Lorenz knots and links

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LORENZ KNOTS AND LINKS

Overview and plan for this talk:

(I). INTRODUCTION to braids, knots, links.

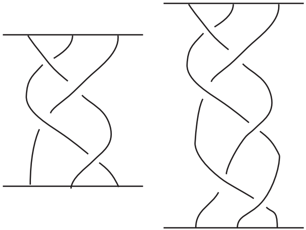
Lorenz links are a special family of knots and links. They appear in several very different ways, in mathematics. Will discuss each of them.

(II). Lorenz knots and links via NON-LINEAR ODE's.

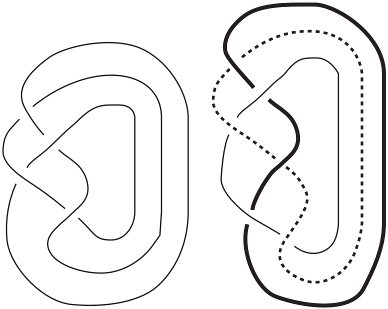
(III). Lorenz links and the GEOMETRY AND TOPOLOGY OF 3-MANIFOLDS. (Joint work with Ilya Kofman).

(IV). Ghys's Theorem: MODULAR KNOTS are Lorenz knots in disguise.

(I) INTRODUCTION: Braids, Knots and Links.

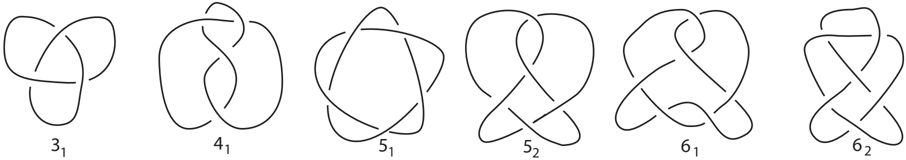


Two 3-strand braids



Two closed 3-braids

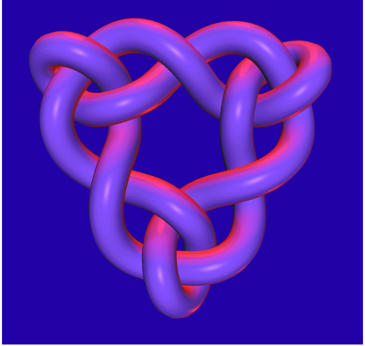
Construction of ‘knot tables’ began with pictures of lowest-crossing knots. Peter Guthrie Tait. End of the 19th century.



Experimental evidence: the problem of distinguishing knots is interesting, difficult. Need computable invariants.

The number of components of a link is an example of a computable invariant. Gauss's linking number another. Both undefined for knots.

$$\int_K \int_K' \left(\frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{3/2}} \right)$$



Example of an alternating knot (picture by Morwen Thistlethwaite)

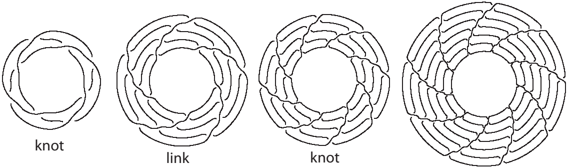
Tait observed that all the knots in his early tables had alternating (and non-alternating projections. He conjectured:

(i)After possibly small modifications, if a knot has an alternating projection, then it realizes min. crossing number' (true).

(ii)Every knot has an alternating projecture (false).

Alternating knots are a special class of knots. Much more tractible that the class of all knots. (i) is one of their special properties.

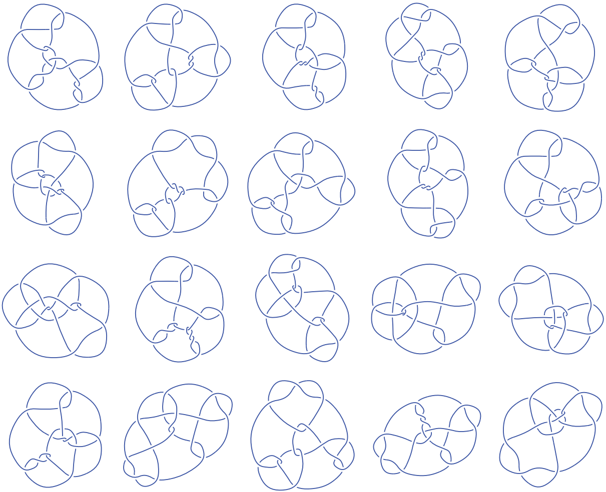
Another special class: Torus knots and links are links that can be embedded on a torus of revolution in $\mathbb{R}^3$. Well-understood class. Classified by 2 integers (p, q). They are not alternating.



The recognition problem for torus links is to decide whether a knot or link is a torus link. It's hard. If you can solve it, then two integers suffice to determine its type.

The Jones polynomial: a more sophisticated tool for distinguishing knots. Defined on all knots and links. The Jones polynomial of a type (p, q), n-component torus link is:

$$V(t) = \frac{t^{\frac{1}{2}(p-1)(q-1)}}{1-t^2} \sum_{j=0}^{n} \binom{n}{j} t^{\frac{p}{n}(1+\frac{q}{n}j)(n-j)} \left(t^{\frac{p}{n}(n-j)} - t^{1+\frac{q}{n}j}\right)$$



20 distinct knots with the same Jones polynomial
(courtesy of Morwen Thistlethwaite)

Like alternating knots and links, and torus knots and links, Lorenz knots and links are a special class.

When we encounter a new and interesting family of knots and links, especially one which arises in several different, seemingly unrelated settings (as do Lorenz knots and links), some questions we would like to answer:

- Which knots and links occur? An interesting question, right now.
- Can we solve the recognition problem for the class? No!!
- How rich is the class? Answered with lots of detail in 1983, but new viewpoints were added in 2006 (Ghys) and 2009 (B-Kofman).
- Does the knotting and linking have meaning as regards the setting in which they were discovered? Partial answers exist for dynamics; new answers by Ghys, as regards number theory.

(II) NON-LINEAR ODE's: Introduce Lorenz links. First appeared in study of certain non-linear ODE's that are of interest in dynamical systems.

E. N. Lorenz, *Deterministic, non-periodic flows*, J. Atmospheric Sciences, 1963. Is weather fundamentally deterministic? Many many variables. Somtimes seems unpredictable. Why? Does deterministic $\Longrightarrow$ ultimately periodic? If so, then weather can't be deterministic.

Starting point: Navier-Stokes equations. Describe dynamics of a viscous, incompressible fluid. Used to model weather, ocean currents, water flow in a pipe, flow around an airfoil, motion of stars inside a galaxy. All very complex problems.

If a problem is too complicated, try to simplify it, preserve its essential features. discard those which are unimportant.

Discarded variables, modified equations, testing each time he changed things via numerical integration. After long (and interesting) search found a very simple system with features that he had noticed, and wanted to retain. Modified equations governed fluid convection in a very thin disc, heated below and cooled above. System of 3 ODE's :

$$\frac{dx}{dt} = 10(y - x), \ \frac{dy}{dt} = 28x - y - xz, \ \frac{dz}{dt} = xy - \frac{8}{3}z$$

Four variables: $x, y, z \in \mathbb{R}$ and $t =$ time.

$y =$ horizontal temperature variation.

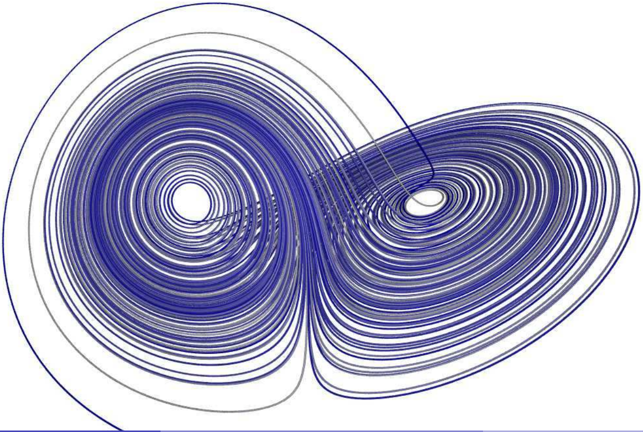
$x =$ vertical temperature variation.

$z =$ rate of convective overturning.

Robust. Vary 10, 28, 8/3 in open set. Behavior doesn't change.

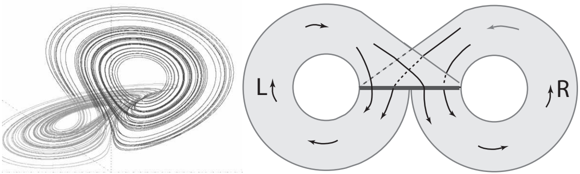
Integrate ODE's in $\mathbb{R}^3$. Orbits determine a flow $\lambda^t : \mathbb{R}^3 \to \mathbb{R}^3$

Lorenz proved there exists a bounded ellipsoid $\mathcal{A} \subset \mathbb{R}^3$ such that forward trajectories $\Longrightarrow \mathcal{A}$ and stay there. $\mathcal{A}$ an Attractor. BUT: orbits extremely sensitive to initial conditions. Start at points that are close, may be very far apart at later time, even though both inside $\mathcal{A}$. 'Butterfly effect'. Prototype for chaos. Here's a picture of what he saw. $\mathcal{A} =$ nbhd of a butterfly-shaped region. Layering of orbits.



Definition: A Lorenz knot is a periodic orbit in the flow on $\mathbb{R}^3$ determined by the Lorenz ODEs. Lorenz Link = finite collection of orbits.

Studied via the *Lorenz template* – Guckenheimer and Williams – an embedded branched surface in $\mathbb{R}^3$ supporting a semi-flow. All closed orbits in flow embed on the template, 1-1, simultaneously.

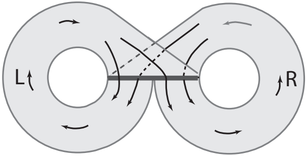


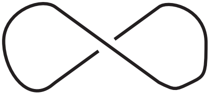
The Lorenz attractor and the Lorenz template

Existence of template supported by numerical evidence in 1970's. Rigorous proofs constructed by Tucker(2002) and Ghys (2006).

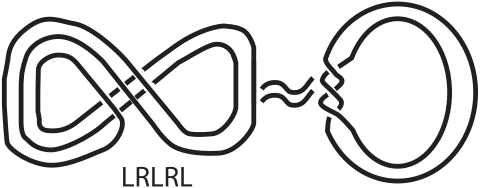
Look carefully at the template. Supports a semiflow. Study typical orbits, embedded in template. Almost all knotted. The template allows us to study Lorenz knots and links, as a class. The first studies of knotted orbits in ODE's were those of Birman and Williams, 1983. Studied knot and link types of Lorenz knots and links.

Can use words in R and L to describe them. Unknot occurs. Trefoil occurs too. For more complicated examples, may be hard to recognize the knot.

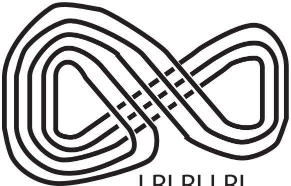




LR



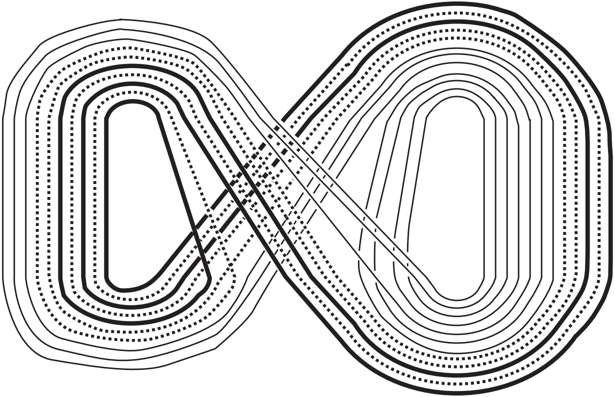
LRLRL



LRLRLLRL

Example: a 3-component Lorenz link, each component described by its word:

Thickest line: U = LRLRL (that was our trefoil)
Dotted line: V = LRLRLRL (a type (3,5) torus knot)
Thinnest line: W = LRLRRRLRRR (type (-3,-7,2) 'pretzel knot'.)



Symbolic dynamics is a tool introduced to study solutions to ODE's like the Lorenz equations, which can't be integrated in closed form.

Regard a Lorenz knot or link to be any link which has a representative that can be embedded on the Lorenz template. (By Tucker and Ghys, nothing lost.) Will use symbolic dynamics to study them.

1983– Williams – Describe orbits on template by cyclic words in R and L (No natural start-point). Don't want our word to be periodic. And for links, we don't want the word for one component to be a power of that for another. Are there other restrictions?

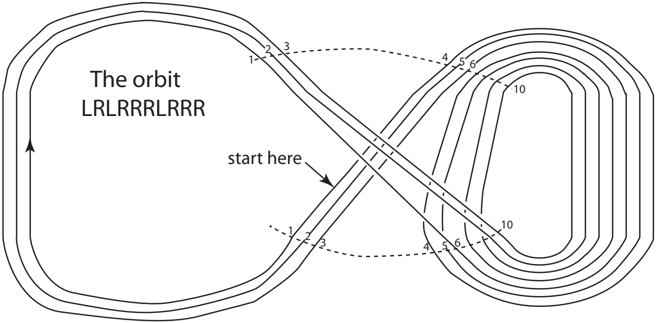
Theorem: [Williams, 1970's] A family $\{W_1, \ldots, W_k\}$ of cyclic words represents a Lorenz link iff no W_i is periodic, and no W_i is (up to cyclic permutation) a power of any W_j.

Call admissible words Lorenz words.

Question: Can we recover orbit from its word? How?

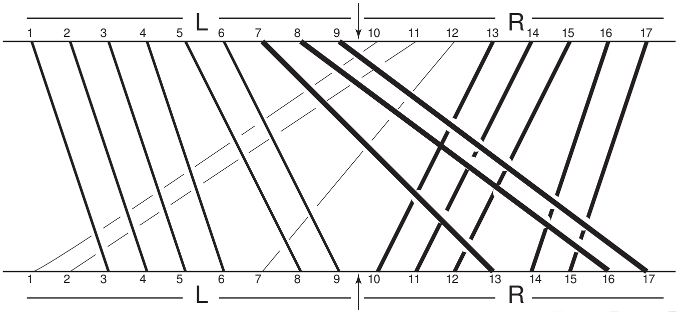
Illustrate via an example. Write down the cyclic permutations of the word. Order lexicographically, using the rule L < R, to recover the knot.

1	LRLRRRLRRR	5	RLRRRLRLRR
6	RLRRRLRRRL	2	LRRRLRLRRR
3	LRRRLRRRLR	9	RRRLRLRRRL
10	RRRLRRRLRL	7	RRLRLRRRLR
8	RRLRRRLRLR	4	RLRLRRRLRR

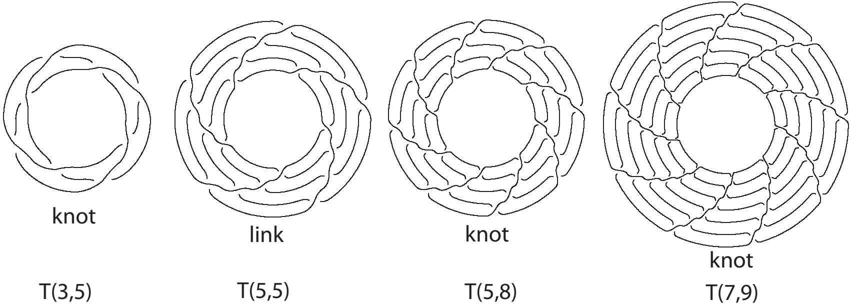


In this way Williams proved {Lorenz words} $\leftrightarrow$ {Lorenz links}.

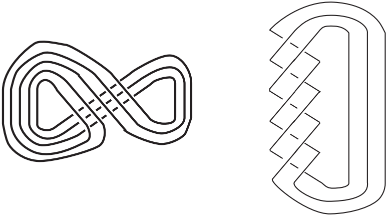
1983: Birman-Williams studied knot types. There are many ways to code Lorenz knots and links. In [B-W] used Lorenz braids. Overcrossing strands determine undercrossing strands. Overcrossing strands travel in "parallel packets". Slope of each packet is determined by the change in x-coordinate. In example describe Lorenz braid by the symbol $(2^4, 3^2, 6, 8^2)$. Every such symbol determines a Lorenz braid, and so a Lorenz link. Note that the overcrossing strands determine the undercrossing strands.



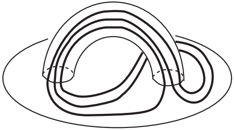
Here is one family of knots and links that embed in the template. It turns out that we know it already, it's the family of torus knots and links.



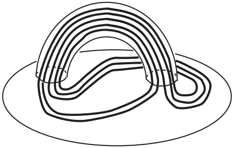
Torus knots and links of all types (p, q), $p \leq q$ occur among Lorenz links. Appear as Lorenz braids, 1 bundle of q parallel overcrossing arcs, subbundles of $p, q - p$ strands.



Can see why, when there is only one parallel packet of strands, a Lorenz knot embeds on a standard torus in 3-space.



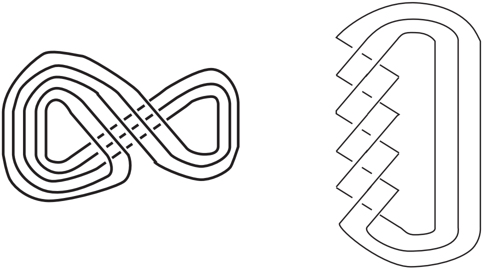
Type (2,3)



Type (3,5)

The Lorenz projections are different from the familiar projections of torus knots and links.

Closed 8-braid vs closed 3-braid for type (5,3) torus knot.



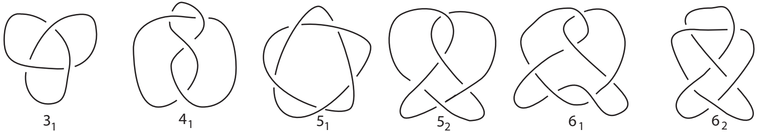
Some special properties of Lorenz knots and links [B-W, 1983]:

- All Lorenz links are prime, and are fibered.
- Link genus determined combinatorially. $2g = c - n + 2 - \mu$.
- Braid index is too. If $W = \prod_{i=1}^{t}(\mathrm{R}^{n_i}\mathrm{L}^{m_i})$, then $t =$ braid index.

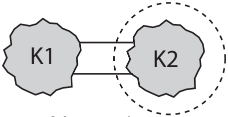
(III) Geometry and Topology of 3-manifolds. Look at Lorenz knots from a new viewpoint.

Early knot tables organized by crossing number = min. for all projections onto a plane. Completed up to 10 crossings, by hand, by 1936. Extended 1998 to ≤16 crossings by 2 teams, working independently, and using computers and all known computable invariants to distinguish them. There are precisely 1,701,936 knots with ≤ 16 crossings.

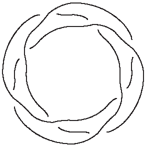
Ghys, Dehornoy,Jablan: 1,701,936 knots, only 20 are Lorenz. Of those, only 7 are non-torus Lorenz. Lorenz knots seem to be an unfamiliar family.



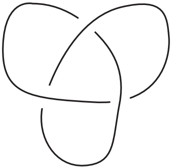
Geometrization Conjecture [Perelman, 2006]. For knots and links [Thurston, 1982]: If a knot is prime, has no essential tori in complement, then admits a finite-volume hyperbolic metric. Mostow Rigidity then says metric is unique.

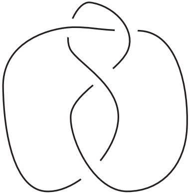


Not prime


Has essential torus

The Trefoil knot (the knot 3_1 in the knot tables), and in fact all torus knots are not hyperbolic. But the Figure-8 knot (the knot 4_1 in the knot tables) is hyperbolic.


3_1


4_1

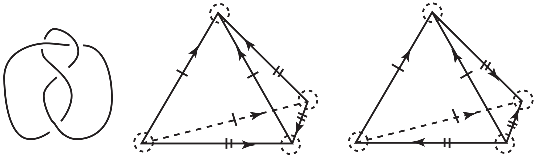
Thurston's Geometrization Conjecture related, initially, to knot complements.

1978 – Famous lecture notes, began with a key example. Studied the complement of the Figure 8 knot.

Poincare sphere gives model for hyperbolic 3-space. Use it to define hyperbolic tetrahedra.

Thurston exhibited hyperbolic structure on $(S^3 \setminus 4_1)$ – by constructing it from 2 ideal hyperbolic tetrahedra (vertices missing, tetrahedral subset of $\mathbb{H}^3$, all dihedral angles $\pi/3$). Later, proved that similar description holds for any hyperbolic knot or link complement.

Thurston's ideal triangulation of the complement of the knot 4_1



While a natural measure of complexity of a knot diagram is crossing number, for a hyperbolic knot K want 'geometrically meaningful complexity'. If $S^3 - K$ is hyperbolic, count minimum number of ideal tetrahedra needed to construct $S^3 - K$. It's two for $S^3 \setminus 4_1$. Good measure of complexity. Hyperbolic volume is a finer measure. Active area of research: the most meaningful measure.

New computer tool: SnapPea. Knot spaces are one-cusped manifolds.

Feed SnapPea a knot diagram, and it determines whether complement has a hyperbolic structure. If so, complement can be built up by an ideal triangulation. 'Ideal' means that vertices are missing (because they are on $\partial\mathbb{H}^3$).

SnapPea does this by solving a system of linear equations (the glueing and consistency equations. Hyperbolic structure exists iff equations have a solution. If they have a solution, SnapPea gives it.

SnapPea ‘census’ (Callahan, Weeks, Dean, Champanerkar, Kofman, Patterson)

There exist exactly 201 hyperbolic knots with ≤ 7 ideal tetrahedra in complementary space.

Identify them by finding diagrams – took lots of patience.

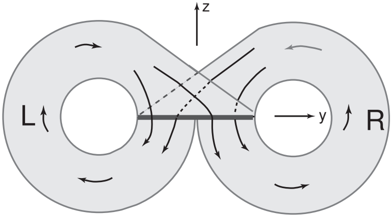
Diagrams show repeatedly: The knots in the census are obtained from low hyperbolic volume knots by adding twists to some of the strands.

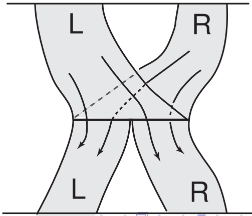
Birman and Kofman (2008) More than half of the simplest hyperbolic knots are Lorenz knots! None of these are torus knots because torus knots are not hyperbolic. In striking contrast to the fact that 7 of the 1,701,936 knots with ≤ 16 crossings are non-torus Lorenz knots.

We want to understand what the data means (but only have limited answers now). Why are so many 'small' hyperbolic knot complements those of Lorenz knots? Need to explain how we solved the Lorenz recognition problem in enough cases to understand the consequences.

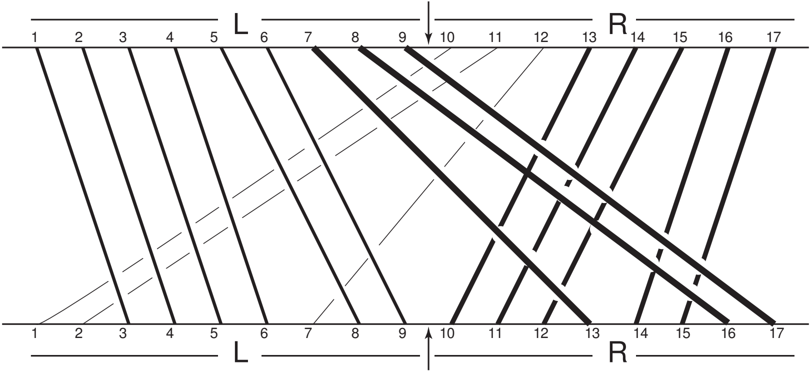
Back to the Lorenz template. By definition, Lorenz knots are all the knots that embed on the template. Think how to study the template, instead of studying orbits, one at a time. Branch line divides naturally into a left side and a right side. Strands fall into 4 groups: types LL, LR. RL and RR. Overcrossing strands are types LL and LR.

There is an associated braid template. Braid strands are also types LL, LR, RL, RR.



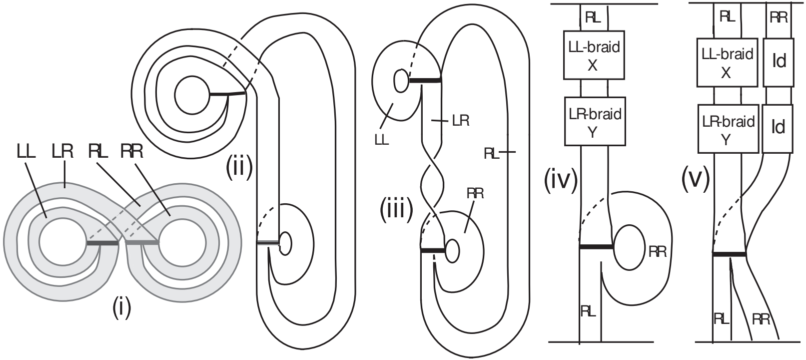


Look at an example of a Lorenz braid. Can see the 4 groups of braid strands. Each packet consists of some number of sets of strands that travel parallel to one-another, in a group.



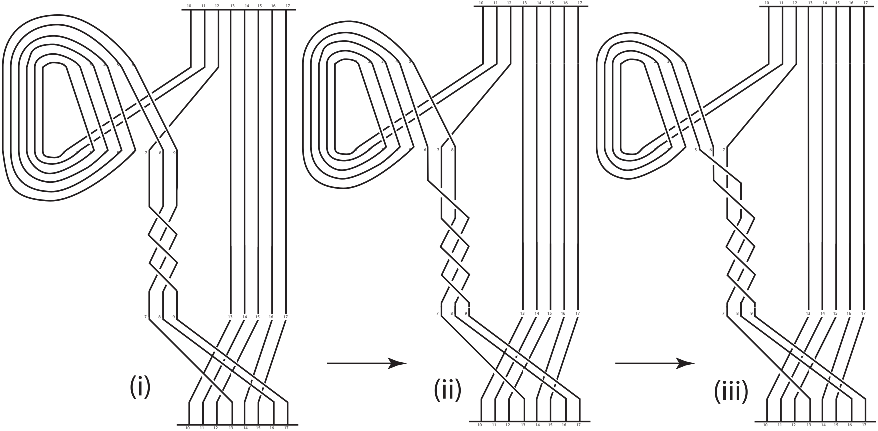
Cut open the template along an orbit, splitting it into 4 pieces: LL strands, LR strands, RL strands and RR strands.

Natural new braid structure. Observe that the LR braid is one full twist of the LR band, so it depends only on the number of LR strands in a Lorenz knot. A miracle is that this number turns out to be the braid index of the Lorenz knot. The LL braid is more complicated.



The LL braid is where all the action occurs.

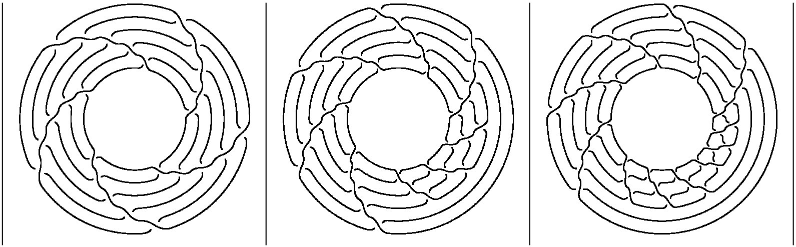
Look at an example.



The uncoiling of bands the LL band in the Lorenz projection introduces twists. The twisting turned out to be a twist of subset of the strands on a basic torus link. This suggested connections with knot census.

Theorem: [Birman-Kofman,2007] The most general Lorenz link is obtained from a torus link by repeatedly twisting special subsets of its strands. The result is a "multiply twisted" torus link. In the general case, this multiply twisted torus link is hyperbolic.

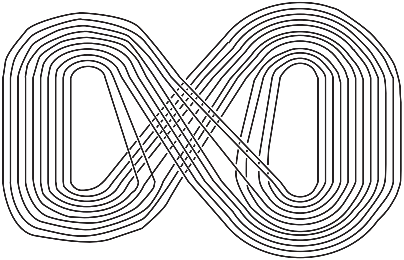
$\langle 6^5 \rangle$ $\quad$ $\langle 6^5, 4^3 \rangle$ $\quad$ $\langle 6^5, 4^3, 3^3 \rangle$



Corollary: Choose any positive integer N. Let $\mathcal{L}$ be a Lorenz link, defined via its embedding on the Lorenz template. Suppose that the associated braid has at most N overcrossing strands. Then the hyperbolic volume of the complement of $\mathcal{L}$ is bounded by a constant that depends only on N.

Proof uses Thurston's 'Dehn Surgery Theorem'

Remark: Lorenz projections seem to be about as far from alternating projections as possible.



Typical orbits

Review our plan for this talk:

(I). INTRODUCTION to braids, knots, links.

(II). Lorenz knots and links via NON-LINEAR ODE's.

(III). Lorenz links and the GEOMETRY AND TOPOLOGY OF 3-MANIFOLDS.

(IV). Ghys''s theorem: MODULAR KNOTS are Lorenz knots in disguise.

IV MODULAR KNOTS

Think of S^3 as the unit sphere in $\mathbb{C}^2$, and the trefoil knot (an 'algebraic knot') as $S^3 \cap \Delta$, where Δ is the algebraic surface:

$$\Delta = \{z_1, z_2 \in \mathbb{C} \ \text{ satisfying } \ z_1^3 - 27z_2^2 = 0\} \subset \mathbb{C}^2. \tag{1}$$

Set $S^3 - \mathcal{T}$ = complement of the trefoil knot. Well-known that $\exists$ isomorphism: $\mathbb{S}^3 \setminus \mathcal{T} \cong PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$.

To prove it, recall that a lattice Λ is an additive subgroup of $\mathbb{C} = \{n_1 z_1 + n_2 z_2 \ | \ n_1, n_2 \in \mathbb{Z}, \ \ z_1, z_2 \in \mathbb{C}$ linearly independent$\}$.

The quantity $z_1^3 - 27z_2^2$ is called the 'discriminant' of the lattice. It vanishes as the lattice squashes down to being degenerate, i.e. to a discrete subgroup of $\mathbb{C}$ with one rather than two generators.

Can identify {non-degenerate lattices $\mathcal{L} \subset \mathbb{C}$} with (a) $S^3 \setminus \mathcal{T}$ and (b) $PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$.

Next, consider a flow, on $S^3 \setminus \mathcal{T}$. (Many any interesting flows.)

Thinking of $S^3 \setminus \mathcal{T}$ as $PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$, define the matrix:

$$H(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

For fixed t it's in $PSL(2,\mathbb{R})$, so it can be used to define a flow $\Phi_t : SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$ defined by $\Phi_t : A \to H(t)A$.

This flow known as the modular flow. What are its closed orbits?

$A \in PSL(2,\mathbb{R})$ in a closed orbit if $\Phi_t : A \to AP$ for some $P \in SL(2,\mathbb{Z})$.

Put it another way, $\exists t$ such that at that t, $A = PA$. Thus $\{$ closed orbits in flow $\} \iff \{$conjugacy classes $APA^{-1} \subset PSL(2,\mathbb{Z})\}$.

Closed orbits represent free homotopy classes in $S^3 \setminus \mathcal{T}$. Since they live in S^3, they determine knots and links.

Look at free homotopy classes in $S^3 \setminus \mathcal{T}$. We know
$G = \pi_1(S^3 \setminus \mathcal{T}) :< U, V; \quad U^2 = V^3 >$

Let $C = U^2 = V^3$. Then $C \in$ center(G) and every free homotopy class is represented by a cyclic word in U and V of the form
$W = C^k UV^{\epsilon_1} UV^{\epsilon_r} \cdots UV^{\epsilon_r}, \quad \epsilon_i = \pm 1.$

Also known: There is a homomorphism $\tau : G \to SL(2, \mathbb{Z})$, with kernel$(\tau)$ generated by C.

Since $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/ \pm I$, conjugacy classes in $PSL(2, \mathbb{Z})$ are represented by words: $W = UV^{\epsilon_1} UV^{\epsilon_r} \cdots UV^{\epsilon_r}, \quad \epsilon_i = \pm 1$, where:

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \; R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Set $L = UV, \quad R = UV^{-1}$. Then W goes over to a word in L, R:

$$UV^{\epsilon_1} UV^{\epsilon_r} \cdots UV^{\epsilon_r} \to L^{\alpha_1} R^{\beta_1} L^{\alpha_2} R^{\beta_2} \cdots L^{\alpha_t} R^{\beta_t}$$

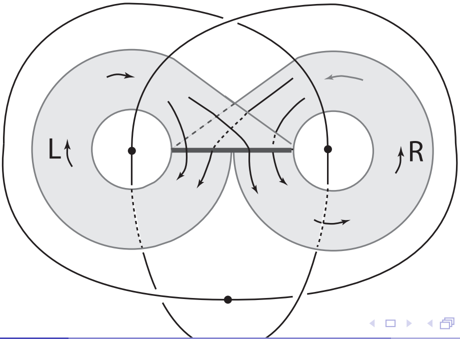
Left-right words begin to look a little bit like Lorenz knots!

Theorem

[Etienne Ghys, 2006] There is a one-to-one correspondence between modular knots and links and Lorenz knots and links.

Ghys' proof is very constructive. He uses structures set up to justify existence of template, as in [B-W]. Those structures appear naturally in work of Ghys, and the template does too.

Proof shows position of 'trefoil with modular eyeglasses'!!



Final remarks, The Rademacher function $\mathcal{R} : PSL(2,\mathbb{Z}) \to \mathbb{Z}$

To define $\mathcal{R}$, let $z \in \mathbb{C}^+$. The Dedekind η function is defined by

$$\eta(z) = e^{\frac{i\pi z}{12}} \prod_{n=1}^{n=\infty} (1 - e^{2\pi inz})$$

It turns out that $\eta(z) \neq 0$ when $z \in \mathbb{C}^+$, and that η^{24} is a modular form of weight 12, i.e. it satisfies the identity:

$$\eta^{24}\left(\frac{az+b}{cz+d}\right) = \eta^{24}(z)(cz+d)^{12}$$

for every $X = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$PSL$(2,\mathbb{Z})$.

To define $\mathcal{R}(X)$ we take logarithms on both sides of:

$$\eta^{24}\left(\frac{az+b}{cz+d}\right) = \eta^{24}(z)(cz+d)^{12}$$

Since η doesn't vanish, there is a holomorphic determination of log η defined on the upper half plane. The third log in what follows is chosen with imaginary part in $(-\pi, \pi)$.

The Rademacher function $\mathcal{R}$ is defined by:

$$24\log(\eta(\frac{az+b}{cz+d})) = +24(\log(\eta(z)) + 6(\log(-cz+d)^2) + 2\pi i\mathcal{R}(X)$$

It depends on $X = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$, and so on a closed orbit in modular flow. Want to know how it changes when we traverse a close path in the modular flow. By Ghys' theorem, this closed path is a Lorenz knot in $S^3 \setminus \mathcal{T}$.

The function $\mathcal{R}(X)$ appears in many ways in the study of modular forms. It seems to be difficult to calculate. This is not surprising, now that we understand that the closed orbits in the modular flow are identical with those in the Lorenz flow. Let's not forget the extreme sensitivity to initial conditions (the butterfly effect).

Choose $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$. Recall: $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ generate $SL(2,\mathbb{Z})$, also $U^2 = V^3 = -identity$. Then

$$X = UV^{\epsilon_1} \cdots UV^{\epsilon_r}, \quad \epsilon_i = \pm 1,$$

$$\mathcal{R}(X) = \epsilon_1 + \cdots + \epsilon_r.$$

Let $L = UV$, $R = UV^{-1}$. Then L, R also generate $SL(2,\mathbb{Z})$, and:

$$X = (L^{\alpha_1} R^{\beta_1}) \cdots (L^{\alpha_t} R^{\beta_t})$$

$$\mathcal{R}(X) = (\beta_1 - \alpha_1) + \cdots + (\beta_t - \alpha_t).$$

Theorem [Ghys, 2006]

$$X = (L^{\alpha_1} R^{\beta_1}) \cdots (L^{\alpha_t} R^{\beta_t})$$
$$\mathcal{R}(X) = (\beta_1 - \alpha_1) + \cdots + (\beta_t - \alpha_t).$$

The Rademacher function associated to the closed orbit X in the modular flow is the difference between the number of passages around the left ear and those around the right ear of the template.

But then, $\mathcal{R}(X) =$ linking number of the closed orbit, on the template, with $\mathcal{T}$.