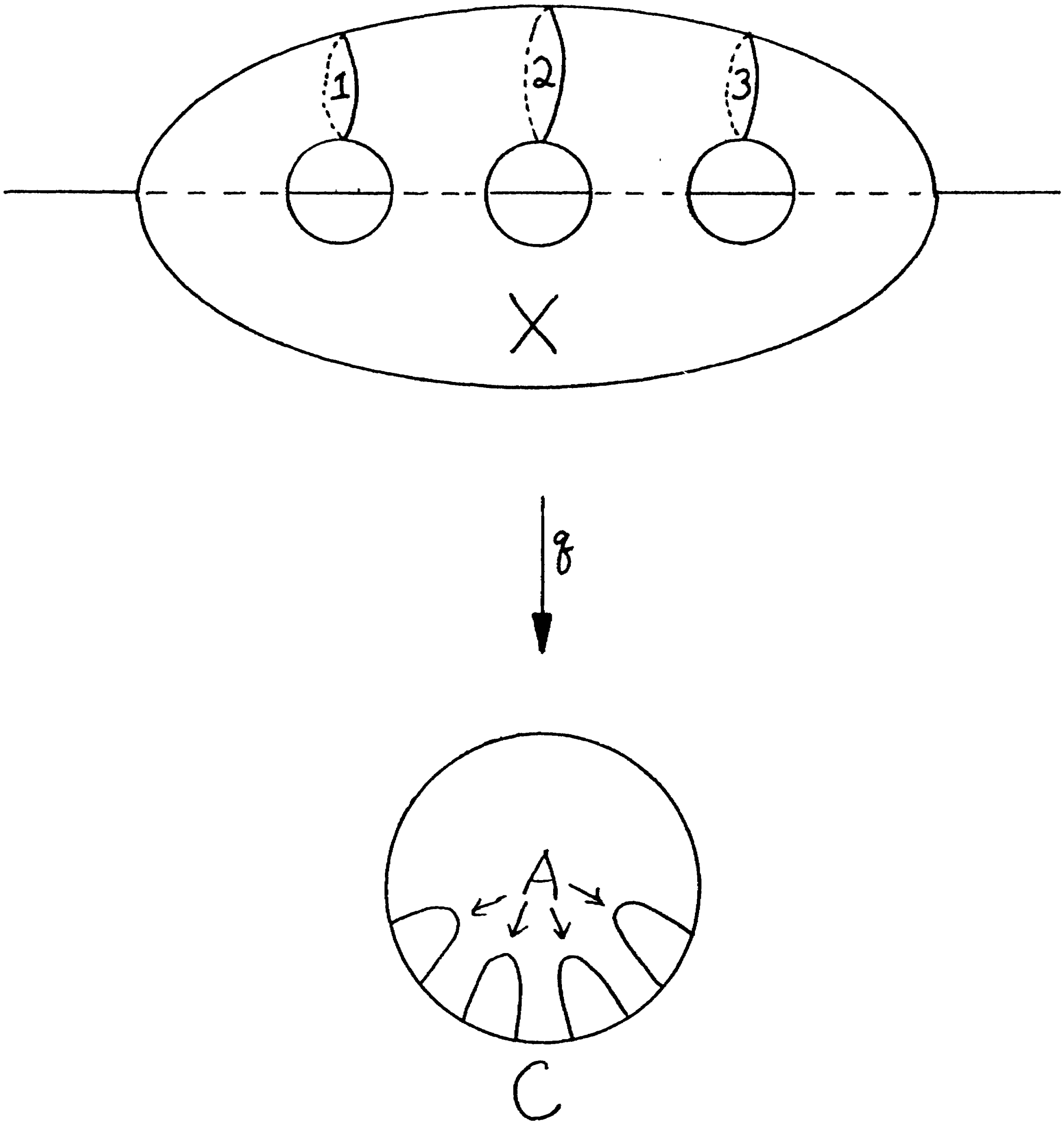

Figure 5

Proof. The surface F of Figure 3 defines a Heegaard splitting of genus 2 for M.

The knot K is strongly-invertible, and using the methods of [3] it can be shown that M is a 2-fold cyclic covering space branched over the knot N of Figure 4. Let $p : M \to S^3$ denote the covering projection.

The 2-sphere S^2 of Figure 4 separates S^3 into two 3-balls B and B′ and there are homeomorphisms $\varphi : (C, A) \to (B, B \cap N)$ and $\varphi' : (C, A) \to (B', B' \cap N)$, where (C, A) is the standard pair shown in Figure 5. Let $q : X \to C$ be the standard 2-fold branched covering of C by a handlebody X of genus 3 (see Figure 5).

There is a commutative diagram

$$\begin{array}{ccc}
(X \underset{i\tilde{\tau}}{\cup} iX, \quad q^{-1} A \underset{i\tilde{\tau}}{\cup} (iq)^{-1} A) & \xrightarrow{\ \alpha\ } & (M, \quad p^{-1} N) \\
\Big\downarrow (q, iq) & & \Big\downarrow p \\
(C \underset{i\tau}{\cup} iC, \qquad A \underset{i\tau}{\cup} iA) & \xrightarrow{\ (\varphi, \varphi')\ } & (S^3, \quad N)
\end{array}$$

where i sends C, X, etc. onto their mirror images iC, iX, etc., τ is an autohomeomorphism of $\partial(C, A)$ and $\tilde{\tau}$ is a lifting of τ, by q. Then

$$(M, p^{-1}S^2; \alpha(i1, i2, i3), \alpha(\tilde{\tau}1, \tilde{\tau}2, \tilde{\tau}3))$$

is a Heegaard diagram for M of genus 3, which can be obtained by a routine but lengthy calculation and is represented in Figure 6, calling $\alpha(i1, i2, i3) = (1, 2, 3)$ and $\alpha(\tilde{\tau}1, \tilde{\tau}2, \tilde{\tau}3) = (x, y, z)$.

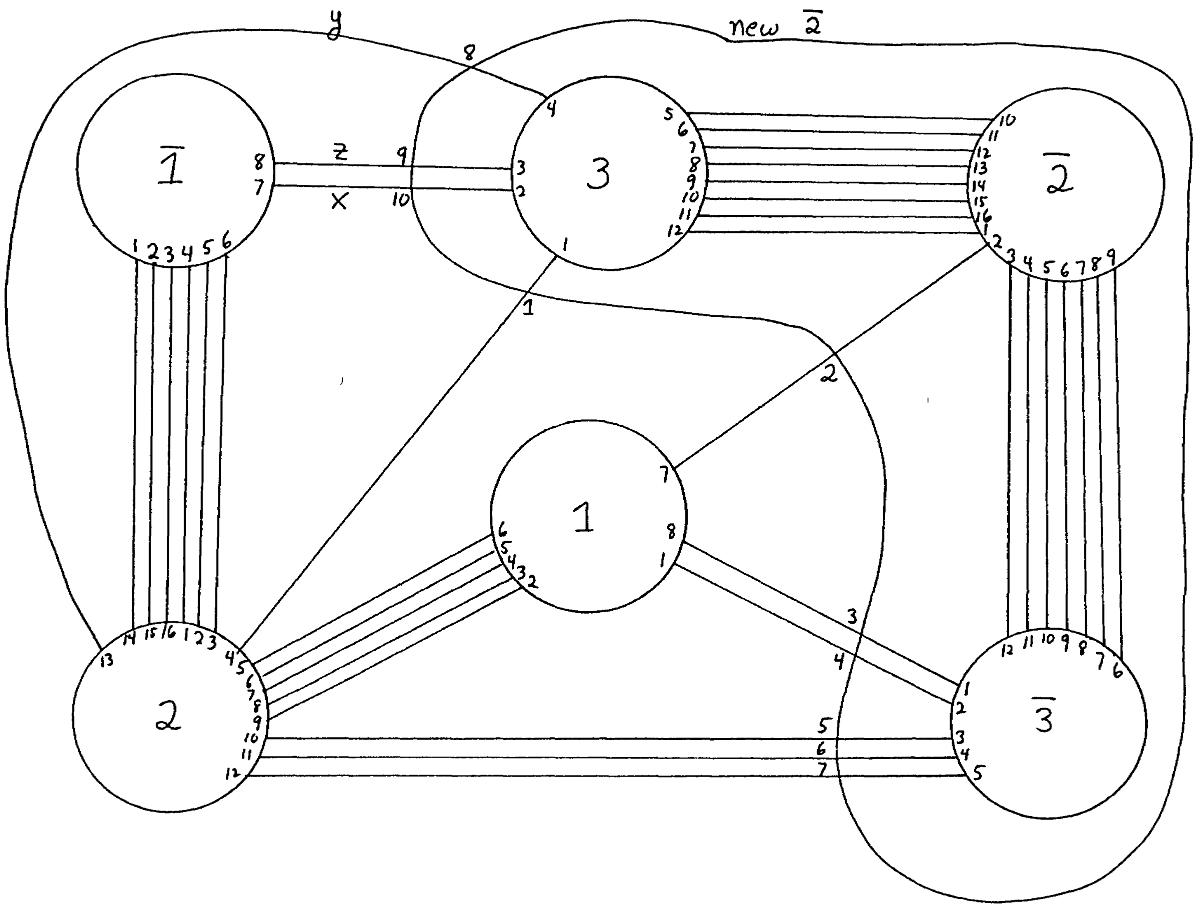

Figure 6

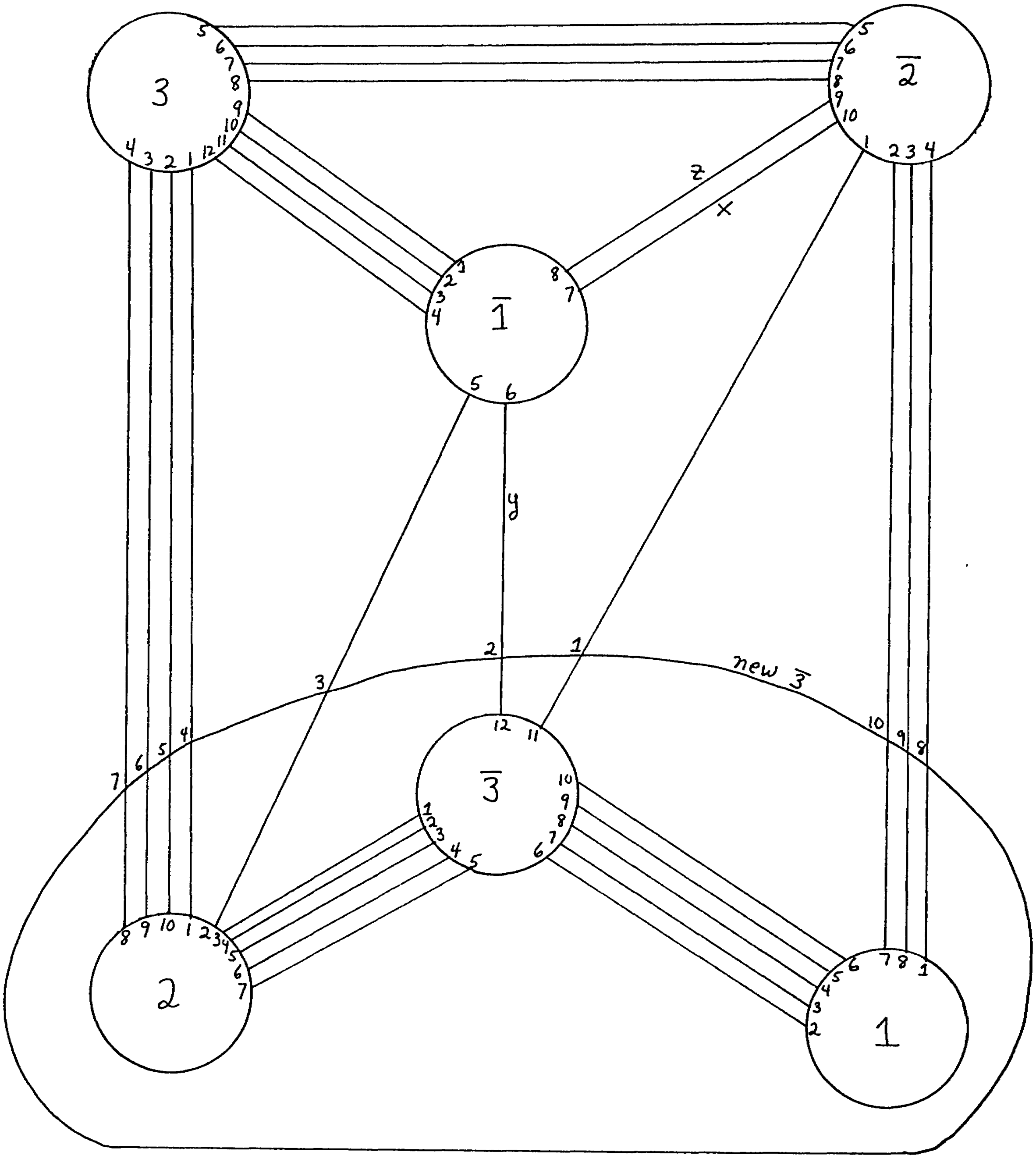

Figure 7

This diagram is not pseudominimal because the geometric T-transformation (see [6]) suggested in Figure 6 decreases its complexity. By successive application of geometric T-transformations we obtain the diagrams of Figures 7 and 8. The last one cannot be further reduced by a geometric T-transformation applied to the system {1, 2, 3}. But the geometric T-transformation which modifies z by a *band move* [5; p. 200] with x (Figure 8) gives us the diagram of Figure 9. This

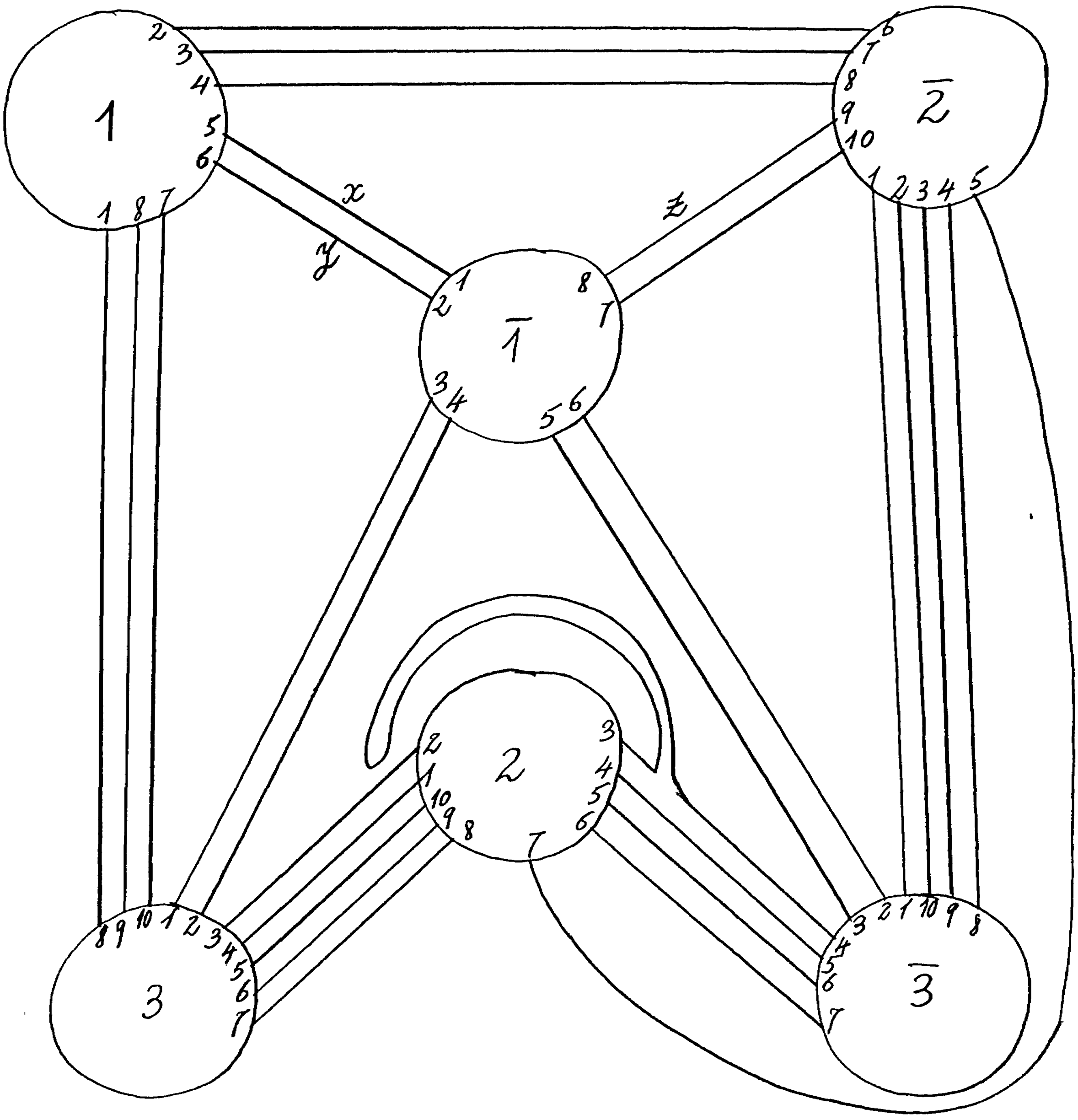

Figure 8

can be reduced to the one of Figure 10 which is pseudominimal (in checking this, the reader is advised that the dual diagram has exactly the same pattern as the original one; this reduces the checking to one system).

Remark. The example given above was discovered at random.

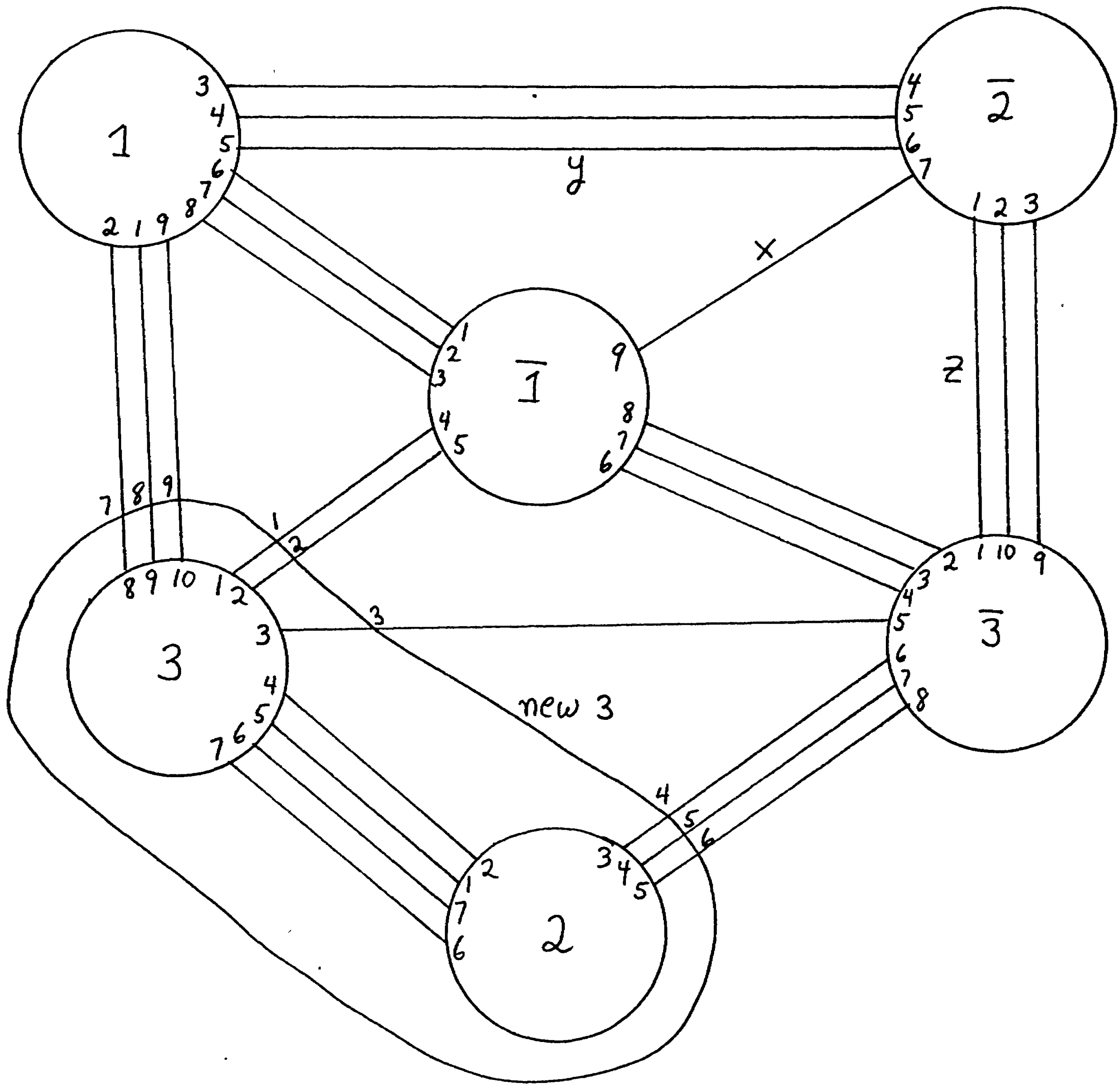

Figure 9

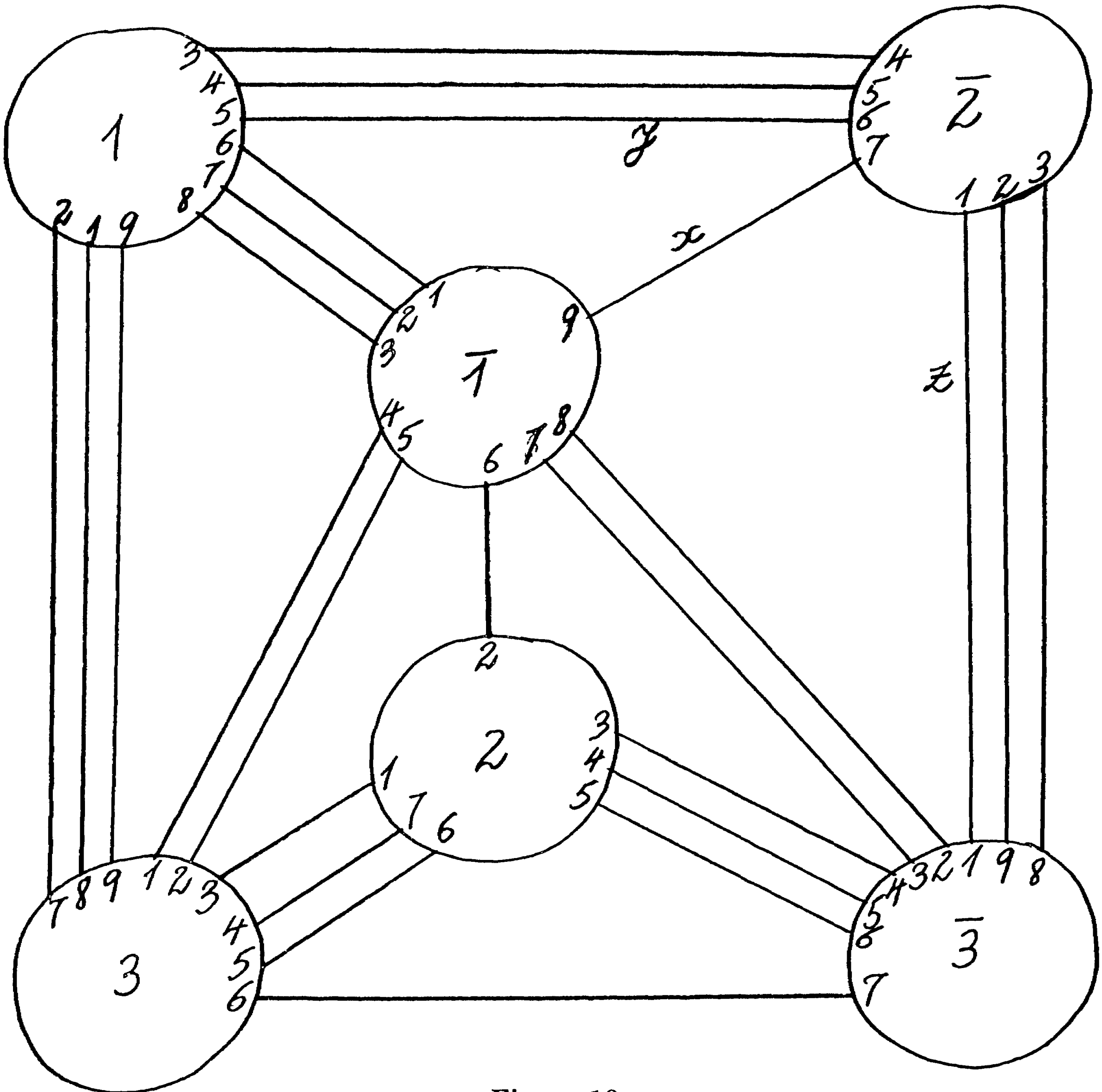

Figure 10

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