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On Siegel's Modular Group

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Siegel's modular group Γ_g is the group of all $2g \times 2g$ matrices with integral entries which satisfy the condition:

$$SJS' = J$$

where $S \in \Gamma_g$, $S' =$ transpose of S, and if I_g and 0_g are the $g \times g$ unit and zero matrices respectively, then:

$$J = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}.$$

Generators for Γ_g were first determined by Hua and Reiner [4], and in 1961 Klingen [5] obtained a characterization of Γ_g for $g \geqq 2$ by a finite system of defining relations. However, Klingen's results have been of somewhat limited use because, while he gives a procedure for finding defining relations, he does not carry it through explicitely, and in fact the explicit determination of such a system involves a fairly tedious and lengthy calculation. Finding ourselves in the position of needing explicit information about Γ_g, we set ourselves the task of reducing Klingen's results to a more useable form. The primary purpose of the paper is to give the results of this calculation (Theorem 1) and to describe the methodology behind our proof.

It is well known that the group Γ_g is a homomorphic image of the mapping class group $M(T_g)$ of a sphere with g handles [7]. In Theorem 2 we determine explicitely the lifts of the generators of Γ_g to $M(T_g)$, and characterize the kernel of the homomorphism as the normal closure of a specified set of elements in $M(T_g)$.

Theorem 1. *Let* $\{Y_i, U_i, Z_j;\ 1 \leqq i \leqq g,\ 1 \leqq j \leqq g-1\}$ *denote the* $2g \times 2g$ *matrices defined in the table. Then these matrices generate Siegel's modular group* Γ_g*, and defining relations are:*

$$(Y_i, U_j, Z_k) \leftrightarrows (Y_r, U_s, Z_t) \quad (i \neq s; j \neq r, t, t+1; k \neq s, s-1; \quad 1 \leqq k, t \leqq g-1; 1 \leqq i, j, r, s \leqq g), \tag{1.1}$$

$$Y_i U_i Y_i = U_i Y_i U_i \quad (1 \leqq i \leqq g), \tag{1.2}$$

$$U_j Z_t U_j = Z_t U_j Z_t \quad (t = j, j-1; 1 \leqq j \leqq g, 1 \leqq t \leqq g-1) \tag{1.3}$$

$$Z_{i+1} = P_i P_{i+1} Z_i P_{i+1}^{-1} P_i^{-1} \quad (1 \leqq i \leqq g-2, g \geqq 3), \tag{1.4}$$

$$P_i=(Y_iU_iY_i)(Y_iU_iZ_iU_{i+1}Y_{i+1})^3(Y_iU_iY_i)^{-1} \quad (1\leqq i\leqq g-1), \tag{1.5}$$

$$(Y_1U_1Y_1)^4=1, \tag{1.6}$$

$$(Y_1U_1Z_1U_2Y_2)^6=1, \tag{1.7}$$

$$(Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1)^2=1, \tag{1.8}$$

$$Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1 \leftrightarrows Y_1, \tag{1.9}$$

$$P_2Z_1P_2^{-1}=(Y_3Y_2Y_1)(Z_1^{-1}U_2Z_2^{-1}U_2^{-1}Y_2^{-1}U_2Z_1U_2Z_2Y_2^{-1}U_2^{-1})\ (g\geqq 3), \tag{1.10}$$

where the symbol ( ) $\leftrightarrows$ ( ) *means that the quantities on the LHS commute pairwise with the quantities on the RHS for appropriate combination of the indices.*

Outline of Proof. The Y_i, U_i, Z_j generate Γ_g, because according to Klingen [5] Γ_g is generated by the matrices:

$$\{O_{i-1}, d_{i-1,i}, d_{i,i-1}, M_1^{(g)}, M_2^{(g)}; 2\leqq i\leqq g\}$$

listed in the table and these matrices can be expressed as the following power products of the Y_i, U_i, Z_j:

$$O_{i-1}=(Y_{i-1}U_{i-1}Y_{i-1})^2, \tag{2.1}$$

$$d_{i-1,i}=Y_{i-1}^{-1}Y_i^{-1}U_i^{-1}Y_i^{-1}Z_{i-1}U_iY_i, \tag{2.2}$$

$$d_{i,i-1}=Y_iY_{i-1}U_{i-1}Y_{i-1}Z_{i-1}^{-1}U_{i-1}^{-1}Y_{i-1}^{-1}, \tag{2.3}$$

$$M_1^{(g)}=U_g, \tag{2.4}$$

$$M_2^{(g)}=(Y_gU_gY_g)^{-1}. \tag{2.5}$$

Following Klingen, defining relations for Γ_g include 4 separate systems of relation which we will designate as sets (3), (4), (5), (6). Set (3) consists of defining relations for the subgroup $\Gamma_2^{(1)}$ of Γ_g generated by $\{Y_1, Y_2, U_1, U_2, Z_1\}$ [1]. This set is not given explicitely by Klingen, who instead refers the reader to the work of Gottschling [3], who gives a procedure which makes it possible to determine defining relative for $\Gamma_2^{(1)}$. We instead determine defining relations for $\Gamma_2^{(1)}$ by utilizing the work of Gold [2], who found defining relation for the quotient group S_2 of $\Gamma_2\simeq\Gamma_2^{(1)}$ obtained by adding the relation that a matrix and its negative are identified. Noting that the matrix

$$Y_1U_1Z_1U_2Y_2^2U_2Z_1U_1Y_1=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

[1] Klingen actually considers a subgroup which he calls $\mathfrak{D}$, and which we will denote by $\Gamma_2^{(g-1)}$ where $\Gamma_2^{(i)}$ = subgroup generated by $\{Y_i, U_i, Z_i, U_{i+1}, Y_{i+1}\}$ $1\leqq i\leqq g-1$. We note that the subgroups $\Gamma_2^{(i)}$, $1\leqq i\leqq g-1$, are all conjugate in Γ_g, and that in view of relations (7.3)–(7.6), which we will show later are a consequence of set (1), if defining relations for any one of them are implied by set (1), then defining relations for the others are too.

Table

$$y_i = \begin{pmatrix} I & -A_i \\ 0 & I \end{pmatrix} \qquad U_i = M_1^{(i)} = \begin{pmatrix} I & 0 \\ A_i & I \end{pmatrix} \qquad Z_i = \begin{pmatrix} I & B_i \\ 0 & I \end{pmatrix}$$

$$O_i = \begin{pmatrix} I-2A_i & 0 \\ 0 & I-2A_i \end{pmatrix} \qquad d_{i,i+1} = \begin{pmatrix} C_i' & 0 \\ 0 & (C_i')^{-1} \end{pmatrix}$$

$$d_{i+1,i} = \begin{pmatrix} C_i' & 0 \\ 0 & C_i^{-1} \end{pmatrix} \qquad M_2^{(i)} = \begin{pmatrix} I-A_i & A_i \\ -A_i & I-A_i \end{pmatrix}$$

$$A_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row} \qquad B_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & -1 & 1 & & \\ & & 1 & -1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row}$$

$$C_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & 1 & \\ & & 0 & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row}$$

where: I is $g \times g$ identity matrix,
0 is $g \times g$ zero matrix,
A_i, B_i, C_i are each $g \times g$ matrices, with all entries not shown explicitely intended to be zero.

is in Γ_2, and also that this matrix is in the center of Γ_2 and has order 2, it follows that defining relations for $\Gamma_2^{(1)}$ can be obtained from those for S_2 by computing the lifts of Gold's defining relations in S_2 to $\Gamma_2^{(1)}$, and adding the relations

$$(Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1)^2 = 1\,, \tag{3.1}$$

$$(Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1) \leftrightarrows Y_1, Y_2, U_1, U_2, Z_1\,. \tag{3.2}$$

Now, Gold's system of defining relations for S_2 are given by Eqs. D_1–D_{16} on p. 27 of [2]. Noting that the elements which Gold denotes by A_1, A_2, B_1, B_2, B_3 lift respectively to $U_1^{-1}, U_2^{-1}, Y_1^{-1}, Y_2^{-1}, Z_1$ in $\Gamma_2^{(1)}$, we find that Gold's system of defining relations for S_2 (after minor changes in notation) lift to the following relations in $\Gamma_2^{(1)}$:

$$Y_1 \leftrightarrows Y_2\,, \tag{3.3}$$

$$U_1 \leftrightarrows Y_2\,, \tag{3.4}$$

$$J_2 = (U_1 Y_1 U_1)(U_2 Y_2 U_2)\,, \tag{3.5}$$

$$A_3 = J_2 Z_1 J_2^{-1}\,, \tag{3.6}$$

$$(U_1, U_2) \leftrightarrows A_3\,, \tag{3.7}$$

$$(U_1 Y_1 U_1)^4 = 1\,, \tag{3.8}$$

$$(U_1 Y_1 U_1)^2 = (U_1 Y_1)^3\,, \tag{3.9}$$

$$P_1 = (U_1 Y_1 U_1)(Y_1 U_1 Z_1 U_2 Y_2)^3 (U_1^{-1} Y_1^{-1} U_1^{-1})\,, \tag{3.10}$$

$$Y_2 = P_1 Y_1 P_1^{-1}\,, \tag{3.11}$$

$$R_3 = (U_1 Y_1 U_1)^{-2}\,, \tag{3.12}$$

$$Y_1 \leftrightarrows R_3\,, \tag{3.13}$$

$$R_4 = R_3 P_1 U_1 U_2^{-1} A_3^{-1} Y_1 U_1^{-1} A_3 Y_1^{-1} P_1^{-1}\,, \tag{3.14}$$

$$Z_1 = R_4 Y_1 R_4^{-1}\,, \tag{3.15}$$

$$P_1^2 = 1\,, \tag{3.16}$$

$$R_4^2 = 1\,, \tag{3.17}$$

$$(P_1 R_3)^4 = 1\,, \tag{3.18}$$

$$J_2^2 = Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1\,, \tag{3.19}$$

$$J_2 \leftrightarrows R_3\,, \tag{3.20}$$

$$J_2 R_4 J_2^{-1} P_1 R_3 R_4^{-1} R_3^{-1} P_1^{-1} = 1\,. \tag{3.21}$$

Set (4) is the set of defining relations for the subgroup $\mathfrak{A} \subset \Gamma_g$ of "rotations", i.e. The subset of Γ_g consisting of all symplectic matrices of the form:

$$\begin{pmatrix} A & 0_g \\ 0_g & B \end{pmatrix}$$

where the symplectic condition implies that $B = (A')^{-1}$, and for these Klingen refers the reader to the work of Magnus [6]. Turning to Magnus' paper, we find that $\mathfrak{A}$ is generated by the elements

$$\{O_{i-1}, d_{i-1,i}, d_{i,i-1}\,; i = 2, 3, \ldots, g\}$$

which are defined in terms of our generators by Eqs. (2.1), (2.2), and (2.3) above, and has defining relations:

$$(O_{i-1}, d_{i-1,i}, d_{i,i-1}) \leftrightarrows (O_{k-1}, d_{k-1,k}, d_{k,k-1}) \qquad \begin{pmatrix} i \neq k, k+1, k-1 \\ i, k = 2, \ldots, g \end{pmatrix}, \tag{4.1}$$

$$\left.\begin{aligned}
&P_{i-1} = O_{i-1}\, d_{i-1,i}^{-1}\, d_{i,i-1}\, d_{i-1,i}^{-1} && (4.2)\\
&P_{i-1}^2 = 1 && (4.3)\\
&O_{i-1}^2 = 1 && (4.4)\\
&P_{i-1} O_{i-1} P_{i-1} = O_i && (4.5)\\
&(O_{i-1}\, d_{i-1,i})^2 = 1 && (4.6)\\
&(O_i\, d_{i-1,i})^2 = 1 && (4.7)\\
&P_{i-1}\, d_{i-1,i} P_{i-1} = d_{i,i-1} && (4.8)
\end{aligned}\right\} \quad (i = 2, 3, \ldots, g),$$

$$\left.\begin{aligned}
&P_i P_{i-1} P_i = P_{i-1} P_i P_{i-1} && (4.9)\\
&(O_{i-1} P_i)^2 = 1 && (4.10)\\
&(P_{i-1} P_i P_{i-1})\, d_{i-1,i} (P_{i-1} P_i P_{i-1}) = d_{i+1,i} && (4.11)\\
&O_{i+1}\, d_{i-1,i} O_{i+1} = d_{i-1,i} && (4.12)\\
&d_{i-1,i}\, d_{i+1,i}\, d_{i-1,i}^{-1}\, d_{i+1,i}^{-1} = 1 && (4.13)\\
&d_{i,i-1}\, d_{i,i+1}\, d_{i,i-1}^{-1}\, d_{i,i+1}^{-1} = 1 && (4.14)\\
&d_{i+1,i}\, d_{i,i-1}\, d_{i+1,i}^{-1}\, d_{i,i-1}^{-1} = P_i d_{i,i-1} P_i^{-1} && (4.15)
\end{aligned}\right\} \quad (i = 2, 3, \ldots, g-1),$$

$$\left.\begin{aligned}
&d_{kr} = d_{kl}\, d_{lr}\, d_{kl}^{-1}\, d_{lr}^{-1} && (4.16)\\
&d_{kl}\, d_{st}\, d_{kl}^{-1}\, d_{st}^{-1} = 1 \qquad (l \neq s,\ k \neq t)\,. && (4.17)
\end{aligned}\right\} \begin{pmatrix} k > l > r, s > t \\ k, l, r, s, t = 1, 2, \ldots, g \end{pmatrix}$$

$$d_{kl} \leftrightarrows (O_{i-1}, d_{i,i-1}, d_{i-1,i}) \qquad (k, l \neq i, i-1; k > l+1; i, k, l = 2, \ldots, g), \tag{4.18}$$

$$\left.\begin{aligned}
&d_{k,i-1} O_{i-1} = O_{i-1}\, d_{k,i-1}^{-1} && (4.19)\\
&d_{k,i-1} O_k = O_k\, d_{k,i-1}^{-1} && (4.20)\\
&d_{k,i-1} \leftrightarrows d_{k,k+1} && (4.21)\\
&d_{k,i-1} \leftrightarrows d_{i-2,i-1} \quad \text{if} \quad i > 2 && (4.22)\\
&d_{k,i-1}\, d_{k-1,k} = d_{k-1,k}\, d_{k,i-1}\, d_{k-1,i-1}^{-1} && (4.23)\\
&d_{k,i-1}\, d_{i-1,i} = d_{i-1,i}\, d_{k,i-1}\, d_{k,i} && (4.24)
\end{aligned}\right\} \begin{pmatrix} k > i \\ i, k = 2, \ldots, g \end{pmatrix}.$$

The third set of relations which Klingen shows is needed to obtain defining relations for Γ_g are given in Eqs. (8), (10), (23), and (27) of [5], and we list them as set (5):

$$M_1^{(\nu)} = U_\nu \qquad (\nu = 1, \ldots, g), \tag{5.1}$$

$$M_2^{(\nu)} = (U_\nu Y_\nu U_\nu)^{-1} \qquad (\nu = 1, \ldots, g), \tag{5.2}$$

$$t_{kl}=(U_k Y_k U_k)\,d_{kl}(U_k Y_k U_k)^{-1} \qquad (k>l;\, k, l=1, \ldots, g)\,, \tag{5.3}$$

$$(P_{i-1}, d_{i,i-1}) \leftrightarrows M_2^{(\nu)} \qquad (\nu=1, \ldots, g;\; i=2, \ldots, g;\; \nu \neq i, i-1)\,, \tag{5.4}$$

$$M_1^{(\nu)} \leftrightarrows M_1^{(\mu)} \qquad (\nu, \mu=1, \ldots, g)\,, \tag{5.5}$$

$$M_1^{(\nu)} \leftrightarrows t_{kl} \qquad (\nu=1, \ldots, g;\; 1 \leqq l<k \leqq g)\,, \tag{5.6}$$

$$t_{kl} \leftrightarrows t_{rs} \qquad (1 \leqq l<k \leqq g;\; 1 \leqq r<s \leqq g)\,. \tag{5.7}$$

Finally, Klingen needs "the relation which corresponds to the matrix equation":

$$\begin{pmatrix} I & 0 \\ T & I \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix}=\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ U'TU & I \end{pmatrix} \tag{6}$$

where we must consider all possible combinations obtained by setting:

$$\begin{pmatrix} I & 0 \\ T & I \end{pmatrix} = U_\nu, t_{kl}\,,$$

$$\begin{pmatrix} U & 0 \\ 0 & (U')^{-1} \end{pmatrix} = O_{i-1}, d_{i-1,i}, d_{i,i-1}\,,$$

$$\begin{pmatrix} I & 0 \\ U'TU & I \end{pmatrix} = \text{some power product of the } U_\nu, t_{kl}$$

for $2 \leqq i \leqq g$, $1 \leqq \nu \leqq g$, $g \geqq k > l \geqq 1$. In order to translate this into an explicit set of relations, it is first necessary to find the appropriate expression for the matrix on the extreme right in (6) for each case. After some experimentation, we find that (6) gives rise to the explicit relations:

$$O_{i-1} \leftrightarrows U_\nu\,, \tag{6.1}$$

$$d_{i-1,i} \leftrightarrows U_\nu \qquad (\nu \neq i-1)\,, \tag{6.2}$$

$$d_{i,i-1} \leftrightarrows U_\nu \qquad (\nu \neq i)\,, \tag{6.3}$$

$$d_{i-1,i}^{-1} U_{i-1}\, d_{i-1,i} = d_{i,i-1}^{-1}\, U_i\, d_{i,i-1} = U_i U_{i-1} t_{i,i-1}\,, \tag{6.4}$$

$$O_{i-1} \leftrightarrows t_{kl} \qquad (i-1 \neq k, l)\,, \tag{6.5}$$

$$O_l t_{kl} O_l = O_k t_{kl} O_k = t_{kl}^{-1}\,, \tag{6.6}$$

$$(d_{i-1,i}, d_{i,i-1}) \leftrightarrows t_{kl} \qquad (k, l \neq i-1, i)\,, \tag{6.7}$$

$$d_{i-1,i}^{-1} t_{kl}\, d_{i-1,i} = t_{k,i-1} t_{ki} \qquad (k \neq i-1, i;\; l=i-1)\,, \tag{6.8}$$

$$= t_{il} \qquad (k=i, l \neq i-1)\,, \tag{6.9}$$

$$= U_i^2 t_{i,i-1} \qquad (k=i, l=i-1)\,, \tag{6.10}$$

$$= t_{i,l} t_{i-1,l} \qquad (k=i-1)\,, \tag{6.11}$$

$$d_{i,i-1}^{-1}\, t_{kl}\, d_{i,i-1} = t_{kl} \qquad (k \neq i-1;\; l = i-1)\,, \tag{6.12}$$

$$= t_{i,l} t_{i-1,l} \qquad (k = i, l \neq i-1)\,, \tag{6.13}$$

$$= U_{i-1}^2 t_{i,i-1} \qquad (k = i, l = i-1)\,, \tag{6.14}$$

$$= t_{i-1,l} \qquad (k = i-1) \tag{6.15}$$

where $2 \leqq i \leqq g$, $1 \leqq v \leqq g$, $1 \leqq l < k \leqq g$.

We will not give details of our proof that relation (3), (4), (5), and (6) are consequences of set (1), however we will try to indicate the procedures which were used. Throughout the calculation, it is necessary to make repeated use of relations (1.1), (1.2), and (1.3). To proceed further, we add certain new relations (set 7), all of which can be verified to be consequences of relations in set (1), by repeated application of (1.1), (1.2), and (1.3). First, we note that the generators Y_i and U_i are related to each other by:

$$Y_i = J_g^{\varepsilon} U_i J_g^{-\varepsilon} \qquad (\varepsilon = \pm 1, i = 1, 2, \ldots, g)\,, \tag{7.1}$$

$$J_g = Y_1 U_1 Y_1 Y_2 U_2 Y_2 \ldots Y_g U_g Y_g\,. \tag{7.2}$$

Thus, for example, one notes that on conjugating both sides of (3.7) by J_g, and using (1.1), it goes over to:

$$(Y_1, Y_2) \leftrightarrows Z_1$$

which is in set (1.1).

Next, we note that the matrices P_i defined by (1.5) play an important role, since:

$$Y_{i+1} = P Y_i P^{-1} \qquad (i = 1, \ldots, g-1)\,, \tag{7.3}$$

$$U_{i+1} = P U_i P^{-1} \qquad (i = 1, \ldots, g-1)\,, \tag{7.4}$$

$$Z_{i+1} = P Z_i P^{-1} \qquad (i = 1, \ldots, g-1)\,, \tag{7.5}$$

$$P = P_1 P_2 \ldots P_{g-1}\,. \tag{7.6}$$

Also, as a consequence of (7.3)–(7.6):

$$d_{i+1,i} = P d_{i,i-1} P^{-1} \qquad (i = 2, \ldots, g-1)\,. \tag{7.7}$$

Using (7.3)–(7.7) one notes that many of the relations in sets (4), (5), and (6) (e.g. (4.2)–(4.15)) need only be established for the case $i = 2$, since all other cases follow by repeated conjugation by P. Moreover, certain of these (e.g. (4.2)–(4.8)) are necessarily consequences of set (1) because they involve only the generators of $\Gamma_2^{(1)}$, and defining relations for $\Gamma_2^{(1)}$ are given by set (3), which is a consequence of set (1).

Another useful relation is:

$$d_{i-1,i} = J_g^{\varepsilon}\, d_{i,i-1}^{-1}\, J_g^{-\varepsilon} \qquad (\varepsilon = \pm 1, i = 2, \ldots, g)\,. \tag{7.8}$$

Using this and (7.1), one finds that certain relations are just conjugates of others by J_g (e.g. (4.14) and (4.13), and many more subtle cases).

The relations which involve the elements d_{kr}, $k>r+1$, are more difficult to handle then those which involve the elements $d_{i,i-1}$ or $d_{i-1,i}$, however these are made easier if we first use the recursive definition of d_{kr} (Eq. (4.16)) in conjunction with (4.8) and (4.15) to establish that:

$$(P_r \dots P_{k-1})\, d_{kr}(P_{k-1} \dots P_r) = d_{r,r+1} \qquad (k>r+1;\ r,k=1,\dots,g).$$

It is then usually possible to reduce a relation involving the d_{kr}, where $k>r+1$, to one involving the $d_{i,i-1}$ and $d_{j,j-1}$ by repeated conjugation by appropriate products of the P_j and by J_g. In this we are aided by noting that:

$$(P_r \dots P_{k-1})^{k-r+1} = 1 \qquad (k>r+1;\ r,k=1,\dots,g),$$

$$P_i \leftrightarrows P_j \qquad (|i-j| \geqq 2;\ i,j=1,\dots,g-1),$$

$$J_g \leftrightarrows P_i \qquad (i=1,\dots,g-1).$$

In handling the relations in sets (5) and (6), one encounters new elements t_{kl}, $k>l+1$, defined by Eq. (5.3), which present the same sort of difficulties as the d_{kl}. Again, we reduced these to relations involving the $d_{i,i-1}$ and $d_{j-1,j}$ by repeated use of relations (7) and the commutativity relations (1.1) and of relations (1.2) and (1.3). In any relations involving d_{kl} and d_{rs} (e.g. (4.17)) or t_{kl} and t_{rs} (e.g. (5.7)) it was necessary to consider separately the various possible arrangements of the integers s, r, l, k. We remark that many of these calculations require considerable patience; also, that a more detailed proof is available from the author's private files.

We now go on to consider the relationship between Γ_g and the mapping class group $M(T_g)$ of a closed, orientable 2-manifold T_g of genus g. Let $\pi_1 T_g$ be the fundamental group of T_g, which admits the well-known presentation:

$$\left\langle W_1, \dots, W_{2g};\ \prod_{i=1}^{g} W_{2i-1} W_{2i} W_{2i-1}^{-1} W_{2i}^{-1} = 1 \right\rangle .$$

The group $M(T_g)$ is defined to be $\mathrm{Aut}^+ \pi_1 T_g / \mathrm{Inn}\, \pi_1 T_g$, where $\mathrm{Aut}^+ \pi_1 T_g$ is the subgroup of $\mathrm{Aut}\, \pi_1 T_g$ consisting of all automorphisms which map the single defining relator into a conjugate of itself (but not of its inverse). It is known (Eqs. (12), (13), (14) of Ref. [1]) that the generators of $M(T_g)$ can be represented by the automorphisms:

$$\{\tilde{Y}_i, \tilde{U}_i, \tilde{Z}_j;\ 1 \leqq i \leqq g,\ 1 \leqq j \leqq g-1\}$$

where:

$$\tilde{Y}_i: W_{2i-1} \to W_{2i-1} W_{2i}^{-1}, \tag{8.1}$$

$$\tilde{U}_i: W_{2i} \to W_{2i} W_{2i-1}^{-1}, \tag{8.2}$$

$$\begin{aligned} \tilde{Z}_i: W_{2i-1} &\to W_{2i-1} W_{2i}^{-1} W_{2i+1} W_{2i+2} W_{2i+1}^{-1}, \\ W_{2i} &\to W_{2i+1} W_{2i+2}^{-1} W_{2i+1}^{-1} W_{2i} W_{2i+1} W_{2i+2} W_{2i+1}^{-1} \\ W_{2i+1} &\to W_{2i+1} W_{2i+2}^{-1} W_{2i+1}^{-1} W_{2i} W_{2i+1} . \end{aligned} \tag{8.3}$$

In the above, all subscripts for the W_i's are modulo $2g$, and it is understood that any generator of $\pi_1 T_g$ which is not given explicitely is mapped onto itself.

The natural homomorphism $\mu: M(T_g) \to \Gamma_g$ maps each automorphism of $\pi_1 T_g$ into the corresponding automorphism of the abelianized fundamental group. One verifies easily on comparing Eqs. (8) with the matrices in the table that:

$$\mu \tilde{Y}_i = Y_i, \tag{9.1}$$

$$\mu \tilde{U}_i = U_i, \tag{9.2}$$

$$\mu \tilde{Z}_j = Z_j. \tag{9.3}$$

Thus $\tilde{Y}_i, \tilde{U}_i, \tilde{Z}_j$ are the lifts of Y_i, U_i, Z_j to $M(T_g)$ for each $1 \leqq i \leqq g$, $1 \leqq j \leqq g-1$; moreover the Y_i, U_i, Z_j generate Γ_g, while the $\tilde{Y}_i, \tilde{U}_i, \tilde{Z}_j$ generate $M(T_g)$.

It is now easy to specify the generators of $\ker \mu$: we calculate which of the relators (1.1)–(1.10) lift to $M(T_g)$; those which fail to lift will be the generators of the subgroup of $M(T_g)$ whose normal closure in $M(T_g)$ is $\ker \mu$. Following this procedure, we obtain:

Theorem 2. *Let* $\{\tilde{U}_i, \tilde{Y}_i, \tilde{Z}_i; 1 \leqq i \leqq g, 1 \leqq j \leqq g-1\}$ *be the generators of* $M(T_g)$ *as specified in Eq.* (8). *Let* $\mu: M(T_g) \to \Gamma_g$ *be the homomorphism specified in Eq.* (9). *Then, if* $g = 2$, $\ker \mu$ *is generated by the normal closure of*:

$$(Y_1 U_1 Y_1)^4$$

in $M(T_2)$. *If* $g \geqq 3$, $\ker \mu$ *is the normal closure of the subgroup generated by*:

$$(Y_1 U_1 Y_1)^4,$$

$$(Y_1 U_1 Z_1 U_2 Y_2)^6,$$

$$[Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1, Y_1],$$

$$(Y_3 Y_2 Y_1 Z_1^{-1} U_2 Z_2^{-1} U_2^{-1} Y_2^{-1} U_2 Z_1 U_2 Z_2 Y_2^{-1} U_2^{-1} P_2 Z_1^{-1} P_2^{-1})$$

in $M(T_g)$ [2].

It is interesting to note that relations (1.1, 2, 3, 6, 7, 8, 9) are defining relations for Γ_2, while the lifts of relations (1.1, 2, 3, 7, 8, 9) are defining relations for $M(T_2)$ [see 1]. Defining relations in $M(T_g)$ are not known if $g \geqq 3$. It is also an unsolved problem to determine whether $\ker \mu$ is finitely generated or presented for any $g \geqq 2$.

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[2] The lift of relator (1.8) to $M(T_g)$ is also not a relator if $g \geqq 3$, however one finds that in $M(T_g)$: $(Y_1 U_1 Z_1 U_2 Y_2)^6 = (Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1)^2$ for every $g \geqq 3$, hence the conjugates of $(Y_1 U_1 Z_1 U_2 Y_2^2 U_2 Z_1 U_1 Y_1)^2$ are included in the normal closure of the elements listed in Theorem 2.

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