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ABELIAN AND SOLVABLE SUBGROUPS OF THE MAPPING CLASS GROUP

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I. Introduction. Let M be an orientable, compact Riemann surface of genus g with b boundary components and c connected components. Assume each connected component of M has negative Euler characteristic. The mapping class group, $\mathcal{M}(M)$, of M is the group of isotopy classes of orientation preserving self-homeomorphisms of M, where if $\partial M \neq \emptyset$, admissible isotopies fix each component of ∂M setwise. (Thus, in particular, the isotopy class of a Dehn twist about a curve which is parallel to a component of ∂M is considered to be trivial.) The reader is referred to [B2], [H2], [T], [FLP], and [G2] for background concerning this group. The main results of this paper will be two theorems about the algebraic structure of $\mathcal{M}(M)$:

THEOREM A. *Let G be an abelian subgroup of $\mathcal{M}(M)$. Then G is finitely generated with torsion free rank bounded by $3g + b - 3c$.*

THEOREM B. *Every solvable subgroup of $\mathcal{M}(M)$ is virtually abelian. Furthermore, if G is a virtually solvable subgroup of $\mathcal{M}(M)$, then G contains an abelian subgroup, A, such that the index of A in G is bounded by $V(M)$, where $V(M)$ is a positive integer depending only upon M.*

We now give examples which illustrate Theorem A.

The most obvious examples of abelian subgroups of $\mathcal{M}(M)$ are the groups generated by Dehn twists about a family of disjoint simple closed curves. These groups are free abelian with rank equal to the cardinality of the family of curves. The maximum cardinality of such a family is $3g + b - 3c$. Therefore, the bound in Theorem A is exact. For example, Figure 1 shows a surface of genus 3, constructed by attaching 3 handles, H_1, H_2, and H_3, to a sphere with 3 holes, P, along the boundary curves, $\gamma_1, \gamma_2, \gamma_3$, of P. The group generated by the Dehn twists about $\gamma_1, \gamma_2, \gamma_3$, β_1, β_2, β_3 has maximal rank. However, this rank can be achieved in other ways as well. For example, if σ_i is a pseudo-Anosov map supported on H_i, then one or more of the Dehn twists about the β_i's can be replaced by the σ_i's to yield various free abelian subgroups of maximal rank.

Next we give examples illustrating Theorem B. Refer to Figure 1 again, only now imagine that each H_i is a copy of $M_{g,1}$, where $g \geqslant 1$. Let τ_{ij}, $1 \leqslant i < j \leqslant 3$ be

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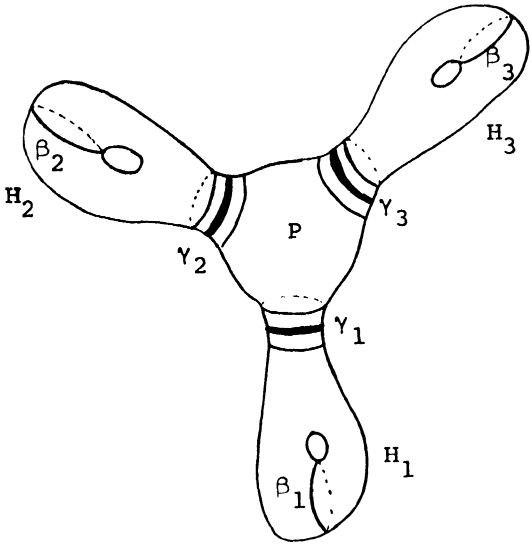

FIGURE 1

an involution on $H_i \cup H_j$ which exchanges H_i and H_j, and extends to a homeomorphism of M which is supported on $H_i \cup H_j \cup P$, so that τ_{ij}^2 is a Dehn twist about γ_k, $k \neq i, j$. The homeomorphisms $\tau_{12}, \tau_{13}, \tau_{23}$ generate a subgroup G_0 of $\mathscr{M}(M)$ which contains the infinite abelian subgroup A_0 generated by $\tau_{12}^2, \tau_{13}^2, \tau_{23}^2$, and $G_0/A_0 \cong S_3$, the symmetric group on 3 letters. Now, in addition, choose an abelian subgroup A of $\mathscr{M}(M_{g,1})$ and "Embed a copy, A_i, in each H_i". Then G_0, A_0, A_1, A_2, A_3 generate a solvable subgroup G of $\mathscr{M}(M)$ which contains an abelian subgroup (the group generated by A_0, A_1, A_2, A_3) of finite index.

The proofs of Theorems A and B use Thurston's classification theorem for single elements in the group $\mathscr{M}(M)$:

THURSTON'S THEOREM ([Th], [FLP]). *To each $\tau \in \mathscr{M}(M)$ there is associated a representative $t \in \mathrm{Diff}^+ M$ and a t-invariant system of disjoint simple closed curves A such that M split along A is the union of two disjoint, compact and in general not connected and possibly empty subsurfaces M_1 and M_2, and up to a permutation of the components of M_1 and of M_2, and isotopy supported in a collar neighborhood of A, t is pseudo-Anosov on M_1 and of finite order on M_2.*

In this theorem, the system A is in general not unique, as is easily seen when t has finite order. We will need something more, and so we introduce in Section 2 the concept of an "essential reduction system". As a by-product we will be able to improve Thurston's Theorem by showing:

THEOREM C. *A system A satisfying the conditions of Thurston's Theorem, which is minimal among such systems, is unique up to isotopy.*

In Section 2 we introduce essential reduction systems, establish their main properties, and prove Theorem C. In Section 3 we prove Theorem A. In Section 4 we prove Theorem B.

Harvey asks in [H2] whether $\mathscr{M}(M)$ has a finite dimensional, faithful, linear representation, and whether $\mathscr{M}(M)$ is arithmetic. In relation to these questions, we observe that $\mathscr{M}(M)$ has various properties in common with the class of finitely generated linear groups. For instance, it is finitely generated [D]; residually finite [Gr], and virtually torsion free (cf. [B-L]). As a consequence of Theorem A, there is a bound to the derived length of solvable subgroups. (See the Remark following the proof of Theorem B.) In [W], it is shown that if Γ is a virtually solvable subgroup of $\mathrm{GL}(n, C)$, then Γ contains a solvable subgroup, G, such that the index of G in Γ is bounded by a function of n. By Theorem B, $\mathscr{M}(M)$ shares this property as well. As a corollary of Theorem B, every solvable subgroup of $\mathscr{M}(M)$ is of bounded Hirsch rank [Hi]. This property and the results of Theorem A are shared by the class of finitely generated arithmetic groups, but not, in general, by the class of finitely generated linear groups. It is interesting to note that Theorem B exhibits a property which is shared by the class of all arithmetic groups acting on hyperbolic spaces. For other relations of this nature see [K].

2. Essential reduction classes. The collection of nonoriented isotopy classes of simple closed curves in M which are not parallel to ∂M and not homotopically trivial in M is denoted by the symbol $\mathscr{S}(M)$.

If $\tau \in \mathscr{M}(M)$ and $\alpha \in \mathscr{S}(M)$, then $\tau(\alpha)$ denotes the class of $t(a)$, where $t \in \tau$ and $a \in \alpha$. Similarly, if $\mathscr{A} \subset \mathscr{S}(M)$, then $\tau(\mathscr{A})$ denotes the collection $\{\tau(\alpha) : \alpha \in \mathscr{A}\}$. We leave it to the reader to verify that if $\tau(\mathscr{A}) = \mathscr{B}$ and $A \in \mathscr{A}$ and $B \in \mathscr{B}$, then there is a representative, $t \in \tau$, such that $t(A) = B$, and this representative is well defined modulo relative isotopy.

A subset, $\mathscr{A} \subset \mathscr{S}(M)$, is *admissible* if a set of representatives, $A \in \mathscr{A}$, can be chosen to consist of pairwise disjoint curves. Similarly, we say that A is an *admissible set of representatives*.

We now introduce some groups which are determined by the choice of an admissible system $\mathscr{A} \subset \mathscr{S}(M)$. The symbol $\mathscr{M}_{\mathscr{A}}(M)$ denotes the stabilizer of $\mathscr{A}$ in $\mathscr{M}(M)$. We write $M_{\mathscr{A}}$ for the natural compactification of $M - A$, where A is any admissible representation of $\mathscr{A}$. If $\tau \in \mathscr{M}_{\mathscr{A}}(M)$, then we can choose an admissible $A \in \mathscr{A}$ and a representative $t \in \tau$ such that $t(A) = A$. Furthermore, $t|_{M-A}$ extends uniquely to $M_{\mathscr{A}}$. Again, we leave it to the reader to verify that this process determines a well-defined class, $\hat{\tau} \in \mathscr{M}(M_{\mathscr{A}})$. We shall refer to this class, $\hat{\tau}$, as the *reduction of* τ *along* $\mathscr{A}$. The assignment, $\tau \to \hat{\tau}$, yields a homomorphism, $\wedge : \mathscr{M}_{\mathscr{A}}(M) \to \mathscr{M}(M_{\mathscr{A}})$, which we shall refer to as the *reduction homomorphism*.

For any simple closed curve, a, in M there is a well known homeomorphism, t_a, which is called a *Dehn twist about* a. It is supported on an annular neighborhood of a, and is defined by splitting M along a, twising one end of the split by 360° and reglueing. If $\alpha \in \mathscr{S}(M)$, then we denote the isotopy class of t_a, where a is any representative of α, by τ_α.

Caution: In general $\wedge$ is not an isomorphism. By our definition of the mapping class group of a bounded surface, each Dehn twist τ_α, $\alpha \in \mathscr{A}$, will be in kernel $\wedge$.

A natural representation, $\partial : \mathscr{M}(M) \to \mathrm{Aut}(\partial M)$, arises from the permutation of boundary components.

Let Z_α be the cyclic subgroup of $\mathscr{M}(M)$ generated by τ_α. If $\mathscr{A}$ is an admissible subset of $\mathscr{S}(M)$, then $Z_{\mathscr{A}}$ will denote the subgroup of $\mathscr{M}(M)$ generated by $\{\tau_\alpha : \alpha \in \mathscr{A}\}$. The following lemma describes the relationship of $Z_{\mathscr{A}}$ to $\wedge$ ([H2], note 1, p. 266).

LEMMA 2.1. *Let $\mathscr{A}$ be an admissible subset of $\mathscr{S}(M)$. Then*:

(1) $\mathrm{kernel}(\wedge) = Z_{\mathscr{A}}$ *is a free abelian group with free basis* $\{\tau_\alpha : \alpha \in \mathscr{A}\}$. *In particular* $\mathrm{rank}(Z_{\mathscr{A}}) = \mathrm{cardinality}(\mathscr{A})$.

(2) $Z_{\mathscr{A}} \subseteq \mathrm{center}(\mathrm{kernel}(\partial \circ \wedge : \mathscr{M}_{\mathscr{A}}(M) \to \mathrm{Aut}(\partial M_{\mathscr{A}}))$.

Proof. Part (1) is well known and we omit the proof. Clearly, therefore $Z_{\mathscr{A}} \subseteq \mathrm{kernel}(\partial \circ \wedge)$. To see that $Z_{\mathscr{A}}$ is contained in the center(kernel($\partial \circ \wedge$)), let $\sigma \in \mathrm{kernel}(\partial \circ \wedge)$. Then, $\sigma(\alpha) = \alpha$. But $\sigma\tau_\alpha\sigma^{-1} = \tau_{\sigma(\alpha)} = \tau_\alpha$. Therefore, τ_α commutes with σ so $Z_{\mathscr{A}}$ is contained in the center of kernel($\partial \circ \wedge$). ||

If $M = \coprod_{i \in I} M_i$, then we will use $\Gamma(M)$ to denote the collection $\{M_i : i \in I\}$ of connected components of M. There is a natural representation, $\varphi : \mathscr{M}(M) \to \mathrm{Aut}(\Gamma(M))$, which arises from permutation of components. Kernel(φ) is naturally isomorphic to $\bigoplus_{i \in I} \mathscr{M}(M_i)$. If $\tau \in \mathscr{M}(M)$, then for some exponent, n, $\tau^n \in \mathrm{kernel}(\varphi)$. For any such exponent, we refer to the elements of $\mathscr{M}(M_i)$ obtained by restriction of τ^n as *restrictions* of τ.

A mapping class, $\tau \in \mathscr{M}(M)$, is *pseudo-Anosov* if $\mathscr{S}(M_i) \neq \emptyset$ for every $i \in I$, and $\tau^n(\alpha) \neq \alpha$ for any $\alpha \in \mathscr{S}(M)$ and any $n \neq 0$. The class, $\tau \in \mathscr{M}(M)$, is said to be *reducible* if there is an admissible set, $\mathscr{A}$, such that $\tau(\mathscr{A}) = \mathscr{A}$. In this event, we shall refer to such a set, $\mathscr{A}$, as a *reduction system for* τ. Each $\alpha \in \mathscr{A}$ is a *reduction class for* τ.

A mapping class, $\tau \in \mathscr{M}(M)$, is *adequately reduced* if each of its restrictions is either finite order or pseudo-Anosov. A reduction system, $\mathscr{A}$, for τ is an *adequate reduction system for* τ if τ reduced along $\mathscr{A}$ is adequately reduced. Using this concept, Thurston's Theorem may now be restated as

THEOREM 2.2 ([Th], [OS]). *Every mapping class, $\tau \in \mathscr{M}(M)$, is either reducible or adequately reduced. If τ is reducible, then there exists an adequate reduction system, $\mathscr{A}$, for τ.*

The function denoted by $i : \mathscr{S}(M) \times \mathscr{S}(M) \to$ is the *geometric intersection form*. It is defined by setting $i(\alpha, \beta)$ equal to the minimum number of points of intersection of a and b, where a and b range over the representatives of α and β respectively.

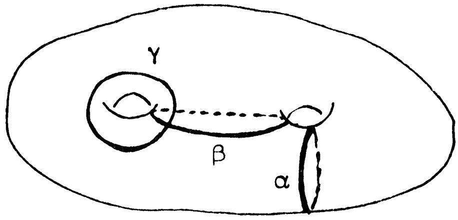

FIGURE 2

Definition. A reduction class, α, for τ is an *essential reduction class for* τ if for each $\beta \in \mathscr{S}(M)$ such that $i(\alpha, \beta) \neq 0$ and for each integer $m \neq 0$, the classes $\tau^m(\beta)$ and β are distinct.

Example. In Figure 2, let $\tau = \tau_\alpha$. Then α is an essential reduction class for τ, (see Appendix, Exposé 4, [FLP]) but β is not, even though $\tau(\beta) = \beta$. The reason is that $i(\beta, \gamma) \neq 0$, but $\tau(\gamma) = \gamma$.

We now establish some of the properties of essential reduction classes and adequate reduction systems.

PROPOSITION 2.3. *Let α, α' be reduction classes for $\tau \in \mathscr{M}(M)$. Suppose α is essential. Then $i(\alpha, \alpha') = 0$.*

Proof. If $i(\alpha, \alpha') \neq 0$, then $\tau^n(\alpha') \neq \alpha'$ for each $n \neq 0$, contradicting the hypothesis that α' is a reduction class. ||

LEMMA 2.4. *Let N be connected where $\chi(N) < 0$. Let δ be an isotopy class of a properly embedded arc which is not homotopic to an arc in ∂N. Let $\tau \in \mathscr{M}(N)$ with $\tau(\delta) = \delta$. Then either*

(i) *N is a pair of pants*

or

(ii) *there exists a class γ in $\mathscr{S}(N)$ such that $\tau(\gamma) = \gamma$ and $i(\gamma, \alpha) \neq 0$ for all classes, $\alpha \in \mathscr{S}(N)$, which intersect δ nontrivially.*

Proof. Let d be a properly embedded arc representing δ. Let $\eta(d)$ be a regular neighborhood of the 1 dimensional subcomplex of N formed by d together with the boundary components of N which meet d at its endpoints (see Figure 3). There are two possible configurations; one corresponds to the situation where the endpoints of d meet distinct boundary components (Figure 3a), the second to the situation where the endpoints meet a common component (Figure 3b). In either case, $\eta(d)$ is a pair of pants (i.e., a sphere with 3 discs removed). It has 3 boundary components. In case (a) two of these are components of ∂N. Let γ denote the isotopy class of the third. If γ is parallel to ∂N, then N is a pair of pants. Otherwise, γ is not homotopically trivial, for if so, then N would be an annulus, which is ruled out. So $\gamma \in \mathscr{S}(N)$. Since $\tau(\delta) = \delta$, then $\tau(\gamma) = \gamma$. If $\alpha \in \mathscr{S}(N)$ intersects δ nontrivially, then $i(\alpha, \gamma) \neq 0$. For if not, then α could be represented by a simple closed curve in $\eta(D)$, so that α would be, necessarily,

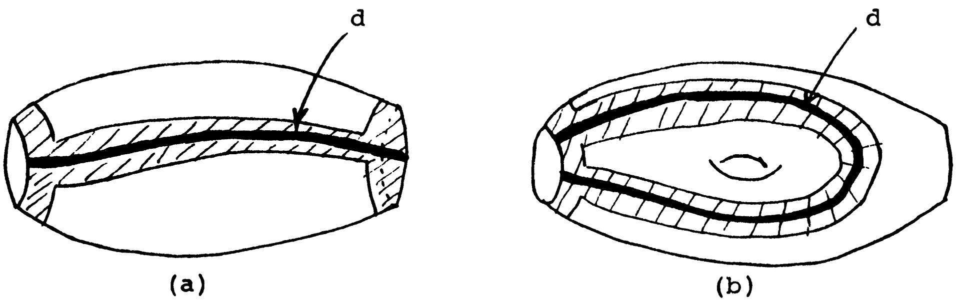

FIGURE 3

parallel to one of the boundary components of $\eta(d)$, and therefore, parallel to ∂N or to γ. Case (b) is similar. $\|$

If $\mathscr{A}$ is an admissible subset of $\mathscr{S}(M)$, then the symbol $\mathscr{S}_{\mathscr{A}}(M)$ will denote the subset of $\mathscr{S}(M)$ consisting of isotopy classes which do not belong to $\mathscr{A}$ and which are representable by curves in $M - A$, where A is any admissible set of representatives for $\mathscr{A}$. It is an easy exercise to show that $\mathscr{S}_{\mathscr{A}}(M)$ is well defined independently of the choice of A.

LEMMA 2.5. *Let $\mathscr{A}$ be an adequate reduction system for τ and let $\alpha \in \mathscr{A}$. Let $\mathscr{A}' = \mathscr{A} - \{\alpha\}$. Then the following are equivalent*:
(1) *α is essential*,
(2) *$\mathscr{A}'$ is not an adequate reduction system for τ^m for any $m \neq 0$.*

Proof. First, we show (1) implies (2). Assume α is essential. Choose $m \neq 0$. If $\tau^m(\mathscr{A}') \neq \mathscr{A}'$, then $\mathscr{A}'$ is not a reduction system for τ^m and we are done. So we may assume $\tau^m(\mathscr{A}') = \mathscr{A}'$ which implies $\tau^m(\alpha) = \alpha$. Let $\hat{\alpha}'$ be the lift of α to $M_{\mathscr{A}'}$. Then $\hat{\alpha}'$ is a reduction class for $\wedge'(\tau^m)$. We will show that $\hat{\alpha}'$ is essential. Choose any $\hat{\gamma}' \in \mathscr{S}_{\mathscr{A}'}(M)$ with $i(\hat{\alpha}', \hat{\gamma}') \neq 0$ and any $n \neq 0$. Then $\hat{\gamma}'$ projects to $\gamma \in \mathscr{S}(M)$ with $i(\alpha, \gamma) \neq 0$. Since α is essential, $\tau^{mn}(\gamma) \neq \gamma$. Lifting back to $M_{\mathscr{A}'}$, it follows that $\wedge'(\tau^{mn})(\hat{\gamma}') \neq \hat{\gamma}'$. Therefore, $\hat{\alpha}'$ is essential for $\wedge'(\tau^m)$. So $\wedge'(\tau^m)$ is not adequately reduced, hence $\mathscr{A}'$ was not an adequate system for τ^m.

Now we show (2) implies (1). We assume α is not essential. We show that $\mathscr{A}'$ is an adequate reduction system for τ^m for some integer $m \neq 0$. Since α is not essential, there exists a curve class $\gamma \in \mathscr{S}(M)$ and an integer $n \neq 0$ such that $i(\alpha, \gamma) \neq 0$ and $\tau^n(\gamma) = \gamma$. Splitting M along $\mathscr{A}$, the class, γ, determines a finite family of pairwise disjoint isotopy classes of properly embedded arcs in $M_{\mathscr{A}}$, which we denote by $\hat{\gamma}$. Since $i(\alpha, \gamma) \neq 0$, we conclude that at least one component of $\hat{\gamma}$ occurs on each component of $M_{\mathscr{A}}$ which "borders on α".

Since $\tau^n(\gamma) = \gamma$, therefore $\hat{\tau}^n(\hat{\gamma}) = \hat{\gamma}$. By choosing a larger exponent, n, if necessary, we can assume, in addition, that $\hat{\tau}^n$ preserves each component of $M_{\mathscr{A}}$, each component of $\partial M_{\mathscr{A}}$ and each component of $\hat{\gamma}$. In particular, the restrictions of $\hat{\tau}^n$ to the components of $M_{\mathscr{A}}$ bordering on α each preserve a nontrivial isotopy class of a properly embedded arc. By Lemma 2.4, for each such component, either the corresponding restriction of $\hat{\tau}^n$ is reducible or the component is a pair of pants. By assumption, $\hat{\tau}^n$ is adequately reduced. Therefore, each restriction of $\hat{\tau}^n$ is either finite order or pseudo-Anosov. Since a pair of pants will not support a pseudo-Anosov mapping class [OS], and since pseudo-Anosov mapping classes are completely reduced, it follows that the particular restrictions in question are each finite order. Hence, by choosing a larger exponent, n, if necessary, we may assume, in addition to the preceding remarks, that the restrictions of $\hat{\tau}^n$ to the components bordering on α are trivial.

We now examine the corresponding situation when we reduce along $\mathscr{A}'$. Let $\wedge'$ be the reduction homomorphism, and let $\hat{\gamma}'$ and $\hat{\alpha}'$ be the lifts of γ and α to $M_{\mathscr{A}'}$. Since $\mathscr{A}$ is an adequate reduction system for τ, it follows that $\hat{\alpha}'$ is an adequate reduction system for $\wedge'(\tau^n)$. Therefore, each restriction of $\wedge'(\tau^n)$ to a component of $M_{\mathscr{A}'}$ is finite order or pseudo-Anosov, except, possibly, the restriction ν to the component N of $M_{\mathscr{A}'}$, which contains $\hat{\alpha}'$.

From the above discussion, it is evident that $\nu(\hat{\alpha}') = \hat{\alpha}'$. Also, since the restrictions of $\hat{\tau}^n$ to the components of $M_{\mathscr{A}}$ bordering on α are trivial, the reduction of ν along $\hat{\alpha}'$ is trivial. We conclude (see Lemma 2.1) that ν is a power of a Dehn twist about $\hat{\alpha}'$.

Now we consider γ again. There are two separate cases to consider:

Case 1. $\gamma \subset M - A'$ where A' represents $\mathscr{A}'$. Then, since $i(\alpha,\gamma) \neq 0$ by our choice of γ, it follows that $i(\hat{\alpha}',\hat{\gamma}') = i(\alpha,\gamma) \neq 0$. So $\hat{\gamma}'$ is a class in $\mathscr{S}(N)$ which intersects $\hat{\alpha}'$ nontrivially, and ν is a power of a Dehn twist about $\hat{\alpha}'$ with $\nu(\hat{\gamma}') = \hat{\gamma}'$. By an argument just like that used in the proof of Lemma 2.1, part (2), we conclude that ν cannot be a nontrivial power. Hence $\nu = \text{identity}$, so $\wedge'(\tau^n)$ is adequately reduced.

Case 2. $i(\gamma,\mathscr{A}') \neq 0$. Then $\hat{\gamma}'$ is a family of arcs, at least one of which passes through N. As before, we may assume that n was chosen so that each component of $\hat{\gamma}'$ is preserved by $\wedge'(\tau^n)$. Applying Lemma 2.4, and the fact that N is not a pair of pants, we conclude that there exists $\delta \in \mathscr{S}(N)$ such that $i(\hat{\alpha}',\delta) \neq 0$ and $\nu(\delta) = \delta$. As before, we conclude that this is possible only if $\nu = \text{identity}$. Therefore, $\wedge'(\tau^n)$ is adequately reduced. ||

LEMMA 2.6. *Let* $\mathscr{A} = \{\alpha \in \mathscr{S}(M) \mid \alpha$ *is an essential reduction class for* $\tau\}$. *Then*

(1) $\sigma(\mathscr{A}_\tau) = \mathscr{A}_{\sigma\tau\sigma^{-1}}$ *for all* $\sigma \in \mathscr{M}(M)$.

(2) $\mathscr{A}_{\tau^m} = \mathscr{A}_\tau$ *for all* $m \neq 0$.

(3) $\mathscr{A}_\tau$ *is an adequate reduction system for* τ.

(4) $\mathscr{A}_\tau \subseteq \mathscr{A}$ *for each adequate reduction system* $\mathscr{A}$ *for* τ.

Proof. (1) Let α be an essential reduction class for τ. Choose a reduction system, $\mathscr{A}$, for τ with $\alpha \in \mathscr{A}$. Then $\sigma(\alpha) \in \sigma(\mathscr{A})$, a reduction system for $\sigma\tau\sigma^{-1}$. If $\beta \in \mathscr{S}(M)$, with $i(\sigma(\alpha), \beta) \neq 0$, then $i(\alpha,\sigma^{-1}(\beta)) \neq 0$. Hence, $\tau^n(\sigma^{-1}(\beta))$

$\neq \sigma^{-1}(\beta)$ for all $n \neq 0$. Hence, $(\sigma\tau\sigma^{-1})^n(\beta) \neq \beta$, so $\sigma(\alpha)$ is an essential reduction class for $\sigma\tau\sigma^{-1}$. Therefore, $\sigma(\mathscr{A}_\tau) \subseteq \mathscr{A}_{\sigma\tau\sigma^{-1}}$. By the same argument, $\sigma^{-1}(\mathscr{A}_{\sigma\tau\sigma^{-1}}) \subseteq \mathscr{A}_\tau$, so $\mathscr{A}_{\sigma\tau\sigma^{-1}} \subseteq \sigma(\mathscr{A}_\tau)$. Hence, $\sigma(\mathscr{A}_\tau) = \mathscr{A}_{\sigma\tau\sigma^{-1}}$.

(2) By Proposition 2.3, $\mathscr{A}_{\tau^m}$ is admissible. By Part (1), $\tau(\mathscr{A}_{\tau^m}) = \mathscr{A}_{\tau^m}$. Hence, every $\alpha \in \mathscr{A}_{\tau^m}$ is a reduction class for τ. Choose $\alpha \in \mathscr{A}_{\tau^m}$, $\beta \in \mathscr{S}(M)$ with $i(\alpha, \beta) \neq 0$, $n \neq 0$. Then $(\tau^m)^n(\beta) \neq \beta$, hence, $\tau^n(\beta) \neq \beta$, hence, α is essential for τ. The reverse inclusion follows similarly.

(3) By Proposition 2.3, $\mathscr{A}_\tau$ is admissible. By part (1) above, $\tau(\mathscr{A}_\tau) = \mathscr{A}_\tau$. Therefore, $\mathscr{A}_\tau$ is a reduction system for τ. Hence, $\tau \in \mathscr{M}_{\mathscr{A}_\tau}(M)$. Let σ be the reduction of τ along $\mathscr{A}_\tau$, and let $\mathscr{B}$ be an adequate reduction system for σ. If the cardinality of $\mathscr{B}$ is 0, then σ is adequately reduced. Otherwise, choose $\beta \in \mathscr{B}$ and let $\mathscr{B}' = \mathscr{B} - \{\beta\}$. $\mathscr{B}$ lifts to an admissible subset, $\mathscr{A}$, of $\mathscr{S}_{\mathscr{A}_\tau}(M)$. Let α be the class of $\mathscr{A}$ such that $\beta = \hat{\alpha}$. We conclude that $\tau(\mathscr{A}) = \mathscr{A}$ and, therefore, that α is a reduction class for τ. Since $\alpha \in \mathscr{S}_{\mathscr{A}_\tau}(M)$, α is not essential for τ. Therefore, we may choose $\gamma \in \mathscr{S}(M)$ and $n \neq 0$ such that $i(\alpha, \gamma) \neq 0$ and $\tau^n(\gamma) = \gamma$. The latter condition implies that $\gamma \in \mathscr{S}_{\mathscr{A}_\tau}(M)$. Hence, γ lifts to $\delta \in \mathscr{S}(M_{\mathscr{A}_\tau})$ with $i(\beta, \delta) \neq 0$ and $\sigma^n(\delta) = \delta$. In other words, β is not an essential reduction class for σ. From Lemma 2.4, we conclude that for some $m \neq 0$, $\mathscr{B}'$ is an adequate reduction system for σ^m. By Part (2), we have $\mathscr{A}_\tau = \mathscr{A}_{\tau^m}$. Since σ^m is the reduction of τ^m along $\mathscr{A}_\tau$, it follows from induction that σ^m is adequately reduced. Therefore, σ is adequately reduced.

(4) Let $\mathscr{A}$ be an adequate reduction system for τ and let α be an essential reduction class. If $\alpha \notin \mathscr{A}$, then Proposition 2.3 asserts that $i(\alpha, \mathscr{A}) = 0$ and, hence $\alpha \in \mathscr{S}_{\mathscr{A}}(M)$. Since $\alpha \in \mathscr{A}_\tau$, therefore $\alpha \in \mathscr{A}_\tau - \mathscr{A}$, which is an admissible set in $\mathscr{S}_{\mathscr{A}}(M)$. Also, $\tau(\mathscr{A}_\tau - \mathscr{A}) = \mathscr{A}_\tau - \mathscr{A}$. Let β be the lift of α to $M_{\mathscr{A}}$, and σ be the reduction of τ along $\mathscr{A}$. By arguments similar to those above, we conclude that β is an essential reduction class for σ, and hence, that σ is not adequately reduced. But this is a contradiction and therefore $\alpha \in \mathscr{A}$. ||

We may now establish Theorem C.

Proof of Theorem C. Let $\tau \in \mathscr{M}(M)$. Then, by Theorem 2.2 either τ is adequately reduced (in which case $A = \emptyset$) or τ is reducible, and if τ is reducible, then there exists an adequate reduction system. Let $\mathscr{A}_\tau = \{\alpha \in \mathscr{S}(M) \mid \alpha \text{ is an essential reduction class for } \tau\}$. By Lemma 2.5, $\mathscr{A}_\tau$ is the intersection of all adequate reduction systems for τ. Hence $\mathscr{A}_\tau$ is canonical and unique. The desired curve system, A, is any representative of $\mathscr{A}_\tau$. ||

3. Abelian subgroups of $\mathscr{M}(M)$. In this section we prove Theorem A.

If G is a subgroup of $\mathscr{M}(M)$ and each $\tau \in G$ is adequately reduced, then G is *adequately reduced*. Let G denote an abelian subgroup of $\mathscr{M}(M)$. Let rank(G) denote the torsion free rank, and let Tor(G) denote the torsion subgroup of G.

LEMMA 3.1. (1) *Let $\mathscr{A}_G$ be the union of the essential reduction systems $\mathscr{A}_\tau$, $\tau \in G$. Then $\mathscr{A}_G$ is an adequate reduction system for each $\tau \in G$.*

(2) *If G is adequately reduced, then* $\operatorname{rank}(G) \leqslant C_0(M)$, *where* $C_0(M)$ *is the number of components of M not homeomorphic to a pair of pants.*

Proof. (1) For each $\tau \in G$ we have $\mathscr{A}_\tau \subseteq \mathscr{A}$. By Lemma 2.6 part (3), $\mathscr{A}_\tau$ is an adequate reduction system for τ. Therefore, it suffices to prove that $\mathscr{A}_G$ is a reduction system for τ, i.e., that $\tau(\mathscr{A}_G) = \mathscr{A}_G$ and that the curves in $\mathscr{A}_G$ are pairwise disjoint. Let $\sigma \in G$. Since G is abelian, $\sigma = \tau\sigma\tau^{-1}$. Hence, from Lemma 2.6, part (1), we conclude that $\tau(\mathscr{A}_\sigma) = \mathscr{A}_\sigma$, hence $\tau(\mathscr{A}_G) = \mathscr{A}_G$. Furthermore, if $\alpha_1, \alpha_2 \in \mathscr{A}_G$, we may choose $\sigma_1, \sigma_2 \in G$ such that α_j is an essential reduction class for σ_j, $j = 1,2$. If $i(\alpha_1, \alpha_2) \neq 0$ then $\sigma_1^n(\alpha_2) \neq \alpha_2$, for each $n \neq 0$. But $\sigma_1(\mathscr{A}_{\sigma_2}) = \mathscr{A}_{\sigma_2}$, hence this is impossible. Therefore, $i(\alpha_1, \alpha_2) = 0$ and $\mathscr{A}_G$ is an admissible system.

(2) Let Γ denote the collection of connected components of $M = \coprod_{i \in I} M_i$, and let φ be the representation

$$\varphi : \mathscr{M}(M) \to \operatorname{Aut}(\Gamma(M)).$$

Let $G' = G \cap \operatorname{kernel}\varphi$. Then $\operatorname{rank}(G) = \operatorname{rank}(G')$. Hence we may assume without loss of generality that $G \subseteq \operatorname{kernel}(\varphi)$. Let $\pi_i : \operatorname{kernel}(\varphi) \to \mathscr{M}(M_i)$ be the natural projection. Let $G_i = \pi_i(G)$ and $H = \oplus G_i$. Then G is contained in H, so $\operatorname{rank}(G) \leqslant \operatorname{rank}(H)$. But $\operatorname{rank}(H) = \sum \operatorname{rank}(G_i)$. In addition, G_i is an adequately reduced abelian subgroup of $\mathscr{M}(M_i)$. If M_i is a pair of pants, then $\mathscr{M}(M_i)$ is finite (see [OS]). Therefore $\operatorname{rank}(G_i) = 0$. If $\operatorname{rank}(G_i) > 0$, then G_i contains a pseudo-Anosov class τ. Since G_i is abelian, G_i is contained in the normalizer, $N(\tau)$, of the cyclic subgroup of $\mathscr{M}(M_i)$ generated by τ. By a result proved in [M], any torsion free subgroup of $N(\tau)$ is infinite cyclic. Hence, $\operatorname{rank}(G_i) = 1$ and the assertion follows immediately. ||

Proof of Theorem A. By Lemma 3.1, $G \subseteq \mathscr{M}_{\mathscr{A}_G}(M)$. Let H be the reduction of G along $\mathscr{A}_G$. Then, by Lemma 2.1, there is a short exact sequence:

$$1 \to G \cap Z_{\mathscr{A}_G} \to G \to H \to 1.$$

From the sequence, $\operatorname{rank}(G) = \operatorname{rank}(G \cap Z_{\mathscr{A}_G}) + \operatorname{rank}(H)$. By Lemma 2.1, $\operatorname{rank}(G \cap Z_{\mathscr{A}_G}) \leqslant$ cardinality of $\mathscr{A}_G$. By Lemma 3.1, part (2), $\operatorname{rank}(H) \leqslant C_0(M_{\mathscr{A}_G})$. For each component of $M_{\mathscr{A}_G}$ not homeomorphic to a pair of pants choose a class, $\beta \in \mathscr{S}(M_{\mathscr{A}_G})$, contained in that component. This forms a collection $\mathscr{B}$ which is admissible and whose cardinality is exactly $C_0(M_{\mathscr{A}_G})$. $\mathscr{B}$ lifts to an admissible subset, $\mathscr{A} \subset \mathscr{S}_{\mathscr{A}_G}(M)$, so that $\mathscr{A}_G \cup \mathscr{A}$ is an admissible subset of $\mathscr{S}(M)$. The cardinality of $\mathscr{A}_G \cup \mathscr{A}$ is exactly the cardinality of $\mathscr{A}_G$ plus $C_0(M_{\mathscr{A}_G})$. Our assertion concerning $\operatorname{rank}(G)$ follows, because the cardinality of an admissible subset of $\mathscr{S}(M)$ is bounded above by $3g + b - 3c$. This proves that the torsion free rank of G is finite. To see that G is finitely generated, therefore, it suffices to show that $\operatorname{Tor}(G)$ is finite. But this follows from the fact that $\mathscr{M}(M)$ contains a torsion free subgroup of finite index.

In order to see this last assertion, we need to reduce to known results. If M is connected and closed, then this result is proved in [H2] and [B-L]. If M is connected and has boundary, then $\mathscr{M}(M)$ is a subgroup of Out(F), where F is a free group of finite rank [N2]. Since Out(F) is virtually torsion free, (cf. [B-L]), the theorem follows immediately. If M is not connected, then, since kernel(φ) is of finite index in $\mathscr{M}(M)$ and since kernel(φ) $\cong \oplus \mathscr{M}(M_i)$, we are reduced to considering the mapping class group, $\mathscr{M}(M_i)$, of a connected component of $\mathscr{M}(M)$, and so we are done.

4. Virtually solvable subgroups of $\mathscr{M}(M)$. In this section we prove Theorem B. Let G be a virtually solvable subgroup of $\mathscr{M}(M)$, and G_0 a normal, solvable subgroup of G, of finite index in G. If G_0 is nontrivial, let H be the last nontrivial term in the derived series for G_0. Since H is a characteristic, normal subgroup of G_0, and G_0 is normal in G, then H is a normal subgroup of G. Furthermore, H is a nontrivial, abelian, normal subgroup of G. We shall say that a subgroup, $H \subseteq G$, is a *preferred subgroup of* G if H is a nontrivial, abelian, normal subgroup of G. In the examples of Section 1, $H = gp\{A_0, A_1, A_2, A_3\}$ is a preferred subgroup. From the above construction, if G does not admit a preferred subgroup, then G_0 is trivial and therefore G is finite. The simplest situation we shall need to consider occurs when M is connected, G is torsion free and G admits an adequately reduced, preferred subgroup. If G satisfies these three conditions, we say that G is *primitive*.

LEMMA 4.1. *If G is primitive, then G is infinite cyclic and generated by a pseudo-Anosov mapping class.*

Proof. Let H be an adequately reduced, preferred subgroup of G. The hypothesis implies that H is torsion free, nontrivial and adequately reduced. Since M is connected, Lemma 3.1 implies that rank$(H) = 1$, and hence H is infinite cyclic and generated by a pseudo-Anosov mapping class, τ. Since H is normal in G, G is a torsion free subgroup of the normalizer, $N(H)$. By a result proved in [M], G is infinite cyclic. Since $\tau \in G$, and a mapping class is pseudo-Anosov if and only if a nontrivial power of it is pseudo-Anosov, then G is also generated by a pseudo-Anosov mapping class. ||

Proof of Theorem B. If G $\subseteq$ kernel(φ) and each nontrivial restriction, $\pi_i(G)$, is primitive, then G *has primitive restrictions*. We shall prove that there exists a subgroup, H, of G, of finite index in G, and an admissible set, $\mathscr{A} \subset \mathscr{S}(M)$, such that $H \subseteq \mathscr{M}_{\mathscr{A}}(M)$ and the reduction, $\wedge(H)$, of H along $\mathscr{A}$ has primitive restrictions. This is the first step. In order to make an estimate on $[G : H]$, the index of H in G, we introduce the following notations. If N is a connected manifold, let $T(N)$ denote a torsion free subgroup of $\mathscr{M}(N)$, of finite index, $t(N)$, in $\mathscr{M}(N)$. (If N is closed, then such a subgroup exists (cf. [H2], [B-L]). If N has boundary, then $\mathscr{M}(N)$ is a subgroup of the outer automorphism group, Out(F), of a free group, F, of finite rank [N2]. Since Out(F) is virtually torsion free, again such a subgroup exists.) Similarly, $T(M)$ denotes $\oplus T(M_i)$, and $t(M)$ its index in $\mathscr{M}(M)$. A simple calculation shows that $t(M) \leqslant [(c(M)!) :$

$(\prod t(M_i))]$. Finally, if H is a subgroup of $\mathscr{M}(M)$, then $H_T = H \cap T(M)$. Note, by our definitions, $H_T \subset \operatorname{kernel}(\varphi)$.

If G_T has primitive restrictions, then by Lemma 4.1, $\oplus \pi_j(G_T)$ is free abelian and hence G_T is free abelian. Also $[G : G_T] \leqslant t(M)$. Hence, in this event we are done. We shall now show that if G_T does not have primitive restrictions, then there exists an admissible set, $\mathscr{A} \subset \mathscr{S}(M)$, with $G_T \subset \mathscr{M}_{\mathscr{A}}(M)$. Hence, assume G_T does not have primitive restrictions; at least one nontrivial restriction, $\pi_i(G_T)$, is not primitive. By construction $\pi_i(G_T)$ is torsion free and virtually solvable; hence $\pi_i(G_T)$ admits a preferred subgroup, H, which is not adequately reduced. By Lemma 3.1, $\mathscr{A}_H \subset \mathscr{S}(M_i)$ is a nontrivial adequate reduction system for H, where $\mathscr{A}_H$ is the union of the essential reduction systems, $\mathscr{A}_\tau$, $\tau \in H$. Since H is normal in $\pi_i(G_T)$, for each $\sigma \in \pi_i(G_T)$ and $\tau \in H$, $\sigma\tau\sigma^{-1} \in H$. Therefore, by Lemma 2.6 (1), $\sigma(\mathscr{A}_H) \subseteq \mathscr{A}_H$. Similarly, $\sigma^{-1}(\mathscr{A}_H) \subseteq \mathscr{A}_H$, so $\mathscr{A}_H \subseteq \sigma(\mathscr{A}_H)$, so $\sigma(\mathscr{A}_H) = \mathscr{A}_H$. If $\sigma \in \pi_j(G_T)$, where $j \neq i$, then clearly $\sigma(\mathscr{A}_H) = \mathscr{A}_H$. Since $G_T \subseteq \oplus \pi_j(G_T)$, it immediately follows that $G_T \subset \mathscr{M}_{\mathscr{A}}(M)$, where $\mathscr{A} = \mathscr{A}_H$.

At this point, the reader may wish to refer to the diagram below as we proceed through the next argument:

$$\begin{tikzcd}
G = G_0 \supseteq & G_1 \arrow[d, "\wedge_1"] & \supseteq & G_2 \arrow[d, "\wedge_1"] & \supseteq & G_3 \arrow[dd, "\wedge_2"] & \supseteq \cdots\cdots \supseteq & G_k \arrow[ddddd, "\wedge_{k-1}"] & \supseteq & G_{k+1} = H \arrow[dddddd, "\wedge_k = \wedge"] \\
& G^1 & \supseteq & G^1_T \arrow[d, "\wedge'_1"] & & & & & & \\
& & & G^2 & \supseteq & G^2_T \arrow[d, "\wedge'_2"] & & & & \\
& & & & & G^3 \supseteq \cdot & & & & \\
& & & & & & \ddots & & & \\
& & & & & & \supseteq & G^{k-1}_T \arrow[d, "\wedge'_{k-1}"] & & \\
& & & & & & & G^k & \supseteq & G^k_T = \wedge(H)
\end{tikzcd}$$

Let $G_1 = G_T$ and $\mathscr{A}_1 = \mathscr{A}$, so that $G_1 \subset \mathscr{M}_{\mathscr{A}_1}(M)$. Let $\wedge_1 : \mathscr{M}_{\mathscr{A}_1}(M) \to \mathscr{M}(M_{\mathscr{A}_1})$ be the reduction homomorphism, and $G^1 = \wedge_1(G_1) \subset \mathscr{M}(M_{\mathscr{A}_1})$. Let $G_2 = \wedge_1^{-1}(G^1_T) \cap G_1$, so $G_2 \subseteq G_1$, $[G_1 : G_2] = [G^1 : G^1_T] \leqslant t(M_{\mathscr{A}_1})$ and $\wedge_1(G_2) =$

G_T^1. If G_T^1 has primitive restrictions, then G_2 is a subgroup of G, of finite index in G, which reduces along $\mathscr{A}_1$ to a group with primitive restrictions. If G_T^1 does not have primitive restrictions, then as before there exists an admissible set, $\mathscr{A} \subset \mathscr{S}(M_{\mathscr{A}_1})$, such that $G_T^1 \subset \mathscr{M}_{\mathscr{A}}(M_{\mathscr{A}_1})$. Let $\mathscr{A}_2$ be the union of $\mathscr{A}_1$ and the projection of $\mathscr{A}$ to $\mathscr{S}(M)$, then $\mathscr{A}_2$ is an admissible set, $\mathscr{A}_1$ is properly contained in $\mathscr{A}_2$ and $G_2 \subseteq \mathscr{M}_{\mathscr{A}_2}(M)$. (In the diagram above, $\wedge_1'$ denotes the reduction homomorphism associated to $\mathscr{M}_{\mathscr{A}}(M_{\mathscr{A}_1})$, and $\wedge_2$, the reduction homomorphism for $\mathscr{M}_{\mathscr{A}_2}(M)$. Clearly, $\wedge_2 = \wedge_1' \circ \wedge_1$.)

Continuing in this manner we construct a descending sequence of subgroups, $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k \supseteq G_{k+1}$, and a strictly increasing sequence of admissible subsets of $\mathscr{S}(M)$

$$\emptyset = \mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots \subset \mathscr{A}_k$$

such that $G_k \subseteq \mathscr{M}_{\mathscr{A}_k}(M)$, $[G_k : G_{k+1}] \leqslant t(M_{\mathscr{A}_k})$, and the reduction, $\wedge_k(G_{k+1})$, of G_{k+1} along $\mathscr{A}_k$ either has primitive restrictions or admits a nontrivial reduction system. In the latter event, we augment the two sequences by introducing G_{k+2} and $\mathscr{A}_{k+1}$ as above. Since $k \leqslant$ cardinality of $\mathscr{A}_k \leqslant 3g + b - 3c$, it is impossible to augment $\mathscr{A}_k$ for $k \geqslant 3g + b - 3c$. Therefore, for some $k \leqslant 3g + b - 3c$, $\wedge_k(G_{k+1})$ has primitive restrictions. Setting $H = G_{k+1}$, $\mathscr{A} = \mathscr{A}_k$ and $\wedge = \wedge_k$, we have completed the first step of the proof; that is, we have obtained a subgroup, H, of G, of finite index in G and an admissible set, $\mathscr{A} \subset \mathscr{S}(M)$, such that $H \subseteq \mathscr{M}_{\mathscr{A}}(M)$ and the reduction, $\wedge(H)$, of H along $\mathscr{A}$ has primitive restrictions. In fact, a simple calculation shows that $[G : H] \leqslant \prod_{i=0}^{k} t(M_{\mathscr{A}_i})$ where $k \leqslant 3g + b - 3c$.

The next step of the proof is to find a subgroup of finite index in H which is free abelian. Let $F = \wedge(H) \subseteq \operatorname{kernel}(\varphi)$, where φ is the natural map $\mathscr{M}(M_{\mathscr{A}}) \to \operatorname{Aut}(\Gamma(M_{\mathscr{A}}))$. By construction of H, F has primitive restrictions. By Lemma 4.1, therefore, F is free abelian. By Lemma 2.1 we have a short exact sequence, $1 \to \mathbf{Z}_{\mathscr{A}} \cap H \to H \xrightarrow{\wedge} F \to 1$, where $\mathbf{Z}_{\mathscr{A}} \cap H$ is also free abelian. If this sequence were a split central extension, then H would be free abelian as well. We must pass to a subgroup of finite index of H to insure that the sequence will be split central. Let $\partial : \mathscr{M}(M_{\mathscr{A}}) \to \operatorname{Aut}(\partial M_{\mathscr{A}})$ be the natural representation, and $F^{\partial} = F \cap \operatorname{kernel}(\partial)$, and $E = \wedge^{-1}(F^{\partial})$. Again, by Lemma 2.1 we have a short exact sequence:

$$1 \to \mathbf{Z}_{\mathscr{A}} \to E \to F^{\partial} \to 1 \qquad (*)$$

Now we shall construct a splitting, $\lambda : F^{\partial} \to E$.

Since F has primitive restrictions and $F^{\partial} \subseteq F$, by Lemma 5.1, the jth restriction, $\pi_j(F^{\partial}) \subset \mathscr{M}(M_{\mathscr{A}_j})$, is trivial or infinite cyclic. If infinite cyclic, then the restriction is generated by the class of a homeomorphism, s_j, of $M_{\mathscr{A}_j}$ which fixes each boundary component of $M_{\mathscr{A}_j}$ pointwise. Viewing $M_{\mathscr{A}_j}$ as a submanifold of M we may extend s_j trivially to a homeomorphism, t_j, of M. If

$j \neq k$, then t_j and t_k will commute, since they are supported on disjoint submanifolds. Let $\sigma_j \in \mathcal{M}(M_{\mathcal{A}_j})$ be the class of s_j and $\tau_j \in \mathcal{M}(M)$ the class of t_j. Since $\oplus \pi_j(F^\partial)$ is free abelian with free basis, $\{\sigma_j\}$, the association $\sigma_j \to \tau_j$ extends uniquely to a homomorphism, $\lambda : \oplus \pi_j(F^\partial) \to \mathcal{M}(M)$. The restriction of λ to F^∂ is, by construction, a splitting, $\lambda : F^\partial \to E$, as desired. Furthermore, since $F^\partial \subseteq$ kernel(∂), by Lemma 2.1 (1), we conclude that $\mathbf{Z}_{\mathcal{A}} \subseteq$ center of E. Hence the sequence, (*), is a split central extension, and E is free abelian.

Let $K = E \cap H$, then K is a free abelian subgroup of H, and $[H : K] = [F : F^\partial] \leqslant b(M_{\mathcal{A}})!$. Therefore, we conclude that

$$[G : K] \leqslant \left[\prod_{i=0}^{k} t(M_{\mathcal{A}_i}) \right] \cdot [b(M_{\mathcal{A}})!] \qquad (**)$$

where

$$k \leqslant 3g + b - 3c$$

It remains to be shown that the right hand term of (**) is bounded by a function of M.

For each i, let $M_{\mathcal{A}_i} = \coprod M_{i,j}$. Then, as mentioned earlier, $t(M_{\mathcal{A}_i}) \leqslant [c(M_{\mathcal{A}_i})!] \cdot [t(M_{i,j})]$. Each $M_{i,j}$ corresponds to a connected submanifold of M with negative Euler characteristic. Since there are only a finite number of possibilities up to homeomorphism, then we may choose a universal bound, $u(M)$, for $\{t(M_{i,j})\}$. Furthermore, $c(M_{\mathcal{A}_i}) \leqslant 2g + b - 2c = |\chi(M)|$; $b(M_{\mathcal{A}}) \leqslant 6g + 3b - 6c$. Together with (**), we obtain an upper bound for $[G : K]$, that is:

$$[G : K] \leqslant \left\{ [(2g + b - 2c)!] [u(M)]^{2g+b-2c} \right\}^{3g+b-3c} \cdot (6g + 3b - 6c)!$$

This proves that every virtually solvable subgroup, G, of $\mathcal{M}(M)$ contains an abelian subgroup, K, of index bounded by $V(M)$.

Remark. We understand the derived length of an abelian subgroup to be 1. Hence, if G is solvable, it is clear that $d(G) \leqslant V(M)$ as well.

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