Optimal Stopping under Model Uncertainty

Ingrid-Mona Zamfirescu

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ABSTRACT

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The aim of this paper is to extend the theory of optimal stopping to cases in which there is model-uncertainty. This means that we are given a set of possible models in the form of a family $\mathcal{P}$ of probability measures, equivalent to a reference probability measure Q on a given measurable space $(\Omega, \mathcal{F})$. We are also given a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ that satisfies the "usual conditions", and a nonnegative adapted reward process Y with RCLL paths. We shall denote by $\mathcal{S}$ the class of $\mathbf{F}-$ stopping times. Our goal is to compute the maximum expected reward under the specified model uncertainty, i.e., to calculate $R = \sup_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P(Y_\tau)$, and to find necessary and/or sufficient conditions for the existence of an optimal stopping time τ^* and an optimal model P^*. We also study the stochastic game with the upper value $\overline{V} = \inf_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P(Y_\tau)$ and the lower value $\underline{V} = \sup_{\tau \in \mathcal{S}} \inf_{P \in \mathcal{P}} E^P(Y_\tau)$; we state conditions under which this game has value, i.e. $\overline{V} = \underline{V} =: V$, and conditions under which there exists a "saddle-point" (τ^*, P^*) of strategies, i.e. $V = E^{P^*}(Y_{\tau^*})$.

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1 Introduction

Throughout this paper we shall consider the reward to be a nonnegative process Y with RCLL paths (Right-Continuous, with Left-hand Limits), on the measurable space $(\Omega, \mathcal{F})$, which is adapted to the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ of sub-σ-algebras of $\mathcal{F}$; this filtration satisfies the "usual conditions" of right-continuity and augmentation by the null sets of $\mathcal{F} = \mathcal{F}_T$. We consider the time horizon $T = \infty$, although the problem can be solved by just the same methods for any fixed constant $T \in (0, \infty)$. Therefore, we interpret $\mathcal{F}_\infty = \sigma(\bigcup_{t \ge 0} \mathcal{F}_t)$ and $Y_\infty = \limsup_{t \to \infty} Y_t$. We denote by $\mathcal{S}$ the class of $\mathbf{F}-$ stopping times with values in $[0, \infty]$; for any stopping time $v \in \mathcal{S}$, we set $\mathcal{S}_v \triangleq \{\tau \in \mathcal{S} \not\ \tau \ge v, a.s.\}$.

On this measurable space we shall consider a family $\mathcal{P}$ of probability measures P, all of them equivalent to a given "reference measure" $Q \in \mathcal{P}$. We shall think of the elements of $\mathcal{P}$ as our different possible "models" or scenarios. For technical reasons, which we shall unveil in the subsequent section, we shall assume that the family of "models" is convex.

The optimal stopping problem under model uncertainty consists of:

- computing the maximum "*expected reward*"

$$R = \sup_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P(Y_\tau), \tag{1.1}$$

and of

- finding necessary and/or sufficient conditions for the existence of an optimal stopping time τ^* and of an optimal model P^* (that attain the supremum in (1.1)).

We shall assume that $0 < R < \infty$, and under this assumption we shall construct a generalized version of the so-called *Snell envelope* of Y. This generalized Snell envelope R^0 is the smallest $\mathcal{P}-$supermartingale with RCLL paths that dominates Y. Here by $\mathcal{P}-$(super)martingale we mean an adapted process, that is $P-$(super)martingale with respect to each measure $P \in \mathcal{P}$. We shall show that a stopping time τ^* and a probability model P^* are optimal, if and only if we have $R^0_{\tau^*} = Y_{\tau^*}$, a.s. (i.e., at time τ^* the "future looks just as bright as the present"), and the stopped P^*-supermartingale $\{R^0_{t\wedge\tau^*}, \mathcal{F}_t\}$ is in fact a P^*-martingale.

In order to prove the existence of an optimal stopping time and of an optimal model, we shall impose the condition: $\sup_{P\in\mathcal{P}} E^P[\sup_{t\geq 0} Y_t] < \infty$. To study the existence of an optimal probability model P^* we shall decompose the generalized Snell envelope into the difference between a $\mathcal{P}-$martingale and a nondecreasing process, and use this decomposition to characterize the optimal model.

The key to this study is provided by the family $\{R_v\}_{v\in\mathcal{S}}$ of random variables

$$R_v \triangleq \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\tau\in\mathcal{S}_v} E^P[Y_\tau|\mathcal{F}_v], \quad v \in \mathcal{S}. \tag{1.2}$$

The random variable R_v is in fact the maximal conditional expected reward that can be achieved by stopping at time v or later, under any of the models. Since any fixed, deterministic time $t \in [0, \infty]$ is also a stopping time, (1.2) also defines a nonnegative adapted process $\mathcal{R} = \{R_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$. For every $v \in \mathcal{S}$, it is tempting to regard (1.2) as the process $\mathcal{R}$ evaluated at the stopping time v. We shall see that there is indeed a modification $\mathcal{R}^0 = \{R^0_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ of the process $\mathcal{R}$, i.e. $P[R^0_t = R_t] = 1$ for all $t \in [0, \infty]$, such that R^0 has RCLL paths, and for each $v \in \mathcal{S}$ we have

$$R_v(\omega) = R^0_{v(\omega)}(\omega) \quad \text{for a.e.} \quad \omega \in \Omega.$$

This process $\mathcal{R}^0$ is the generalized Snell envelope of the reward process Y.

In Section 3, we shall study some aspects of the non-cooperative version of the optimal stoping problem. There are many examples when it is interesting to solve the optimization problem assuming that "nature" is working against us. Namely, we are going to study the stochastic game with lower value

$$\underline{V} \triangleq \sup_{\tau \in \mathcal{S}} \inf_{P \in \mathcal{P}} E^P(Y_\tau) \tag{1.3}$$

and upper value

$$\overline{V} \triangleq \inf_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P(Y_\tau). \tag{1.4}$$

We are going to establish that this stochastic game has a value, i.e., that the upper and lower values are in fact the same: $\overline{V} = \underline{V} =: V$. We are also going to give necessary and sufficient conditions under which this game will have a "saddle-point", that is, a strategy (τ^*, P^*) such that $V = E^{P^*}(Y_{\tau^*})$. It is well known (and easy to check) that the existence of a saddle-point implies that the game has a value.

Let us recall from the theory of optimal stopping the following notation for the Snell envelope, under a given probability “scenario” P:

$$\widehat{V}_P(t) \triangleq \operatorname*{esssup}_{\tau\in\mathcal{S}_t} E^P\left[Y_\tau|\mathcal{F}_t\right], \quad 0 \le t \le \infty. \tag{1.5}$$

The same theory also states that the Snell envelope $\widehat{V}_P(\cdot)$ is the smallest P-supermaringale dominating the reward process $Y(\cdot) \geqq 0$, and that if we impose appropriate conditions (i.e., if Y is assumed quasi-left-continuous, and if $E^P[\sup_{0\le t} Y_t] < \infty$), then $\widehat{V}_P$ is P-martingale up to the stopping time

$$\hat{\rho_P}(t) \triangleq \inf\{u \ge t \;/\; \widehat{V}_P(u) = Y_u\}, \tag{1.6}$$

which attains the supremum in (1.5).

The key to the study of our stochastic game problem is given by the the family of random variables

$$\overline{V}_\tau \triangleq \operatorname*{essinf}_{P\in\mathcal{P}} \widehat{V}_P(\tau), \quad \tau\in\mathcal{S}, \tag{1.7}$$

the “upper-value” process of the stochastic game, and by the family of stopping times

$$\rho_\tau \triangleq \inf\{u \ge \tau \;/\; \overline{V}_u = Y_u\}, \quad \tau\in\mathcal{S}. \tag{1.8}$$

The random variable $\overline{V}_\tau$ is the maximum expected reward for stopping at time τ or later. We notice that we must have $\rho_\tau \le \hat{\rho_P}(\tau)$, since

$$Y_u \le \underline{V}_u \le \overline{V}_u \le \widehat{V}_P(u) \le R_u, \text{ for all } u\in[0,\infty].$$

Here $R_u \triangleq \operatorname{esssup}_{P\in\mathcal{P}} \widehat{V}_P(u)$ is the value of the cooperative game, which we shall study extensively in the next section.

We shall show that the stochastic game has a value, namely $V_\tau := \overline{V}_\tau = \underline{V}_\tau$, for all $\tau \in \mathcal{S}$. Under appropriate conditions on the family of "scenarios" $\mathcal{P}$, we shall also show that there exists a stopping time τ^* and a probability model P^* which constitute a saddle-point under the following necessary and sufficient conditions: that $V_{\tau^*} = Y_{\tau^*}$, a.s. (i.e., at time τ^* the "future looks just as bright as the present"), and that the stopped P^*-supermartingale $\{V_{t\wedge\tau^*}, \mathcal{F}_t\}$ is a P^*-martingale.

2 Cooperative Game

2.1 Generalized Snell Envelope

We now begin to study the properties of the family of random variables defined in (1.2). Since $\mathcal{P}$ contains equivalent probability measures, we can define the likelihood ratio $Z_t^P \triangleq \frac{dP}{dQ}|_{\mathcal{F}_t}$ for any $t \in [0, \infty]$ and any $P \in \mathcal{P}$. Obviously Z^P is a Q-martingale with $Z_0^P = 1$ and $Z_t^P > 0$ a.s., for all $t \in [0, \infty]$. Therefore, we can define the set $\mathcal{Z}$ of Q-martingales

$$Z_t^P = \frac{dP}{dQ}\Big|_{\mathcal{F}_t}, \qquad 0 \le t \le \infty \quad \text{for } P \in \mathcal{P}. \tag{2.1}$$

Because of the Bayes' rule (see [7]: Lemma 5.3/page 193), we can rewrite $E^P[Y_\tau|\mathcal{F}_v]$ as

$$E^P\left[Y_\tau|\mathcal{F}_v\right] = E^Q\left[\frac{Z_\tau^P}{Z_v^P} Y_\tau \Big| \mathcal{F}_v\right] = \frac{1}{Z_v^P}\; E^Q\left[Z_\tau^P Y_\tau|\mathcal{F}_v\right], \tag{2.2}$$

for all $\tau \in \mathcal{S}_v$, and R_v becomes

$$R_v = \operatorname*{esssup}_{Z \in \mathcal{Z}} \operatorname*{esssup}_{\tau \in \mathcal{S}_v} \Gamma(v|\tau, Z), \tag{2.3}$$

where we have set

$$\Gamma(v|\tau, Z) \triangleq \frac{1}{Z_v}\; E^Q\left[Z_\tau Y_\tau|\mathcal{F}_v\right]. \tag{2.4}$$

Clearly, the random variable of (2.4) depends only on the restriction of the process Z to the stochastic interval $[\![v, \tau]\!]$. We shall denote by $\mathcal{Z}_{v,\tau}$ the restriction of $\mathcal{Z}$ to this interval.

Lemma 2.1. *Suppose the set of Q-martingales $\mathcal{Z}$ in* (2.1) *is convex. Then, for any $v \in \mathcal{S}$, the family of random variables $\{\Gamma(v|\tau, Z) \ / \ \tau \in \mathcal{S}_v, \ Z \in \mathcal{Z}_{v,\tau}\}$ is stable under the operations of supremum and infimum. That is, for all $v \in \mathcal{S}$, there is a sequence $\{(\tau_k, Z^k)\}_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{S}_v$ and $Z^k \in \mathcal{Z}_{v,\tau_k}$ such that the sequence $\{\Gamma(v|\tau_k, Z^k)\}_{k\in\mathbb{N}}$ is increasing, and*

$$R_v = \lim_{k\to\infty} \uparrow \Gamma(v|\tau_k, Z^k). \tag{2.5}$$

Proof: Let $\tau_1, \tau_2 \in \mathcal{S}_v$ and $Z^1, Z^2 \in \mathcal{Z}$, and consider the event

$$A = \{\Gamma(v|\tau_2, Z^2) \geq \Gamma(v|\tau_1, Z^1)\} \in \mathcal{F}_v.$$

We can also define:

$$\tau = \tau_1 1_{A^c} + \tau_2 1_A$$
$$Z_u = Z_u^1 \ Q(A^c|\mathcal{F}_u) + Z_u^2 \ Q(A|\mathcal{F}_u)$$

Since $A \in \mathcal{F}_v$, the random time τ is a stopping time with $\tau \in \mathcal{S}_v$, and moreover because $\mathcal{Z}$ is convex, we have that $Z \in \mathcal{Z}$. Therefore:

$$\begin{aligned}
\Gamma(v|\tau, Z) &= E^Q\left[\frac{Z_\tau}{Z_v} Y_\tau \middle| \mathcal{F}_v\right] \\
&= E^Q\left[\frac{Z^1_{\tau_1}}{Z^1_v} Y_{\tau_1} \middle| \mathcal{F}_v\right] 1_{A^c} + E^Q\left[\frac{Z^2_{\tau_2}}{Z^2_v} Y_{\tau_2} \middle| \mathcal{F}_v\right] 1_A \\
&= \Gamma(v|\tau_1, Z^1) 1_{A^c} + \Gamma(v|\tau_2, Z^2) 1_A \\
&= \Gamma(v|\tau_1, Z^1) \vee \Gamma(v|\tau_2, Z^2).
\end{aligned}$$

In other words, the set $\{\Gamma(v|\tau, Z) \ / \ \tau \in \mathcal{S}_v, Z \in \mathcal{Z}_{v,\tau}\}$ is closed under pairwise maximization. The fundamental property of the essential supremum (see [8]: Theorem

A.3 /page 324), assures us that for all $v \in \mathcal{S}$, there is a sequence $\{(\tau_k, Z^k)\}_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{S}_v$ and $Z^k \in \mathcal{Z}_{v,\tau_k}$ for each $k \in \mathbb{N}$, and such that

$$R_v = \lim_{k\to\infty} \uparrow \Gamma(v|\tau_k, Z^k), \quad \text{a.s.}$$

$\diamond$

Remark 2.2. *It seems that a simpler version of Lemma* 2.1 *can be proved; i.e., if $\mathcal{Z}$ is convex, then the family $\mathcal{Z}_\tau \triangleq \{Z_\tau \;/\; Z \in \mathcal{Z}\}$ is stable under the operations of supremum and infimum. Hence, under the same fundamental property of the essential supremum we can conclude that there exist an increasing sequence $\{Z^k_\tau\} \in \mathcal{Z}_\tau$ and an decreasing sequence $\{\tilde{Z}^k_\tau\} \in \mathcal{Z}_\tau$ such that*

$$\operatorname*{esssup}_{Z\in\mathcal{Z}} Z_\tau = \lim_{k\to\infty} \uparrow Z^k_\tau, \tag{2.6}$$

and

$$\operatorname*{essinf}_{Z\in\mathcal{Z}} Z_\tau = \lim_{k\to\infty} \downarrow \tilde{Z}^k_\tau. \tag{2.7}$$

Proof. Let $Z^1_\tau, Z^2_\tau \in \mathcal{Z}_\tau$, and consider the event $A \triangleq \{Z^2_\tau \geq Z^1_\tau\} \in \mathcal{F}_\tau$. We can also define

$$Z_u = Z^1_u \; Q(A^c|\mathcal{F}_u) + Z^2_u \; Q(A|\mathcal{F}_u)$$

Since $A \in \mathcal{F}_\tau$, and $\mathcal{Z}$ is convex, we have that $Z \in \mathcal{Z}$. Therefore, $Z_\tau = Z^1_\tau \vee Z^2_\tau \in \mathcal{Z}_\tau$. The same argument stands for pairwise minimization as well. $\diamond$

Since Lemma 2.1 and Remark 2.2 will be extensively applied throughout the rest of this work, we shall assume from now on that the family of probability measures $\mathcal{P}$ is convex, without specifying it all the time.

Proposition 2.3. *For any $v \in \mathcal{S}$, $\tau \in \mathcal{S}_v$ we have:*

$$\operatorname*{esssup}_{P\in\mathcal{P}} E^P[R_\tau|\mathcal{F}_v] \quad = \quad \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P[Y_\sigma|\mathcal{F}_v], \qquad a.s. \tag{2.8}$$

as well as

$$E^P[R_\tau|\mathcal{F}_v] \quad \leq \quad R_v, \qquad a.s. \tag{2.9}$$

for all $P \in \mathcal{P}$.

Proof: Let $P \in \mathcal{P}$ be arbitrary, and denote $Z = Z^P$. Let $\mathcal{N}^Z_{v,\tau} = \{M \in \mathcal{Z}_{v,\tau} /\ M_u = Z_u,\ \forall u \in [v,\tau]\}$ be the set of Q-martingales in $\mathcal{Z}$ which agree with Z on the stochastic interval $[\![v,\tau]\!]$. Then, according to Lemma 2.1 and without any loss of generality, we may write:

$$R_\tau = \lim_{k\to\infty} \uparrow \Gamma(\tau|\tau_k, M^k),$$

where $\tau_k \in \mathcal{S}_\tau$ and $M^k \in \mathcal{N}^Z_{v,\tau} \bigcap \mathcal{Z}_{\tau,\tau_k}$ for every $k \in \mathbb{N}$, and obtain with the help of Fatou's lemma:

$$\begin{aligned}
E^P[R_\tau|\mathcal{F}_v] \quad &= \quad E^Q\left[\frac{Z_\tau}{Z_v} R_\tau \middle| \mathcal{F}_v\right] \\
&= \quad E^Q\left[\frac{Z_\tau}{Z_v} \lim_{k\to\infty} E^Q\left[\frac{M^k_{\tau_k}}{M^k_\tau} Y_{\tau_k}\middle|\mathcal{F}_\tau\right] \middle| \mathcal{F}_v\right] \\
&= \quad E^Q\left[\lim_{k\to\infty} E^Q\left[\frac{Z_\tau}{Z_v}\frac{M^k_{\tau_k}}{M^k_\tau} Y_{\tau_k}\middle|\mathcal{F}_\tau\right] \middle| \mathcal{F}_v\right] \\
&\leq \quad \lim_{k\to\infty} E^Q\left[\frac{M^k_{\tau_k}}{M^k_v} Y_{\tau_k}\middle|\mathcal{F}_v\right] \\
&\leq \quad \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P[Y_\sigma|\mathcal{F}_v].
\end{aligned}$$

But this is valid for arbitrary $P \in \mathcal{P}$, so we have:

$$\operatorname*{esssup}_{P\in\mathcal{P}} E^P[R_\tau|\mathcal{F}_v] \quad \leq \quad \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P[Y_\sigma|\mathcal{F}_v], \quad \text{a.s.} \tag{2.10}$$

The reverse inequality follows immediately, since for all $\tau \in \mathcal{S}_v$ and all $\sigma \in \mathcal{S}_\tau$ we have:

$$R_\tau \quad \geq \quad E^P[Y_\sigma|\mathcal{F}_\tau], \quad \text{a.s.} \tag{2.11}$$

from (1.2). Therefore, we can say that for all $\sigma \in \mathcal{S}_\tau$, and for all $P \in \mathcal{P}$ we have

$$E^P[R_\tau|\mathcal{F}_v] \quad \geq \quad E^P[E^P[[Y_\sigma|\mathcal{F}_\tau]|\mathcal{F}_v] = E^P[Y_\sigma|\mathcal{F}_v],$$

and after taking the essential supremum we obtain the desired inequality:

$$\operatorname*{esssup}_{P\in\mathcal{P}} E^P[R_\tau|\mathcal{F}_v] \quad \geq \quad \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P[Y_\sigma|\mathcal{F}_v], \quad \text{a.s.} \tag{2.12}$$

From (2.10) and (2.12), the equation (2.8) now follows. Finally, the proof of (2.9) is immediate, since for any $P' \in \mathcal{P}$ we have:

$$\begin{aligned} E^{P'}[R_\tau|\mathcal{F}_v] &\leq \operatorname*{esssup}_{P\in\mathcal{P}} E^P[R_\tau|\mathcal{F}_v] \\ &= \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P[Y_\sigma|\mathcal{F}_v] \\ &\leq \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_v} E^P[Y_\sigma|\mathcal{F}_v] = R_v, \quad \text{a.s.} \end{aligned}$$

thanks to (2.8) and (1.2). $\diamond$

As mentioned earlier, we may take the stopping time v in (1.2) to be equal to a constant $t \in [0, \infty]$, and thereby obtain a nonnegative, adapted process $\{R_t, \mathcal{F}_t; 0 \leq$

$t \leq \infty\}$. From (2.9) we see that this process is a $\mathcal{P}-$supermartingale. For our purposes, this is not enough; we should like to see if there is at least a RCLL modification of this process.

Theorem 2.4. *There exists an adapted process $\{R_t^0, \mathcal{F}_t; 0 \leq t \leq \infty\}$ with RCLL paths, that satisfies*

$$R_t = R_t^0 \quad a.s. \tag{2.13}$$

for every $t \in [0, \infty]$, and is a $\mathcal{P}-$supermartingale.

Definition 2.5. *Let X^1, X^2 be arbitrary processes. We say that X^1 dominates X^2 if $P\{X_t^1 \geq X_t^2, \forall t \in [0, \infty]\} = 1$ holds for some (then also for all) $P \in \mathcal{P}$.*

Remark 2.6. *If X^1, X^2 are right-continuous processes and for each $t \in [0, \infty]$ we have $X_t^1 \geq X_t^2$ a.s., then X^1 dominates X^2.*

Remark 2.7. *If X^1, X^2 are right-continuous processes and X^1 dominates X^2, then $X_\tau^1 \geq X_\tau^2$ a.s., for any $\tau \in \mathcal{S}$. This is because every stopping time can be approximated from above by a decreasing sequence of stopping times that take values in a countable set.*

Proof of Theorem 2.4: Let $\mathbf{D} = \mathbf{Q}_+$. Because R is $\mathcal{P}$-supermartingale, we have the following well known facts (see, for instance, [7]: Prop 3.14/page 16):

(i) the limits $R_{t+}(\omega) = \lim_{s \in \mathbf{D}, s \downarrow t} R_s(\omega)$ and $R_{t-}(\omega) = \lim_{s \in \mathbf{D}, s \uparrow t} R_s(\omega)$ exist for all $t \geq 0$ (respectively, $t > 0$).

(ii) $E^P[R_{t+}|\mathcal{F}_t] \le R_t$, P -a.s, $\forall t \ge 0$.

(iii) $\{R_{t+}, \mathcal{F}_{t+}, 0 \le t \le \infty\}$ is $\mathcal{P}-$supermartingale with $P-$almost every path RCLL, for any $P \in \mathcal{P}$.

Now observe that, since the filtration $\mathbf{F}$ is right-continuous (i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$ for all t), we must have that $\{R_{t+}, \mathcal{F}_t, 0 \le t \le \infty\}$ is $\mathcal{P}$-supermartingale. Also, because we assume that the process Y has RCLL paths, we have $R_{t+} \ge Y_{t+} = Y_t$ a.s for any given $t \in [0, \infty)$. Hence, Remark 2.6 gives us that R_{t+} *dominates* Y_t; therefore, due to Remark 2.7, the property (iii), and the right-continuity of the filtration and of the processes involved, we must have:

$$R_t = \operatorname*{esssup}_{P \in \mathcal{P}} \operatorname*{esssup}_{\tau \in \mathcal{S}_t} E^P(Y_\tau|\mathcal{F}_t) \le \operatorname*{esssup}_{P \in \mathcal{P}} \operatorname*{esssup}_{\tau \in \mathcal{S}_t} E^P(R_{\tau+}|\mathcal{F}_{t+}) \le R_{t+}, \quad \text{a.s.}$$

On the other hand, property (ii) and the fact that the filtration $\mathbf{F}$ is right-continuous, imply

$$R_{t+} = E^P[R_{t+}|\mathcal{F}_t] \le R_t, \quad \text{a.s.},$$

and we conclude that $R_t = R_{t+}$, a.s. Therefore, our choice of the process R^0 has to be the process $t \mapsto R_{t+}$, that is, we should take $R^0_t \equiv R_{t+}$. This process R^0 is indeed a RCLL modification of R. $\diamond$

Theorem 2.8. *The generalized Snell envelope R^0 of Y satisfies $R_v = R^0_v$ a.s. for all $v \in S$. Moreover, R^0 is the smallest $\mathcal{P}-$supermartingale with RCLL paths, which dominates Y in the sense of the Definition* 2.5.

Proof: Let us show that $R_v = R_v^0$ holds a.s., for all $v \in S$. We have already seen that $R_t = R_t^0$ a.s. for all t, therefore $R_v = R_v^0$ a.s. for all stopping times v with values in the set $\mathcal{D}$ of dyadic rationals. Also, we know that for any $v \in \mathcal{S}$, there is a decreasing sequence $\{v_n\}$ of stopping times with values in $\mathcal{D}$, such that $v = \lim_{n\to\infty} \downarrow v_n$.

Step 1: We prove first $R_v^0 \leq R_v$, a.s., that is, for any $v \in \mathcal{S}$, $A \in \mathcal{F}_v$, and for any $P \in \mathcal{P}$, we have $\int_A R_v^0 dP \leq \int_A R_v dP$.

Because of the $\mathcal{P}-$supermartingale property of R we can say that, for any $A \in \mathcal{F}_v$, and for any $P \in \mathcal{P}$:

$$\int_A R_{v_n} dP \leq \int_A R_v dP \tag{2.14}$$

holds almost surely for all positive integers n. Therefore, the sequence $\{\int_A R_{v_n} dP\}_{n\in\mathbb{N}}$ is nondecreasing and bounded from above by $\int_A R_v dP$; hence $\lim_{n\to\infty} \int_A R_{v_n} dP \leq \int_A R_v dP$. The same argument can be made about R^0, so we also have:

$$\lim_{n\to\infty} \int_A R_{v_n}^0 dP \leq \int_A R_v^0 dP.$$

The reverse of this inequality follows from Fatou's lemma and the right-continuity of R^0. Coupling these observations, we obtain:

$$\int_A R_v^0 dP = \lim_{n\to\infty} \int_A R_{v_n}^0 dP = \lim_{n\to\infty} \int_A R_{v_n} dP \leq \int_A R_v dP. \tag{2.15}$$

Step 2: We need to prove that for any $P \in \mathcal{P}$ and $A \in \mathcal{F}_v$, we have:

$$\int_A Y_\tau dP \leq \lim_{n\to\infty} \int_A R_{v_n} dP, \quad \forall \tau \in \mathcal{S}_v. \tag{2.16}$$

For this, let us take an arbitrary $\tau \in \mathcal{S}_v$. On the event $\{\tau = v\}$ things are easy, since Fatou's lemma and the right-continuity of Y guarantee that:

$$\begin{aligned}\int_{A\cap\{\tau=v\}} Y_\tau dP &\le \lim_{n\to\infty}\int_{A\cap\{\tau=v\}} Y_{v_n} dP \\ &\le \lim_{n\to\infty}\int_{A\cap\{\tau=v\}} R_{v_n} dP, \quad \text{a.s.}\end{aligned} \tag{2.17}$$

For establishing this inequality on the event $\{\tau > v\}$, we define the following sequence of stopping times:

$$\tau_n = \left\{ \begin{array}{ccc} \tau & , & \text{if} \quad v_n < \tau \\ \infty & , & \text{if} \quad v_n \ge \tau \end{array} \right\} \in \mathcal{S}_{v_n}, \quad \forall\, n \in \mathbb{N},$$

and observe

$$\begin{aligned} &\int_{\{v_n<\infty\}\cap A\cap\{\tau>v\}} Y_{\tau_n} dP - \int_{\{\tau\le v_n<\infty\}\cap A\cap\{\tau>v\}} Y_\infty dP \\ = &\int_{\{v_n<\tau\}\cap A\cap\{\tau>v\}} Y_{\tau_n} dP = \int_{\{v_n<\tau\}\cap A\cap\{\tau>v\}} E^P[Y_{\tau_n}|\mathcal{F}_{v_n}] dP \\ \le &\int_{\{v_n<\tau\}\cap A\cap\{\tau>v\}} R_{v_n} dP \le \int_{\{v<\infty\}\cap A\cap\{\tau>v\}} R_{v_n} dP, \quad \forall\, A \in \mathcal{F}_v. \end{aligned}$$

Now $Y_{\tau_n} = Y_\tau$ holds a.e. on $\{\tau > v\}$ for all n large enough, and we have $1_{\{v_n<\infty\}\cap A} \uparrow 1_{\{v<\infty\}\cap A}$ almost surely; letting $n \to \infty$ in the previous inequality, we obtain

$$\int_{\{v<\infty\}\cap A\cap\{\tau>v\}} Y_\tau dP \le \lim_{n\to\infty}\int_{\{v<\infty\}\cap A\cap\{\tau>v\}} R_{v_n} dP. \tag{2.18}$$

due to the right-continuity of Y. This leads directly to

$$\int_{A\cap\{\tau>v\}} Y_\tau dP \le \lim_{n\to\infty}\int_{A\cap\{\tau>v\}} R_{v_n} dP. \tag{2.19}$$

Summing (2.19) and (2.17), we conclude that (2.16) holds.

Step 3: We know from *Step 1*, that $R_v^0 \leq R_v$ holds $P-$a.s., for all $v \in \mathcal{S}$ and $P \in \mathcal{P}$. We shall argue by contradiction that the reverse inequality also holds. Suppose then, that there exists a $v \in \mathcal{S}$, such that $P(R_v^0 < R_v) > 0$, for some $P \in \mathcal{P}$. In other words, there exists an $\varepsilon > 0$, such that $P(R_v^0 \leq R_v - \varepsilon) > 0$. Setting $A_\varepsilon \triangleq \{R_v^0 \leq R_v - \varepsilon\} \in \mathcal{F}_v$, we must have $P(A_\varepsilon) > 0$ for all $P \in \mathcal{P}$, since $\mathcal{P}$ contains equivalent probability measures. Because (2.16) is satisfied for all $A \in \mathcal{F}_v$, it must be satisfied for A_ε, too. From this inequality, (2.15), and the definition of A_ε, we must have for all $\tau \in \mathcal{S}_v$:

$$\int_{A_\varepsilon} Y_\tau dP \leq \lim_{n\to\infty} \int_{A_\varepsilon} R_{v_n} dP = \int_{A_\varepsilon} R_v^0 dP \leq \int_{A_\varepsilon} (R_v - \varepsilon) dP. \tag{2.20}$$

Let us take an arbitrary $\tau \in \mathcal{S}_v$ and an arbitrary set $A \in \mathcal{F}_v$. Then

$$\begin{aligned} \int_A Y_\tau dP &= \int_{A_\varepsilon \cap A} Y_\tau dP + \int_{A_\varepsilon^c \cap A} Y_\tau dP \\ &\leq \int_{A_\varepsilon \cap A} (R_v - \varepsilon) dP + \int_{A_\varepsilon^c \cap A} R_v dP \\ &\leq \int_A (R_v - \varepsilon 1_{A_\varepsilon}) dP. \end{aligned} \tag{2.21}$$

In other words, $E^P(Y_\tau | \mathcal{F}_v) \leq R_v - \varepsilon 1_{A_\varepsilon}$, for all $P \in \mathcal{P}$. And after taking the essential supremum with respect to $P \in \mathcal{P}$ and $\tau \in \mathcal{S}_v$, we obtain

$$R_v \leq R_v - \varepsilon 1_{A_\varepsilon}, \quad \text{a.s.} \tag{2.22}$$

But this contradicts our assumption, that $P(A_\epsilon) > 0$ holds for all $P \in \mathcal{P}$. Therefore, we must have $R_v^0 = R_v$, a.s, for all $v \in \mathcal{S}$.

Finally, let $\tilde{R}$ be a RCLL supermartingale dominating Y. Then, for any $t \in [0, \infty]$ and $\tau \in \mathcal{S}_t$, the optional sampling theorem implies that $E^P[Y_\tau|\mathcal{F}_t] \leq E^P[\tilde{R}_\tau|\mathcal{F}_t] \leq \tilde{R}_t$, a.s. Therefore, for each $t \in [0, \infty]$, we have

$$R^0_t = R_t = \operatorname*{esssup}_{P \in \mathcal{P}} \operatorname*{esssup}_{\tau \in \mathcal{S}_t} E^P[Y_\tau|\mathcal{F}_t] \leq \tilde{R}_t, \quad \text{a.s.}$$

$\diamond$

Theorem 2.9 (Necessary And Sufficient Conditions For Optimality). *A stopping time τ^* and a probability measure P^* are optimal in* (1.1), *i.e.,*

$$E^{P^*}[Y_{\tau^*}] = R^0_0 = \sup_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P[Y_\tau] \tag{2.23}$$

holds, if and only if:

(i) $R^0_{\tau^*} = Y_{\tau^*}$, and

(ii) the stopped $\mathcal{P}-$supermartingale $\{R^0_{t \wedge \tau^*}, \mathcal{F}_t, 0 \leq t \leq T\}$ is a P^*-martingale.

Proof of Necessity: Suppose τ^* and P^* are optimal, i.e., that (2.23) holds. We can use then (2.8) with $\tau = \tau^*$ and $v = 0$, and obtain:

$$\begin{aligned} E^{P^*}(R^0_{\tau^*}) &\leq \sup_{P \in \mathcal{P}} E^P[R^0_{\tau^*}] \\ &= \sup_{P \in \mathcal{P}} \sup_{\sigma \in \mathcal{S}_{\tau^*}} E^P[Y_\sigma] \\ &\leq \sup_{P \in \mathcal{P}} \sup_{\sigma \in \mathcal{S}} E^P[Y_\sigma] \\ &= E^{P^*}[Y_{\tau^*}] \\ &\leq E^{P^*}(R^0_{\tau^*}), \end{aligned}$$

since R^0 dominates Y, and (i) follows.

In order to prove (ii), we start by noticing that for all $\sigma \in \mathcal{S}$ we have:

$$E^{P^*}[Y_{\tau^*}] = R_0^0 = \sup_{P\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^P[Y_\tau] = \sup_{P\in\mathcal{P}} \sup_{\tau\in\mathcal{S}_{\sigma\wedge\tau^*}} E^P[Y_\tau].$$

On the other hand, we can use (2.8) with $\tau = \sigma \wedge \tau^*$ and $v = 0$, to obtain:

$$\sup_{P\in\mathcal{P}} E^P[R^0_{\sigma\wedge\tau^*}] = \sup_{P\in\mathcal{P}} \sup_{\tau\in\mathcal{S}_{\sigma\wedge\tau^*}} E^P[Y_\tau] = E^{P^*}[Y_{\tau^*}],$$

and the supermartingale property of R^0 gives

$$E^{P^*}[R^0_{\sigma\wedge\tau^*}] \le \sup_{P\in\mathcal{P}} E^P[R^0_{\sigma\wedge\tau^*}] = E^{P^*}[Y_{\tau^*}] \le E^{P^*}[R^0_{\tau^*}] \le E^{P^*}[R^0_{\sigma\wedge\tau^*}].$$

Consequently, $E^{P^*}[R^0_{\tau^*}] = E^{P^*}[R^0_{\sigma\wedge\tau^*}]$, so the expectation $E^{P^*}[R^0_{\sigma\wedge\tau^*}]$ does not depend on σ; this shows that the process $\{R^0_{t\wedge\tau^*}, \mathcal{F}_t, 0 \le t \le \infty\}$ is indeed a P^*-martingale (see [7], Problem 1.3.26).

Proof of Sufficiency: Conversely, (i) and (ii) give:

$$E^{P^*}[Y_{\tau^*}] = E^{P^*}[R^0_{\tau^*}] = R^0_0 = \sup_{P\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^P[Y_\tau],$$

thus, the pair (τ^*, P^*) is optimal. $\diamond$

2.2 Existence Of Optimal Stopping Times

Having characterized optimal stopping times in Theorem 2.9, we now seek to establish their *existence*. We begin by constructing a family of "approximately optimal" stopping times. For $\alpha \in (0,1)$ and $v \in \mathcal{S}$, define the stopping time:

$$U_v^\alpha \triangleq \inf\{t \geq v \ / \ \alpha R_t^0 \leq Y_t\} \in \mathcal{S}_v. \tag{2.24}$$

Proposition 2.10. *For any $\alpha \in (0,1)$ and $v \in \mathcal{S}$, we have*

$$R_v^0 = \operatorname*{esssup}_{P \in \mathcal{P}} E^P[R^0_{U_v^\alpha} | \mathcal{F}_v], \quad a.s. \tag{2.25}$$

Proof:

Step 1: For fixed $\alpha \in (0,1)$, define the family of nonnegative random variables suggested by the right-hand side of (2.25), namely

$$J_v^\alpha \triangleq \operatorname*{esssup}_{P \in \mathcal{P}} E^P[R^0_{U_v^\alpha} | \mathcal{F}_v], \tag{2.26}$$

and note that we have $J_v^\alpha \leq R_v^0$, a.s. from Theorem 2.8. We want to show that, for any $P \in \mathcal{P}$ and $\tau \in \mathcal{S}_v$, we also have the supermartingale property

$$E^P[J_\tau^\alpha | \mathcal{F}_v] \leq J_v^\alpha, \quad \text{a.s.} \tag{2.27}$$

or, to put it in an equivalent form:

$$E^Q\left[\frac{Z_\tau}{Z_v} J_\tau^\alpha \middle| \mathcal{F}_v \right] \leq J_v^\alpha, \quad \text{a.s.} \tag{2.28}$$

for any given $Z \in \mathcal{Z}$ and $\tau \in \mathcal{S}_v$. It is very easy to show, following a proof similar to that of Lemma 2.1, that there exists a sequence $\{M^k\}_{k\in\mathbb{N}} \subseteq \mathcal{Z}_{\tau,U^\alpha_\tau}$ such that:

$$J^\alpha_\tau = \lim_{k\to\infty} \uparrow E^Q\left[\frac{M^k_{U^\alpha_\tau}}{M^k_\tau} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_\tau\right] \quad \text{a.s.}$$

Again without any loss of generality, we can assume $\{M^k\}_{k\in\mathbb{N}} \subseteq \mathcal{N}^Z_{v,\tau}$ (recall the notation established in the proof of Proposition 2.3). Therefore, for any $Z \in \mathcal{Z}$ and $\tau \in \mathcal{S}_v$, we have:

$$\begin{aligned}
E^Q\left[\frac{Z_\tau}{Z_v} J^\alpha_\tau\middle|\mathcal{F}_v\right] &= E^Q\left[\frac{Z_\tau}{Z_v}\cdot\lim_{k\to\infty} E^Q\left[\frac{Z_\tau}{Z_v}\cdot\frac{M^k_{U^\alpha_\tau}}{M^k_\tau} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_\tau\right]\middle|\mathcal{F}_v\right] \\
&\leq \lim_{k\to\infty} E^Q\left[\frac{Z_\tau}{Z_v}\cdot\frac{M^k_{U^\alpha_\tau}}{M^k_\tau} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_v\right] \\
&\leq \operatorname*{esssup}_{M\in\mathcal{N}^Z_{v,\tau}} E^Q\left[\frac{M_{U^\alpha_\tau}}{M_v} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_v\right] \\
&\leq \operatorname*{esssup}_{M\in\mathcal{Z}} E^Q\left[\frac{M_{U^\alpha_\tau}}{M_v} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_v\right] \\
&= \operatorname*{esssup}_{M\in\mathcal{Z}} E^Q\left[\frac{M_{U^\alpha_v}}{M_v}\cdot E^Q\left[\frac{M_{U^\alpha_\tau}}{M_{U^\alpha_v}} R^0_{U^\alpha_\tau}\middle|\mathcal{F}_{U^\alpha_v}\right]\middle|\mathcal{F}_v\right] \\
&\leq \operatorname*{esssup}_{M\in\mathcal{Z}} E^Q\left[\frac{M^k_{U^\alpha_v}}{M^k_v} R^0_{U^\alpha_v}\middle|\mathcal{F}_v\right] = J^\alpha_v, \quad \text{a.s.}
\end{aligned}$$

This proves (2.28), and we conclude that the process $\{J^\alpha_t, \mathcal{F}_t;\ 0 \leq t \leq \infty\}$ is indeed a $\mathcal{P}-$supermartingale, for all $\alpha \in (0,1)$.

Step 2: We need to show that $J^\alpha_v \geq R^0_v$ holds a.s. for any $\alpha \in (0,1)$, $v \in \mathcal{S}$. Since we have already shown the reverse inequality, this will prove (2.25).

For a fixed $\alpha \in (0,1)$, consider the $\mathcal{P}-$supermartingale $\alpha R^0 + (1-\alpha)J^\alpha$. With $\tau \in \mathcal{S}_v$ arbitrary but fixed, we have the following inequalities:

- On $\{U^\alpha_\tau = \tau\}$ we have $J^\alpha_\tau = R^0_\tau$ from (2.26) and thus, for any $P \in \mathcal{P}$:

$$\alpha R^0_\tau + (1-\alpha)J^\alpha_\tau = \alpha R^0_\tau + (1-\alpha)R^0_\tau = R^0_\tau \geq Y_\tau, \quad \text{a.s.}$$

- On the other hand, (2.24) implies

$$\alpha R^0_\tau + (1-\alpha)J^\alpha_\tau \geq \alpha R^0_\tau > Y_\tau, \quad \text{a.s. on} \quad \{U^\alpha_\tau > \tau\}.$$

Therefore, for all $\tau \in \mathcal{S}_v$ we have:

$$\alpha R^0_\tau + (1-\alpha)J^\alpha_\tau \geq Y_\tau, \quad \text{a.s.},$$

which leads immediately to:

$$E^P[Y_\tau|\mathcal{F}_v] \leq E^P[\alpha R^0_\tau + (1-\alpha)J^\alpha_\tau|\mathcal{F}_v] \leq \alpha R^0_v + (1-\alpha)J^\alpha_v, \quad \text{a.s.},$$

thanks to (2.27) and to Theorem 2.8. And after taking the supremum, we obtain the inequality

$$R^0_v = \operatorname*{esssup}_{P\in\mathcal{P}} \operatorname*{esssup}_{\tau\in\mathcal{S}_v} E^P[Y_\tau|\mathcal{F}_v] \leq \alpha R^0_v + (1-\alpha)J^\alpha_v,$$

which leads immediately to $J^\alpha_v \geqq R^0_v$. ◇

For fixed $v \in \mathcal{S}$, the family of stopping times $\{U^\alpha_v\}_{\alpha\in(0,1)}$ is nondecreasing in α, so we may define the limiting stopping time:

$$U^*_v \triangleq \lim_{\alpha\uparrow 1} U^\alpha_v, \quad \text{a.s.} \tag{2.29}$$

It seems now that U_0^* is a good candidate for optimal stopping time. To prove this we shall need some sort of "left-continuity" property for Y; it turns out that it is enough to assume what we will call "quasi-left-continuity", in order to prove the optimality of U_0^*.

Definition 2.11. *We shall say that a process Y is quasi-left-continuous, if for any increasing sequence $\{\tau_n\}_{n\in N} \subseteq \mathcal{S}$ we have*

$$\limsup_{n\to\infty} Y_{\tau_n} \leq Y_\tau, \quad a.s.,$$

where $\tau \triangleq \lim_{n\to\infty} \tau_n \in \mathcal{S}$.

Theorem 2.12. *Assume that $Y^* := \sup_{0\leq t<\infty} Y_t$ is integrable with respect to all $P \in \mathcal{P}$, and that the process Y is quasi-left-continuous. Then for each $v \in \mathcal{S}$, the stopping time U_v^*, defined by (2.29), satisfies:*

$$R_v^0 = \operatorname*{esssup}_{P\in\mathcal{P}} E^P[Y_{U_v^*}|\mathcal{F}_v], \quad a.s. \tag{2.30}$$

In particular, the stopping time U_0^* attains the supremum in the initial problem of (1.1). Furthermore, for all $v \in \mathcal{S}$:

$$U_v^* = V_v \triangleq \inf\{t \geq v / R_t^0 = Y_t\}, \quad a.s. \tag{2.31}$$

Proof: From the supermartingale property of R_v^0 we obtain immediately that

$$E^P[Y_{U_v^*}|\mathcal{F}_v] \leq E^P[R^0_{U_v^*}|\mathcal{F}_v] \leq R_v^0, \quad \text{a.s.},$$

for every $P \in \mathcal{P}$, which implies

$$\operatorname*{esssup}_{P\in\mathcal{P}} E^P[Y_{U_v^*}|\mathcal{F}_v] \leq R_v^0, \quad \text{a.s.} \tag{2.32}$$

We shall prove the reverse inequality by contradiction. Suppose that for all $\varepsilon > 0$ the event $A_\varepsilon \triangleq \{R_v^0 - \varepsilon \geq E^P[Y_{U_v^*}|\mathcal{F}_v]\} \in \mathcal{F}_v$ is such that $P(A_\varepsilon) > 0$, for all $P \in \mathcal{P}$. Hence, for all $A \in \mathcal{F}_v$ we have:

$$\begin{aligned}
\int_A Y_{U_v^*} dP &= \int_A E^P[Y_{U_v^*}|\mathcal{F}_v] dP \\
&= \int_{A\cap A_\varepsilon} E^P[Y_{U_v^*}|\mathcal{F}_v] dP + \int_{A\cap A_\varepsilon^c} Y_{U_v^*} dP \\
&\leq \int_{A\cap A_\varepsilon} (R_v^0 - \varepsilon) dP + \int_{A\cap A_\varepsilon^c} R_{U_v^*}^0 dP \\
&\leq \int_{A\cap A_\varepsilon} (R_v^0 - \varepsilon) dP + \int_{A\cap A_\varepsilon^c} R_v^0 dP \\
&= \int_A (R_v^0 - \varepsilon 1_{A_\varepsilon}) dP
\end{aligned}$$

That is, we can say that $E^P[Y_{U_v^*}|\mathcal{F}_v] \leq R_v^0 - \varepsilon 1_{A_\varepsilon}$, holds almost surely. Since Y is quasi-left-continuous we have the following inequalities.

$$\begin{aligned}
\lim_{\alpha\uparrow 1} E^P[R_{U_v^\alpha}^0|\mathcal{F}_v] &\leq \lim_{\alpha\uparrow 1} E^P\left[\frac{1}{\alpha} Y_{U_v^\alpha} \middle| \mathcal{F}_v\right] \\
&\leq E^P\left[\lim_{\alpha\uparrow 1}\left(\frac{1}{\alpha} Y_{U_v^\alpha}\right)\right] \\
&\leq E^P[Y_{U_v^*}|\mathcal{F}_v],
\end{aligned}$$

for all $P \in \mathcal{P}$. Therefore, the exists an $\alpha_0 \in (0,1)$ such that

$$E^P[R_{U_v^{\alpha_0}}^0|\mathcal{F}_v] \leq R_v^0 - \varepsilon 1_{A_\varepsilon},$$

holds almost surely, for all $P \in \mathcal{P}$. At last, taking supremum with respect to all $P \in \mathcal{P}$, we arrive to a contradiction. Hence,

$$R_v^0 \leq \operatorname*{esssup}_{P \in \mathcal{P}} E^P[Y_{U_v^*} | \mathcal{F}_v], \quad \text{a.s.} \tag{2.33}$$

Then (2.33) and (2.32) lead to

$$R_v^0 = \operatorname*{esssup}_{P \in \mathcal{P}} E^P[Y_{U_v^*} | \mathcal{F}_v], \quad \text{a.s.,}$$

and this shows that the stopping time U_v^* is optimal.

Now, we turn to (2.31). Because of the definition of U_v^α, we have:

$$\alpha R_{U_v^\alpha}^0 \leq Y_{U_v^\alpha} \quad \text{a.s., hence} \quad \alpha \cdot E^P(R_{U_v^\alpha}^0) \leq E^P(Y_{U_v^\alpha}) \text{ for all } P \in \mathcal{P}.$$

We also know that $\{U_v^\alpha\}_{\alpha \in (0,1)}$ is an increasing family of stopping times, therefore $U_v^\alpha \leq U_v^*$ for all $\alpha \in (0,1)$, and

$$\alpha \cdot E^P(R_{U_v^*}^0) \leq \alpha \cdot E^P(R_{U_v^\alpha}^0) \leq E^P(Y_{U_v^\alpha}) \quad \text{for all } \alpha \in (0,1),\ P \in \mathcal{P}$$

from the $\mathcal{P}-$supermartingale property of R^0 (Theorem 2.8). Taking the limit $\alpha \uparrow 1$, and using the fact that Y is quasi-left-continuous to apply the dominated convergence theorem, we obtain:

$$E^P(R_{U_v^*}^0) \leq \liminf_{\alpha \uparrow 1} E^P(R_{U_v^\alpha}^0) \leq \lim_{\alpha \uparrow 1} E^P(Y_{U_v^\alpha}) \leq E^P(Y_{U_v^*}) \quad \text{for all } P \in \mathcal{P}.$$

But Y is dominated by R^0, so we must have $R_{U_v^*}^0 = Y_{U_v^*}$ a.s.; therefore,

$$U_v^* \geq V_v \quad \text{as on the right-hand-side of (2.31).} \tag{2.34}$$

For the reverse inequality, note that on the event $\{v < U_v^*\}$ we must have $v < U_v^{\alpha_0}$ for some $\alpha_0 \in (0,1)$. Moreover, this is true for all $\alpha \geq \alpha_0$ since the family of stopping times is increasing. Therefore, because of the definition of U_v^α, on the event $\{v < U_v^*\}$ we have :

$$Y_t < \alpha R_t^0 \leq R_t^0 \quad \text{for all} \quad t \in [v, U_v^\alpha), \quad \text{and for all} \quad \alpha \in [\alpha_0, 1).$$

Therefore, we must have $U_v^\alpha \leq V_v$ for all $\alpha \in [\alpha_0, 1)$ on $\{v < U_v^*\}$, thus

$$U_v^* = \lim_{\alpha \uparrow 1} U_v^\alpha \leq V_v, \quad \text{on} \quad \{v < U_v^*\}.$$

On the other hand, on $\{v = U_v^*\}$ the inequality $U_v^* \leq V_v$ is obviously satisfied, so we can conclude that:

$$U_v^* \leq V_v, \quad \text{a.s.} \tag{2.35}$$

Putting (2.34) and (2.35) together, the theorem is completely proved. $\diamond$

Remark 2.13. *If Y is not quasi-left-continuous, then we can only guarantee that each U_0^α is α-optimal, for every $\alpha \in (0,1)$.*

2.3 Existence Of Optimal Models

In what follows we shall attempt to find conditions on the family $\mathcal{P}$, under which there is an optimal model for our problem. It turns out that under suitable conditions for the family $\mathcal{P}$, we can decompose our generalized Snell envelope in a "universal fashion", i.e., in the form $R_t^0 = R_0^0 + X_t - C_t$, where X is a $\mathcal{P}-$martingale with RCLL paths and $X_0 = 0$, and C is an adapted process with non-decreasing, RCLL paths, $C_0 = 0$, and $E^P(C_\infty) < \infty$ for all $P \in \mathcal{P}$.

Obviously, the conditions we need to impose on the family $\mathcal{P}$ are required in order to ensure the existence of such a special $\mathcal{P}-$martingale X.

Thus, we are looking for a $\mathcal{P}-$martingale X with RCLL paths, such that $\langle Z, X\rangle = 0$ for every $Q-$martingale Z in the class $\mathcal{Z}$ of (2.1). This can be achieved in the spirit of the Kunita-Watanabe work on square-integrable martingales (see [5]).

Let us denote by $\mathcal{M}_2$ the set of all zero-mean, square-integrable Q-martingales. It was proved in the above-mentioned paper that $\{\mathcal{M}_2, \|\cdot\|_t\}$ is a complete separable space, where the semi-norms $\|\cdot\|_t$ are defined by the formula: $\|X\|_t = \sqrt{E^Q(X_t^2)}$, for $X \in \mathcal{M}_2$; see also Proposition 1.5.23 in [7].

Definition 2.14. *A subset $\mathcal{N}$ of $\mathcal{M}_2$ is called a subspace of $\mathcal{M}_2$, if it satisfies the following conditions:*

- *$X, Y \in \mathcal{N}$ then $X + Y \in \mathcal{N}$.*

- *If $X \in \mathcal{N}$ and ϕ is predictable, then $\int \phi dX \in \mathcal{N}$.*
- *$\mathcal{N}$ is closed in $\{\mathcal{M}_2, \|\cdot\|_t\}$.*

Remark 2.15. *If $\mathcal{N}$ is a subspace of $\mathcal{M}_2$ in the sense of Definition* 2.14*, then any element of $\mathcal{M}_2$ can be decomposed uniquely as $W = W' + W''$, where $W' \in \mathcal{N}$ and $W'' \in \mathcal{N}^{\perp}$. This decomposition is generally referred to as the "Kunita-Watanabe decomposition"; see Proposition 3.4.14 in* [7].

Notation: For every $Q-$local martingale N, we shall denote by $\mathcal{E}(N)$ the Doléans-Dade exponential of N, that is, the solution of the Stochastic Differential Equation: $dU_t = U_{t-}dN_t$, with $U_0 = 1$. Also recall that if two probability measures P and Q are equivalent, then there is a $Q-$local martingale N, such that $\frac{dP}{dQ}|_{\mathcal{F}_t} = \mathcal{E}(N)_t$ (see [9]).

Consider now, as $\mathcal{N}$ the set of all $Q-$martingales N, for which $\mathcal{E}(N)$ belongs in the set $\mathcal{Z}$ of (2.1).

Remark 2.16. *If $\mathcal{N}$, as considered above, is closed in $\{\mathcal{M}_2, \|\cdot\|_t\}$, we can prove that $\mathcal{Z}$ of* (2.1) *is also closed in the set of all square-intergable martingales with mean 1, under the semi-norms $\|\cdot\|_t$ defined by the following formula: $\|Z\|_t = \sqrt{E^Q(Z_t - 1)^2}$.*

Theorem 2.17. *Suppose that $\mathcal{N}$ is a subspace of $\mathcal{M}_2$ in the sense of the Definition* 2.14*. Then any $\mathcal{P}-$supermartingale S with RCLL paths and of class DL with respect to the reference probability measure $Q \in \mathcal{P}$(see the Definition 4.8 in* [7]*/page*

24), can be decomposed as:

$$S_t = S_0 + X_t - C_t, \quad 0 \leq t < \infty,$$

where $X \in \mathcal{M}_2$ is a RCLL $\mathcal{P}$-martingale, and C is an optional increasing process with RCLL paths and $C_0 = 0$.

In contrast to the standard Doob-Meyer decomposition, the process C is in general *not* predictable but only optional, and is *not* uniquely determined. However, the decomposition is *"universal"*, in the sense that it holds simultaneously for all $P \in \mathcal{P}$. The existence of such an *"optional decomposition"* was shown by El Karoui and Quenez, [9], and Kramkov, [2], for a special class of models, $\mathcal{P}$, while solving the problem of hedging contingent claims in incomplete security markets. Also, Fölmer and Kabanov, [4], have proved such a decomposition in a more general framework that El Karoui - Quenez and Kramkov. However, their result is insufficient for the purposes of our paper, since our set of probability models is more general than the one Fölmer and Kabanov are working with. Theorem 2.17 suits our problem much better, since it provides conditions under which there is such a special $\mathcal{P}$-supermartingale X; moreover, we *do not* assume that $\mathcal{P}$ is the class of *all* equivalent martingale measures of X, as it was the case in the above cited papers.

Proof of Theorem 2.17*:*

Since S is supermartingale of class DL under Q, it admits the Doob-Meyer decomposition $S = S_0 + M - A$, where M is a Q-martingale and A is an increasing predictable process with $A_0 = 0$. Also, because $\mathcal{N}$ is a *subspace* of $\mathcal{M}_2$, the martingale M admits the Kunita-Watanabe decomposition: $M = L + X$, where $L \in \mathcal{N}$, and $X \in \mathcal{N}^{\perp}$, i.e., $\langle N, X \rangle = 0$ for all $N \in \mathcal{N}$. This leads to $\langle \mathcal{E}(N), X \rangle = 0$ for all $N \in \mathcal{N}$; in other words, we have identified X as the $\mathcal{P}$-martingale in our decomposition. Hence, we can say that:

$$S_t = S_0 + X_t + L_t - A_t, \quad 0 \leq t < \infty, \quad Q- \text{ a.s.} \tag{2.36}$$

Now, the square-integrable martingale $L \in \mathcal{N}$ can be decomposed further as the sum of its continuous martingale part L^c, and its purely discontinuous martingale part L^d; moreover $\langle L^c, L^d \rangle = 0$ (see [1] /page 367).

Step 1: We want to show now that $L^c = 0$.

First, observe that because S is $\mathcal{P}$-supermartingale, we obtain as a consequence of the Girsanov theorem (see Corollary 1A (ii) in the Appendix of [9], also see [1], VII.45. page 255).

$$\langle N, L + X \rangle - A \quad \text{ is a decreasing process,} \quad \forall N \in \mathcal{N}. \tag{2.37}$$

And because $\langle N, X \rangle = 0$ for every $N \in \mathcal{N}$, we can rewrite (2.37) as:

$$A - \langle N, L \rangle \text{ is an increasing process,} \quad \forall N \in \mathcal{N}. \tag{2.38}$$

Since (2.38) is satisfied for all $N \in \mathcal{N}$, we can try and use it for a conveniently chosen $N \in \mathcal{N}$. We shall take:

$$N_t = \int_0^t \eta_s dL_s^c \qquad 0 \leq t < \infty \tag{2.39}$$

where η is a bounded predictable process to be chosen shortly in a convenient way. Since $\mathcal{N}$ is a subspace of $\mathcal{M}_2$, it is closed under stochastic integration, and we must have $N \in \mathcal{N}$. Therefore, (2.38) implies that

$$A - \int \eta_s d\langle L^c \rangle_s \quad \text{is increasing.} \tag{2.40}$$

Now let us decompose the measure dA_t with respect to $d\langle L^c \rangle_t$, and try to choose a convenient η. The process $\langle L^c \rangle$ is integrable, because $L \in \mathcal{M}_2$. By the Lebesgue decomposition theorem, there exists a positive predictable process $h \in L^1([0, \infty) \times \Omega, d\langle L^c \rangle dQ)$ and an integrable predictable increasing process B, such that

$$dA_t = h_t d\langle L^c \rangle_t + dB_t \tag{2.41}$$

and such that, Q almost surely, the measure dB_t is singular with respect to $d\langle L^c \rangle_t$. For each integer p, we can write the following version of (2.41):

$$dA_t = h_t 1_{\{h_t \le p\}} d\langle L^c \rangle_t + dB_t^p, \tag{2.42}$$

where, Q almost surely, the measure dB_t^p is singular with respect to $1_{\{h_t \le p\}} d\langle L^c \rangle_t$. An immediate consequence of the singularity of the measure dB_t^p with respect to $1_{\{h_t \le p\}} d\langle L^c \rangle_t$ is that on the event $\{h \le p\}$ we have

$$A_t = \int_0^t h_s 1_{\{h_s \le p\}} d\langle L^c \rangle_s. \tag{2.43}$$

Let us select $\eta_t \triangleq (1 + h_t) 1_{\{h_t \le p\}}$. According to (2.40), on the event $\{h \le p\}$, the process $A - \int (1 + h_s) 1_{\{h_s \le p\}} d\langle L^c \rangle_s$ is increasing. Therefore, due to (2.43), we conclude that on the event $\{h \le p\}$:

$$A_t - \int_0^t \eta_s d\langle L^c\rangle_s = \int_0^t h_s 1_{\{h_s \le p\}} d\langle L^c\rangle_s - \int_0^t (1+h_s) 1_{\{h_s \le p\}} d\langle L^c\rangle_s$$

is an increasing process. Hence, Q almost surely, $-\langle L^c\rangle_t$ is increasing on $\{h \le p\}$, for all p, and this yields the equality:

$$\langle L^c\rangle_\infty = 0, \quad Q\text{-a.s.}$$

Hence, $L^c = 0$, and thus $L = L^d$.

Step 2: Next we need to show that not only is L a purely discontinuous martingale, it also has only negative jumps. To prove this, let us decompose L with respect to the sign of its jumps: $L = L^+ + L^-$, where L^+ (respectively, L^-) is the compensated integral of $1_{\{\Delta L > 0\}}$ (respectively, $1_{\{\Delta L < 0\}}$) with respect to L. Notice that L^+ and L^- are both square-integrable martingales. Following the same pattern as in *Step 1* for $N_t = \int_0^t \eta_s dL_s^+$, we prove that $L^+ = 0$.

Step 3: We have proved so far, that the L process in the decomposition (2.36) is indeed a purely discontinuous martingale with negative jumps. Hence, the process $C_t = A_t - L_t$ is indeed increasing, although not necessarily predictable, only optional. Since A is predictable, the failure of $A - L$ to be predictable must be a result of L being just optional (RC instead of LC). Thus, we conclude is that we can decompose S, in a “universal fashion”

$$S_t = S_0 + X_t - C_t, \quad \forall \quad t \in [0, \infty] \tag{2.44}$$

almost surely, where X is RCLL, mean zero, $\mathcal{P}$-martingale, and C is RCLL, adapted, optional increasing process with $C_0 = 0$. $\diamond$

Corollary 2.18. *If $\mathcal{N}$ is a subspace of $\mathcal{M}_2$ in the sense of the Definition* 2.14*, and if* $\sup_{P\in\mathcal{P}} E^P(Y^*) < \infty$ *holds for* $Y^* \triangleq \sup_{t\geq 0} Y_t$*, then the generalized Snell envelope* R^0 *of the process* Y *admits a "universal" optional decomposition; i.e., there exists a RCLL, mean zero, uniformly integrable* $\mathcal{P}$*-martingale* X*, and an optional increasing process* C *with RCLL paths and* $C_0 = 0$*, such that almost surely:*

$$R^0_t = R^0_0 + X_t - C_t, \quad \forall \quad t \in [0,\infty]. \tag{2.45}$$

Proof: In order to use Theorem 2.17, we need to show that the process R^0 is of class DL with respect to the reference probability $Q \in \mathcal{P}$. We shall see that in fact we have more; due to the condition $\sup_{P\in\mathcal{P}} E^P(Y^*) < \infty$, we can prove that R^0 is of class D, with respect to any probability measure $P \in \mathcal{P}$.

Thus, let us prove the uniform integrability of the family $\{R^0_\tau\}_{\tau\in\mathcal{S}}$. From (2.8), and Theorem 2.8 for $v = 0$, we obtain

$$E^P(R^0_\tau) \leq \sup_{P\in\mathcal{P}} E^P(R^0_\tau) \leq \sup_{P\in\mathcal{P}} \sup_{\sigma\in\mathcal{S}_\tau} E^P(Y_\sigma) \leq \sup_{P\in\mathcal{P}} E^P(Y^*) < \infty,$$

for all $\tau \in \mathcal{S}$. Given $P \in \mathcal{P}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$A \in \mathcal{F}_\infty \text{ and } P(A) < \delta \quad \Longrightarrow \quad \int_A Y^* dP < \varepsilon.$$

Let $\alpha > \frac{1}{\delta} \sup_{P\in\mathcal{P}} E^P(Y^*)$ be given. Then we have

$$P\{R^0_\tau > \alpha\} \leq \frac{1}{\alpha} \sup_{P\in\mathcal{P}} E^P(Y^*) < \delta$$

and

$$E^P(R^0_\tau 1_{\{R^0_\tau>\alpha\}}) \leq \sup_{P\in\mathcal{P}} E^P(Y^* 1_{\{R^0_\tau>\alpha\}}) = \sup_{P\in\mathcal{P}} \int_{\{R^0_\tau>\alpha\}} Y^* dP < \varepsilon, \quad \forall \quad \tau \in \mathcal{S}.$$

Therefore, the family $\{R^0_\tau\}_{\tau\in\mathcal{S}}$ is uniformly-integrable, under any $P \in \mathcal{P}$, which makes R^0 a $\mathcal{P}$-supermartingale of class D under all $P \in \mathcal{P}$. Now we are free to use the Theorem 2.17, so the desired decomposition holds. Moreover, we can use a similar argument as the one presented in [7] for the proof of Theorem 1.4.10, to argue that if R^0 is of class D, then the $\mathcal{P}$-martingale X must be $\mathcal{P}-$uniformly integrable. $\diamond$

Since all of our subsequent results depend on the existence of this "universal" optional decomposition, we shall assume from now on, that the set $\mathcal{N}$ is a subspace of $\mathcal{M}_2$, that the reward process Y has quasi-left-continuous paths, and that $\sup_{P\in\mathcal{P}} E^P(Y^*) < \infty$*, without specifying it all the time.*

Theorem 2.19. *If $R^0_\tau = E^{P^*}(Y_{U^*_\tau}|\mathcal{F}_\tau)$ a.s. for some $\tau \in \mathcal{S}$, and $P^* \in \mathcal{P}$, then $R^0_\tau = E^P(Y_{U^*_\tau}|\mathcal{F}_\tau)$ a.s. for any $P \in \mathcal{P}$.*

Proof: Because of (2.31) we can say that:

$$E^{P^*}(R^0_\tau) = E^{P^*}(Y_{U^*_\tau}) = E^{P^*}(R^0_{U^*_\tau}).$$

Then, the decomposition (2.45) and the optional sampling theorem, lead to $E^{P^*}(C_\tau) = E^{P^*}(C_{U^*_\tau})$. But C is a non-decreasing process, so we must have $C_\tau = C_{U^*_\tau}$, P^*-a.e., thus also P-a.e. for all $P \in \mathcal{P}$. Since the decomposition of R^0 is universal, that is it holds for all $P \in \mathcal{P}$, we conclude that $E^P(R^0_\tau) = R^0_0 + E^P(X_\tau) + E^P(C_\tau) = R^0_0 + E^P(X_{U^*_\tau}) + E^P(C_{U^*_\tau}) = E^P(R^0_{U^*_\tau})$. Consequently, we get

$$R^0_\tau = E^P(R^0_{U^*_\tau}|\mathcal{F}_\tau), \quad P\text{-a.s.} \tag{2.46}$$

$\diamond$

Remark 2.20. *For $\tau = 0$, Theorem 2.19 gives the following result. If an optimal model P^* exists, then any $P \in \mathcal{P}$ is optimal.*

In other words, in this context it is enough to find conditions for the existence of an optimal model P^*. A good place to start is to look for conditions under which $C_\tau = C_{U^*_\tau}$ holds $P-$a.s., for any $P \in \mathcal{P}$ and $\tau \in \mathcal{S}$.

If the non-decreasing process C of the optional decomposition in (2.45) is in fact predictable, we obtain very interesting necessary and sufficient conditions for the existence of an optimal model P^*. Very recent work of Delbaen and Protter, [3], shows that the process C of the optional decomposition is predictable, if all the martingales in the set $\mathcal{Z}$ of (2.1) have continuous paths.

Theorem 2.21. *Suppose that the non-decreasing process C in the optional decomposition of (2.45) is predictable. Then we have $R^0_\tau = E^{P^*}(Y_{U^*_\tau}|\mathcal{F}_\tau)$ a.s. for any $\tau \in \mathcal{S}$ as well as $P^* \in \mathcal{P}$, if and only if C is "flat" away from the set $\mathcal{H}(\omega) = \{t \geq 0 \;/\; R^0_t(\omega) = Y_t(\omega)\}$, that is, $\int_0^\infty 1_{\{R^0_t(\omega) > Y_t(\omega)\}} dC_t = 0$ a.s.*

Proof of Necessity: If C is "flat" away from the set $\mathcal{H}$, then $C_t = C_{U^*_t}$ a.s. Therefore, we must have (2.46) as argued in the previous proof.

Proof of Sufficiency:

Step 1: We can show that, if P^* is optimal, then R^0 is *regular*, i.e., we have $E^P(R^0_v) = \lim_{n\to\infty} E^P(R^0_{v_n})$ for any nondecreasing sequence $\{v_n\}_{n\in\mathbb{N}}$ of stopping times in $\mathcal{S}$ with $v = \lim_{n\to\infty} v_n$ a.s. for all $P \in \mathcal{P}$.

The inequality $E^P(R^0_v) \leq \lim_{n\to\infty} E^P(R^0_{v_n})$ follows from the supermartingale property of R^0. For the reverse inequality, observe that the sequence $\{U^*_{v_n}\}_{n\in\mathbb{N}}$ of stopping times is also nondecreasing; moreover, $\lim_{n\to\infty} U^*_{v_n} \in \mathcal{S}_v$. Then we must have, due to Theorem 2.19: $E^P(R^0_{v_n}) = E^P(Y_{U^*_{v_n}})$, for any $P \in \mathcal{P}$. Therefore, Fatou's lemma, the quasi-left-continuity of Y, and the supermartingale property R^0, lead to:

$$\begin{aligned}
\limsup_{n\to\infty} E^P(R^0_{v_n}) &= \limsup_{n\to\infty} E^P(Y_{U^*_{v_n}}) \\
&\leq E^P(\limsup_{n\to\infty} Y_{U^*_{v_n}}) \\
&\leq E^P(Y\{\lim_{n\to\infty} U^*_{v_n}\}) \\
&\leq E^P(R^0\{\lim_{n\to\infty} U^*_{v_n}\}) \\
&\leq E^P(R^0_v).
\end{aligned}$$

Thus R^0 is regular. Also, because the process C is assumed predictable, we can prove as in [7] (Th.4.14/ page 28) that C is actually continuous.

Step 2: Define the family of stopping times:

$$p_t = \inf\{s \geq t \ / \ C_t < C_s\} \quad \text{for} \quad t \in [0, \infty]. \tag{2.47}$$

Then, due to Theorem 2.19:

$$E^P(R^0_{p_t}) = E^P(Y_{U^*_{p_t}}) = E^P(R^0_{U^*_{p_t}}), \quad \text{for any} \quad P \in \mathcal{P}.$$

Also, after applying the optional sampling theorem in the optional decomposition, we conclude that: $E^P(C_{p_t}) = E^P(C_{U^*_{p_t}})$, for any $P \in \mathcal{P}$. And since C is nondecreasing, we must have $C_{p_t} = C_{U^*_{p_t}}$ a.s.

The definition of p_t shows that $p_t(\omega) \in \mathcal{H}(\omega)$. Moreover, we can say that for P-a.e. $\omega \in \Omega$, we have

$$\{p_q(\omega) \;/\; q \in [0,\infty) \cap \mathbf{Q}\} \subseteq \mathcal{H}(\omega). \tag{2.48}$$

We now fix an ω for which (2.48) holds, for which the mapping $t \mapsto C_t(\omega)$ is continuous, and for which the mappings $t \mapsto R^0_t(\omega)$, $t \mapsto Y_t(\omega)$ are RCLL. To understand the set on which C is “flat”, we define

$$J(\omega) \triangleq \{t \geq 0 \;/\; (\exists)\epsilon > 0 \text{ with } C_{t-\epsilon}(\omega) = C_{t+\epsilon}(\omega)\}. \tag{2.49}$$

It is apparent that $J(\omega)$ is open, and thus can be written as a countable union of open intervals whose closures are disjoint:

$$J(\omega) = \cup_i(\alpha_i(\omega), \beta_i(\omega)). \tag{2.50}$$

We are interested in the set

$$\hat{J}(\omega) = \cup_i[\alpha_i(\omega), \beta_i(\omega)) = \{t \geq 0 \;/\; (\exists)\epsilon > 0 \text{ with } C_t(\omega) = C_{t+\epsilon}(\omega)\},$$

and in its complement $\hat{J}^c(\omega)$. The function $t \mapsto C_t(\omega)$ is "flat" on $\hat{J}(\omega)$, in the sense that

$$\int_0^\infty 1_{\hat{J}(\omega)}(t)dC_t(\omega) = \sum_i [C_{\beta_i(\omega)} - C_{\alpha_i(\omega)}] = 0. \tag{2.51}$$

Our task is to show that:

$$\hat{J}^c(\omega) \equiv \{t \geq 0 \;/\; (\forall)s > t, C_t(\omega) < C_s(\omega)\} \subseteq \mathcal{H}(\omega). \tag{2.52}$$

Let $t \in \hat{J}^c(\omega)$ be given. Then there is a strictly decreasing sequence $\{t_n\}_{n=1}^\infty$ such that $\{C_{t_n}\}_{n=1}^\infty$ is also strictly decreasing, and

$$t = \lim_{n\to\infty} t_n, \quad C_t(\omega) = \lim_{n\to\infty} C_{t_n}(\omega).$$

For each n, let q_n be a rational number in (t_n, t_{n+1}). Then $t \leq p_{q_n}(\omega) \leq t_{n+1}$ and $t = \lim_{n\to\infty} p_{q_n}(\omega)$. From (2.48) we have $R^0_{p_{q_n}(\omega)}(\omega) = Y_{p_{q_n}(\omega)}(\omega)$, and letting $n \to \infty$, using the right-continuity of R^0 and Y, we discover that $t \in \mathcal{H}(\omega)$, so (2.52) is proved. $\diamond$

Going back to the study of the process C we observe

$$E^P[R^0_{U_v^*}|\mathcal{F}_v] = R^0_v - E^P[C_{U_v^*} - C_v|\mathcal{F}_v], \quad \text{a.s.}$$

from (2.45). After taking the supremum with respect to P, we obtain:

$$\operatorname*{essinf}_{P\in\mathcal{P}} E^P[C_{U_v^*} - C_v|\mathcal{F}_v] = 0. \tag{2.53}$$

Using again the fundamental property of essential supremum/infimum we get a sequence $\{Z^k\} \subseteq \mathcal{Z}$ such that:

$$\operatorname*{essinf}_{Z\in\mathcal{Z}} E^Q\left[\frac{Z_{U_v^*}}{Z_v}\cdot(C_{U_v^*}-C_v)\Big|\mathcal{F}_v\right] = \lim_{k\to\infty}\downarrow E^Q\left[\frac{Z^k_{U_v^*}}{Z^k_v}\cdot(C_{U_v^*}-C_v)\Big|\mathcal{F}_v\right] = 0.$$

Then we use Fatou's lemma for conditional expectations to obtain:

$$0 \le E^Q\left[\lim_{k\to\infty}\downarrow \frac{Z^k_{U_v^*}}{Z^k_v}\cdot(C_{U_v^*}-C_v)\Big|\mathcal{F}_v\right] \le \lim_{k\to\infty}\downarrow E^Q\left[\frac{Z^k_{U_v^*}}{Z^k_v}\cdot(C_{U_v^*}-C_v)\Big|\mathcal{F}_v\right] = 0.$$

Therefore,

$$E^Q\left[\lim_{k\to\infty}\downarrow \frac{Z^k_{U_v^*}}{Z^k_v}\cdot(C_{U_v^*}-C_v)\Big|\mathcal{F}_v\right] \equiv 0.$$

But, because C is nondecreasing, if $\lim_{k\to\infty}\downarrow\left(\frac{Z^k_{U_v^*}}{Z^k_v}\right) > 0$, we must have $C_v = C_{U_v^*}$, $P-$a.s.

Lemma 2.22. *If $\mathcal{Z}$ is closed (as in Remark 2.16), then*

$$\operatorname*{essinf}_{Z\in\mathcal{Z}}\left(\frac{Z_\mu}{Z_\tau}\right) > 0, \;\; a.s.$$

for all stopping times $\tau, \mu \in \mathcal{S}$.

Proof. The Remark 2.16 tells us that $\mathcal{Z}$ is closed, and implied that $\mathcal{Z}_\sigma = \{Z_\sigma \,/ Z\in\mathcal{Z}\}$ being closed, with respect to the $\mathbf{L^2}$ norm, for all stopping times

$\sigma \in \mathcal{S}$. Hence, due to Remark 2.2 we must have that there exist processes $\underline{Z}, \overline{Z} \in \mathcal{Z}$, such that almost surely,

$$\begin{aligned} \underline{Z}_\mu &\triangleq \operatorname*{essinf}_{Z \in \mathcal{Z}} Z_\mu = \lim_{k\to\infty} \downarrow \tilde{Z}^k_\mu \in \mathcal{Z}_\mu, \text{ and} \\ \overline{Z}_\tau &\triangleq \operatorname*{esssup}_{Z \in \mathcal{Z}} Z_\tau = \lim_{k\to\infty} \uparrow Z^k_\tau \in \mathcal{Z}_\tau. \end{aligned} \tag{2.54}$$

It is obvious then, such that for all $Z \in \mathcal{Z}$

$$\frac{Z_\mu}{Z_\tau} \geq \frac{\underline{Z}_\mu}{\overline{Z}_\tau} > 0 \quad \text{a.e.}$$

Recall that for all $Z \in \mathcal{Z}$ we have $Z > 0$, a.s., and also $Z < \infty$, a.s., since the processes Z are Radon-Nikodym derivatives for an equivalent pair of probability measure. Hence, the above ratios ar not only well defined, but they must also be strictly positive. Therefore, after taking essential infimum with respect to Z, the Lemma is proved. $\diamond$

Recall now that we have assumed $\mathcal{N}$ to be a subspace of $\mathcal{M}_2$, therefore it is a closed set, and this implies that $\mathcal{Z}$ is closed as well. Thus, by using the Lemma 2.22, we have that $C_v = C_{U^*_v}$, a.s., which leads immediately to:

$$E^P[R^0_{U^*_v} | \mathcal{F}_v] = R^0_v, \quad \text{a.s.} \tag{2.55}$$

and therefore, due to the Theorem 2.9, the following theorem is proved.

Theorem 2.23. *There exists an optimal model P^* for our optimization problem. Moreover, any model is then optimal.*

2.4 Deterministic Approach to the Optimal Stopping Problem Under Model-Uncertainty

We have already seen in the previous subsection that, under appropriate conditions, we can decompose our Snell envelope R^0 into $R^0_t = R^0_0 + X_t - C_t$, where X is a $\mathcal{P}$-martingale, and C is an adapted process with nondecreasing, RCLL paths, $C_0 = 0$, and $E^P(C_\infty) < \infty$ for all $P \in \mathcal{P}$. The focus of our analysis was mostly on C, which has the obvious interpretation as the loss of available reward due to failure to stop at the right times. Here we show that the martingale component X also has a clear interpretation; namely, that the (non-adapted) process Λ defined by $\Lambda_t = X_\infty - X_t$ is the Lagrange multiplier enforcing the constraint that the modified process $Y + \Lambda$ must be stopped at stopping times rather than at general random times.

Theorem 2.24. *We have*

$$R = \sup_{P \in \mathcal{P}} \sup_{\tau \in \mathcal{S}} E^P(Y_\tau) = E^P \left[\sup_{0 \le t \le \infty} (Y_t + \Lambda_t) \right], \tag{2.56}$$

for all $P \in \mathcal{P}$, *where* $\Lambda_t \triangleq X_\infty - X_t$.

Proof: With $M_t \triangleq R^0_0 + X_t = R^0_t + C_t$, it is obvious that $\Lambda_t = M_\infty - M_t$. Define $Q_t \triangleq Y_t + \Lambda_t$, and $\nu_t \triangleq \sup_{s \ge t} Q_s$. We shall prove that

$$E^P[\nu_t | \mathcal{F}_t] = R^0_t \quad \text{a.s.} \tag{2.57}$$

holds for every $t \in [0, \infty]$ and $P \in \mathcal{P}$, from which (2.56) will follow immediately by taking $t = 0$. Observe that,

$$\begin{aligned} Q_s &= Y_s + M_\infty - M_s \\ &= Y_s + (R^0_\infty + C_\infty) - (R^0_s + C_s) \\ &= R^0_\infty - (R^0_s - Y_s) + (C_\infty - C_s). \end{aligned}$$

But $R^0_s - Y_s \geq 0$, and $C_\infty - C_s \leq C_\infty - C_t$ for $s \geq t$, so clearly we have:

$$Q_s \leq R^0_\infty + C_\infty - C_t = M_\infty - C_t.$$

Therefore, if we take the supremum over all $s \geq t$, we obtain:

$$\nu_t \leq M_\infty - C_t, \quad \text{for all } t \in [0, \infty]. \tag{2.58}$$

On the other hand we have, for any $\alpha \in (0, 1)$:

$$\begin{aligned} Q_{U^\alpha_t} &\geq R^0_\infty - R^0_{U^\alpha_t}(1 - \alpha) + (C_\infty - C_{U^\alpha_t}) \\ &= M_\infty - R^0_{U^\alpha_t}(1 - \alpha) - C_{U^\alpha_t} \\ &= M_\infty - C_t - R^0_{U^\alpha_t}(1 - \alpha). \end{aligned}$$

We have used the fact that the non-decreasing process C is flat on the interval $[t, U^*_t]$, which consists of all U^α_t for all $0 < \alpha < 1$. Letting $\alpha \uparrow 1$ we conclude that

$$\nu_t \geq M_\infty - C_t, \quad \text{for all } t \in [0, \infty], \tag{2.59}$$

since $C_t = C_{U^*_t} = \lim_{\alpha \uparrow 1} C_{U^\alpha_t}$ for all $t \in [0, \infty]$. Therefore:

$$\nu_t = M_\infty - C_t. \tag{2.60}$$

Now let us take the conditional expectations with respect to any P, given $\mathcal{F}_t$, and obtain:

$$E^P[\nu_t|\mathcal{F}_t] = E^P[M_\infty - C_t|\mathcal{F}_t] = R_t^0.$$

$\diamond$

Another interesting result, inspired by Rogers [10], is very useful in the context of dealing with the rather cumbersome task of determining the martingale X of the optional decomposition.

Theorem 2.25. *We have*

$$R = \inf_{\lambda \in \mathcal{L}} E^P \left(\sup_{t \in [0,\infty]} (Y_t + \lambda_t) \right), \tag{2.61}$$

for all $P \in \mathcal{P}$. Here $\mathcal{L}$ is the class of measurable processes $\lambda : [0,\infty] \times \Omega \to \mathbb{R}$, for which $E^P(\sup_{0 \le t \le \infty} |\lambda_t|) < \infty$ and $E^P(\lambda_\tau) = 0$, hold for every $P \in \mathcal{P}$ and for any $\tau \in \mathcal{S}$. The infimum is attained by $\lambda = \Lambda$ (where Λ is the Lagrange multiplier from Theorem 2.24).

Proof:

From the optional sampling theorem, the process $\Lambda_. \equiv X_\infty - X_.$ from Theorem 2.24 satisfies $E^P(\Lambda_\tau) = E^P(X_\infty) - E^P(X_\tau) = 0$, for any $\tau \in \mathcal{S}$, thus $\Lambda \in \mathcal{L}$, and we have:

$$R = E^P \left[\sup_{t \ge 0} (Y_t + \Lambda_t) \right] \ge \inf_{\lambda \in \mathcal{L}} E^P \left[\sup_{t \ge 0} (Y_t + \lambda_t) \right],$$

for every $P \in \mathcal{P}$. To prove the reverse inequality, it suffices to show that

$$E^P(Y_\tau) \leq E^P\left[\sup_{t \geq 0}(Y_t + \lambda_t)\right],$$

for any $\tau \in \mathcal{S}$ holds for every $\lambda \in \mathcal{L}$. But this is obvious. $\diamond$

3 Non-cooperative Game

3.1 The Value of the Stochastic Game

We shall begin in the study of the stochastic game by recalling a key result of the theory of optimal stopping, for a given probability "scenario" P. It was proved in [8] (see page 358, Th D.12) that if we assume that the "reward" process Y is quasi-left-continuous, and that $Y^* \triangleq \sup_{t\in[0,\infty]} Y_t$ is $P-$integrable, the Snell envelope has the following "martingale" property:

$$E^P\left[Y_{\hat{\rho}_P(\tau)}|\mathcal{F}_\tau\right] = E^P\left[\widehat{V}_P(\hat{\rho}_P(\tau))|\mathcal{F}_\tau\right] = \widehat{V}_P(\tau) \tag{3.1}$$

a.s., for every $P \in \mathcal{P}$. Moreover, since the Snell envelope $\widehat{V}_P$ is $P-$supermatingale, (3.1) is also true if we replace $\hat{\rho}_P(\tau)$, with any stopping time ρ with values in the stochastic interval $[\tau, \hat{\rho}_P(\tau)]$. This property of the Snell envelope will help us prove the following two "key" results about the upper-value process $\overline{V}$ of (1.7).

Proposition 3.1. *For all $P \in \mathcal{P}$, and any stopping time ρ with values in the stochastic interval $[\tau, \rho_\tau]$, we have:*

$$E^P\left[\overline{V}_\rho|\mathcal{F}_\tau\right] \geq \overline{V}_\tau, \quad a.s. \tag{3.2}$$

In particular, the stopped process $\{\overline{V}_{t\wedge\rho_0}, \mathcal{F}_t\}_{0\leq t\leq\infty}$ is a $\mathcal{P}-$submartingale.

Proof. As a consequence of the fundamental property of the essential infimum, we can show that for a certain sequence $\{P_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}$ we have $\overline{V}_t = \lim_{k\to\infty} \widehat{V}_{P_k}(t)$.

Also, notice that since $0 \leq \widehat{V}_{P_k}(t) \leq R_t$, holds a.s. for all $P_k \in \mathcal{P}$, hence the sequence $\{\widehat{V}_{P_k}(t)\}_{k\in\mathbb{N}}$ is dominated by the random variable R_t, random variable which is integrable if we assume that the cooperative version of the game has value, i.e. $R_0 < \infty$. (Recall that the $\mathcal{P}-$supermartingale property of R gives us that $E^P(R_t) \leq R_0$; the maximum expected reward of the cooperative game.)

Consider under the same notation as in the Subsection 2.1, the processes $Z_t^k = \frac{dP_k}{dQ}\big|_{\mathcal{F}_t}$ and $Z_t = \frac{dP}{dQ}\big|_{\mathcal{F}_t}$ in $\mathcal{Z}$.

Clearly, for $\sigma \leq \rho$, the random variable $\widehat{V}_{P_k}(\rho)$ does not depend on the values of Z_σ^k; therefore, we may assume, without any loss of generality, that $\{Z^k\}_{k\in\mathbb{N}} \subseteq \mathcal{Z} \cap \mathcal{N}_{\tau,\rho}^Z$, i.e., that Z^k agrees with Z on the stochastic interval $[\![\tau, \rho_\tau]\!]$.

Hence, due to dominated convergence theorem and because $\rho \leq \rho_\tau \leq \hat{\rho}_P(\tau)$ for all P, we have:

$$
\begin{aligned}
E^P\left[\overline{V}_\rho|\mathcal{F}_\tau\right] &= E^P\left[\lim_{k\to\infty}\widehat{V}_{P_k}(\rho)|\mathcal{F}_\tau\right]\\
&= \lim_{k\to\infty} E^P\left[\widehat{V}_{P_k}(\rho)|\mathcal{F}_\tau\right]\\
&= \lim_{k\to\infty} E^Q\left[\frac{Z_\rho}{Z_\tau}\cdot\widehat{V}_{P_k}(\rho)|\mathcal{F}_\tau\right]\\
&= \lim_{k\to\infty} E^Q\left[\frac{Z_\rho^k}{Z_\tau^k}\cdot\widehat{V}_{P_k}(\rho)|\mathcal{F}_\tau\right]\\
&= \lim_{k\to\infty} E^{P_k}\left[\widehat{V}_{P_k}(\rho)|\mathcal{F}_\tau\right]\\
&\geq \operatorname*{essinf}_{P\in\mathcal{P}} E^P\left[\widehat{V}_P(\rho)|\mathcal{F}_\tau\right] = \operatorname*{essinf}_{P\in\mathcal{P}}\widehat{V}_P(\tau) = \overline{V}_\tau.
\end{aligned}
$$

$\diamond$

Another consequence of Proposition 3.1 is that, under the *worst case scenario*, had the player not stopped the game before time τ, should wait until time ρ_τ to do so.

Proposition 3.2. $\overline{V}_\tau = \operatorname{essinf}_{P\in\mathcal{P}} E^P\left[Y(\rho_\tau)|\mathcal{F}_\tau\right]$ *a.s.*

Proof. From to Proposition 3.1, we have

$$E^P\left[Y_{\rho_\tau}|\mathcal{F}_\tau\right] = E^P\left[\overline{V}_{\rho_\tau}|\mathcal{F}_\tau\right] \geq \overline{V}_\tau, \quad \text{a.s.}$$

for all $P \in \mathcal{P}$, hence $\overline{V}_\tau \leq \operatorname{essinf}_{P\in\mathcal{P}} E^P\left[Y(\rho_\tau)|\mathcal{F}_\tau\right]$ a.s. The reverse inequality follows immediately, since:

$$\overline{V}_\tau = \operatorname*{essinf}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P\left[Y_\sigma|\mathcal{F}_\tau\right] \geq \operatorname*{essinf}_{P\in\mathcal{P}} E^P\left[Y(\rho_\tau)|\mathcal{F}_\tau\right], \quad \text{a.s.}$$

$\diamond$

Theorem 3.3. *The stochastic game of* (1.3), (1.4) *has value, i.e., for all* $\tau \in \mathcal{S}$*,*

$$V_\tau = \underline{V}_\tau = \overline{V}_\tau, \quad a.s.$$

Proof. The proof follows immediately from Proposition 3.2, which gives

$$\overline{V}_\tau = \operatorname*{essinf}_{P\in\mathcal{P}} E^P\left[Y(\rho_\tau)|\mathcal{F}_\tau\right] \leq \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} \operatorname*{essinf}_{P\in\mathcal{P}} E^P\left[Y_\sigma|\mathcal{F}_\tau\right] \triangleq \underline{V}_\tau,$$

and from the definitions (1.3), (1.4) which give $\overline{V}_\tau \geq \underline{V}_\tau$, a.s. $\diamond$

From now on we shall deal only with the "value" process V.

3.2 Properties Of The "*Value*" Process

We have established so far the behavior of the value process V on stochastic intervals of the type $[\![\tau, \rho_\tau]\!]$. It is not clear, however, what happens with this process the rest of the time. The following propositions attempt to answer this very question.

Proposition 3.4. *For all stopping times $v \in \mathcal{S}$ and $\tau \in \mathcal{S}_v$, we have:*

$$\operatorname*{essinf}_{P\in\mathcal{P}} E^P\left[V_\tau|\mathcal{F}_v\right] \leq \operatorname*{essinf}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P\left[Y_\sigma|\mathcal{F}_v\right]. \tag{3.3}$$

Moreover, the reverse inequality holds for all τ in the stochastic interval $[\![v, \rho_v]\!]$.

Proof. From Proposition 3.1 we have $V_\tau \leq E^P\left[V_{\rho_\tau}|\mathcal{F}_\tau\right]$ for all $P \in \mathcal{P}$. Therefore, after taking conditional expectations, we obtain

$$\begin{aligned} E^P\left[V_\tau|\mathcal{F}_v\right] &\leq E^P\left[E^P\left[V_{\rho_\tau}|\mathcal{F}_\tau\right]|\mathcal{F}_v\right] = E^P\left[V_{\rho_\tau}|\mathcal{F}_v\right] \\ &= E^P\left[Y_{\rho_\tau}|\mathcal{F}_v\right] \leq \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P\left[Y_\sigma|\mathcal{F}_v\right], \quad \forall P \in \mathcal{P}. \end{aligned}$$

for all $\tau \in \mathcal{S}_v$. Taking essential infimum with respect to P, we arrive at the inequality (3.3).

Moreover, from the same Proposition 3.1 (the submartingale property of V on $[\![v, \rho_v]\!]$), we deduce that for all τ in the stochastic interval $[\![v, \rho_v]\!]$, we have

$$\begin{aligned} E^P\left[V_\tau|\mathcal{F}_v\right] &\geq V_v \\ &= \operatorname*{essinf}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_v} E^P\left[Y_\sigma|\mathcal{F}_v\right] \\ &\geq \operatorname*{essinf}_{P\in\mathcal{P}} \operatorname*{esssup}_{\sigma\in\mathcal{S}_\tau} E^P\left[Y_\sigma|\mathcal{F}_v\right], \quad \forall P \in \mathcal{P}. \end{aligned}$$

Taking again the essential infimum with respect to $P \in \mathcal{P}$ we obtain the desired inequality for all τ in the stochastic interval $[\![v, \rho_v]\!]$. $\diamond$

The following result offers additional insight into the nature of the "value" process. Although it does not necessarily prove that V is a supermartingale, it shows that indeed, under the *worst case scenario*, the expected "value" decreases with time.

Proposition 3.5. *For all stopping times τ, μ such that $\tau \leq \mu$, we have*

$$\inf_{P \in \mathcal{P}} E^P(V_\tau) \geq \inf_{P \in \mathcal{P}} E^P(V_\mu). \tag{3.4}$$

Proof. We use again the fundamental property of essential infimum; from Proposition 3.1, we have

$$\begin{aligned}
V_\tau &= \operatorname*{essinf}_{P \in \mathcal{P}} \operatorname*{esssup}_{\sigma \in \mathcal{S}_\tau} E^P \left[Y_\sigma | \mathcal{F}_\tau \right] \\
&\geq \operatorname*{essinf}_{P \in \mathcal{P}} E^P \left[Y_{\rho_\mu} | \mathcal{F}_\tau \right] \\
&= \operatorname*{essinf}_{P \in \mathcal{P}} E^P \left[V_{\rho_\mu} | \mathcal{F}_\tau \right] \\
&= \operatorname*{essinf}_{P \in \mathcal{P}} E^P \left[E^P \left[V_{\rho_\mu} | \mathcal{F}_\mu \right] | \mathcal{F}_\tau \right] \\
&\geq \operatorname*{essinf}_{P \in \mathcal{P}} E^P \left[V_\mu | \mathcal{F}_\tau \right] \\
&= \lim_{k \to \infty} E^{P_k} \left[V_\mu | \mathcal{F}_\tau \right] \\
&= \lim_{k \to \infty} E^Q \left[\frac{Z_\mu^k}{Z_\tau^k} \cdot V_\mu | \mathcal{F}_\tau \right],
\end{aligned}$$

for a certain sequence $\{P_k\}_{k \in \mathbb{N}} \subseteq \mathcal{P}$.

Let us have a probability model $P \in \mathcal{P}$ fixed. Once again we can select the sequence $\{P_k\}_{k\in\mathbb{N}}$, such that $\{Z^k\}_{k\in\mathbb{N}} \subseteq \mathcal{Z} \cap \mathcal{N}^Z_{0,\tau}$, therefore due to dominated convergence theorem

$$\begin{aligned} E^P(V_\tau) &\geq E^Q\left[Z_\tau \cdot \lim_{k\to\infty} E^Q\left[\frac{Z^k_\mu}{Z^k_\tau} \cdot V_\mu | \mathcal{F}_\tau\right]\right] \\ &= \lim_{k\to\infty} E^Q\left[Z^k_\mu V_\mu\right] \\ &= \lim_{k\to\infty} E^{P_k}(V_\mu) \\ &\geq \inf_{P\in\mathcal{P}} E^P(V_\mu), \end{aligned}$$

and after taking the infimum with respect to P, the inequality (3.4) is proved. $\diamond$

Corollary 3.6. *For all stopping times $\tau \in \mathcal{S}$ we have*

$$\inf_{P\in\mathcal{P}} E^P(V_\tau) = \inf_{P\in\mathcal{P}} E^P(V_{\rho_\tau}). \tag{3.5}$$

Proof. From Proposition 3.1 we know that $E^P(V_\tau) \leq E^P(V_{\rho_\tau})$ holds for all $P \in \mathcal{P}$, therefore after taking infimum we obtain:

$$\inf_{P\in\mathcal{P}} E^P(V_\tau) \leq \inf_{P\in\mathcal{P}} E^P(V_{\rho_\tau}).$$

But $\tau \leq \rho_\tau$ a.e., and thus the reverse inequality follows from Proposition 3.5. $\diamond$

In this subsection we have used the term *worst case scenario* rather loosely, since we do not know, yet, whether such a scenario exists, and moreover, if at any time the same probability model will make for the *worst case scenario*. The next proposition helps us shed some light into this mater.

Proposition 3.7. *If* $E^{P^*}(V_{\rho_\tau}) = \inf_{P\in\mathcal{P}} E^P(V_{\rho_\tau})$ *for some* $P^* \in \mathcal{P}$ *and* $\tau \in \mathcal{S}$, *then we must have* $E^{P^*}(V_\tau) = E^{P^*}(V_{\rho_\tau})$.

Proof. The previous Corollary and Proposition 3.1 give

$$E^{P^*}(V_{\rho_\tau}) = \inf_{P\in\mathcal{P}} E^P(V_{\rho_\tau}) = \inf_{P\in\mathcal{P}} E^P(V_\tau) \leq E^{P^*}(V_\tau) \leq E^{P^*}(V_{\rho_\tau}).$$

$\diamond$

3.3 Characterization of a Saddle-Point

In this subsection we present results about the characterization of a saddle-point for the stochastic game defined by (1.4) and (1.3). First we shall need some preliminary facts.

Lemma 3.8. *For all stopping times τ, μ with $\mu \geq \tau$, we have*

$$\{Z^*_\tau = \operatorname*{essinf}_{Z \in \mathcal{Z}} Z_\tau\} \subseteq \{Z^*_\mu = \operatorname*{essinf}_{Z \in \mathcal{Z}} Z_\mu\}. \tag{3.6}$$

Proof. The proof is based on the fact that the processes $Z \in \mathcal{Z}$ are $Q-$martingales, dominated by an integrable random variable in the following sense: for all $Z \in \mathcal{Z}$ we have $Z_\sigma \leq \tilde{Z}_\sigma \triangleq \operatorname{esssup}_{Z \in \mathcal{Z}} Z_\sigma$, where $\tilde{Z}_\sigma$ is $Q-$integrable, since

$$E^Q(\tilde{Z}_\sigma) = E^Q\left(\lim_{k \to \infty} \uparrow Z^k_\sigma\right) = \lim_{k \to \infty} E^Q(Z^k_\sigma) = 1$$

thanks to Remark 2.2 and the Monotone Convergence Theorem, for some increasing sequence $\{Z^k_\sigma\}_{k \in \mathbb{N}}$. Therefore, we may apply the Dominated Convergence Theorem to a conveniently chosen sequence $\{Z^k_\mu\}_{k \in \mathbb{N}}$, for which we have

$$\operatorname*{essinf}_{Z \in \mathcal{Z}} Z_\mu = \lim_{k \to \infty} \downarrow Z^k_\mu$$

as explained in Remark 2.2.

Notice now that the event $A \triangleq \{Z^*_\tau = \operatorname{essinf}_{Z \in \mathcal{Z}} Z_\tau\}$ belongs to $\mathcal{F}_\tau$, hence the

Dominated Convergence Theorem and the martingale property of Z give us

$$\begin{aligned}
\int_A \operatorname*{essinf}_{Z\in\mathcal{Z}} Z_\mu dQ &= \int_A \lim_{k\to\infty} \downarrow Z^k_\mu dQ \\
&= \lim_{k\to\infty} \downarrow \int_A Z^k_\mu dQ \\
&= \lim_{k\to\infty} \downarrow \int_A Z^k_\tau dQ \\
&\geq \int_A Z^*_\tau dQ \\
&= \int_A Z^*_\mu dQ \\
&\geq \int_A \operatorname*{essinf}_{Z\in\mathcal{Z}} Z_\mu dQ.
\end{aligned}$$

Therefore, we must have $Z^*_\mu = \operatorname{essinf}_{Z\in\mathcal{Z}} Z_\mu$ on A, i.e.

$$\{Z^*_\tau = \operatorname*{essinf}_{Z\in\mathcal{Z}} Z_\tau\} \subseteq \{Z^*_\mu = \operatorname*{essinf}_{Z\in\mathcal{Z}} Z_\mu\} \tag{3.7}$$

for $\mu \in \mathcal{S}_\tau$, and the Lemma is proved. $\diamond$

In the following proposition we investigate in more detail the nature of a saddle point (τ^*, P^*). We notice that if at time τ^* the "reward" process Y becomes null, there is no room for improvement in the future. If the probability P^* can be interpreted as the "scenario" that returns the smallest expected reward at a certain time, then we can be sure that the same probability model is going to be *worst case scenario* at any other stopping time from then on as well.

Proposition 3.9. *If (τ^*, P^*) is a saddle-point (i.e. $E^{P^*}(Y_\tau) \leq E^{P^*}(Y_{\tau^*}) \leq E^P(Y_{\tau^*})$, holds for all stopping times $\tau \in \mathcal{S}$, and probability models $P \in \mathcal{P}$), then* *(i)* $\{Y_{\tau^*} = 0\} \subseteq \bigcap_{\mu \in \mathcal{S}_{\tau^*}} \{Y_\mu = 0\}$*;*

(ii) $E^{P^*}(Y_\mu) \leq E^P(Y_\mu)$ *for all* $P \in \mathcal{P}$ *and for all* $\mu \in \mathcal{S}_{\tau^*}$*.*

Proof. To prove (i) notice that for all stopping times $\mu \in \mathcal{S}_{\tau^*}$ the first of the saddle-point inequalities can be written as

$$Y_{\tau^*} \geq E^{P^*}(Y_\mu | \mathcal{F}_{\tau^*}), \quad \text{a.s.}$$

Therefore, $0 = Y_{\tau^*} 1_{\{Y_{\tau^*}=0\}} \geq E^{P^*}(Y_\mu 1_{\{Y_{\tau^*}=0\}} | \mathcal{F}_{\tau^*}) \geq 0$, a.s. Then we must have $\{Y_{\tau^*} = 0\} \subseteq \{Y_\mu = 0\}$, for all stopping times $\mu \in \mathcal{S}_{\tau^*}$.

To prove (ii) we shall follow a similar argument as the one presented in the proof of the Lemma 3.8 and we obtain that

$$\begin{aligned}
E^{P^*}(Y_{\tau^*}) &= E^Q(Z^*_{\tau^*} \cdot Y_{\tau^*}) \\
&\geq E^Q\left(\operatorname*{essinf}_{Z \in \mathcal{Z}} Z_{\tau^*} \cdot Y_{\tau^*}\right) \\
&= E^Q\left(\lim_{k\to\infty} \downarrow Z^k_{\tau^*} \cdot Y_{\tau^*}\right) \\
&= \lim_{k\to\infty} \downarrow E^Q(Z^k_{\tau^*} \cdot Y_{\tau^*}) \geq E^Q(Z^*_{\tau^*} \cdot Y_{\tau^*}) \\
&= E^{P^*}(Y_{\tau^*}).
\end{aligned}$$

Therefore,

$$E^Q\left(\operatorname*{essinf}_{Z \in \mathcal{Z}} Z_{\tau^*} \cdot Y_{\tau^*}\right) = E^Q(Z^*_{\tau^*} \cdot Y_{\tau^*}). \tag{3.8}$$

This means that on the event $\{Y_{\tau^*} > 0\}$ we must have $Z^*_{\tau^*} = \operatorname{essinf}_{Z \in \mathcal{Z}} Z_{\tau^*}$. Therefore, due to Lemma 3.8, we must have $Z^*_\mu = \operatorname{essinf}_{Z \in \mathcal{Z}} Z_\mu$ on this same event, for all $\mu \in \mathcal{S}_{\tau^*}$. From the above observation, and from part (i) of this proposition, it follows that, for an arbitrary $\mu \in \mathcal{S}_{\tau^*}$, and $P \in \mathcal{P}$ (setting $Z_\sigma \triangleq \frac{dP}{dQ}|_{\mathcal{F}_\sigma}$ for $\sigma \in \mathcal{S}$), we have

$$\begin{aligned}
E^{P^*}(Y_\mu) &= E^Q(Z^*_\mu \cdot Y_\mu) \\
&= E^Q\left(Z^*_\mu \cdot Y_\mu 1_{\{Y_{\tau^*}=0\}} + Z^*_\mu \cdot Y_\mu 1_{\{Y_{\tau^*}>0\}}\right) \\
&\leq E^Q\left(Z^*_\mu \cdot Y_\mu 1_{\{Y_\mu=0\}}\right) + E^Q\left(Z^*_\mu \cdot Y_\mu 1_{\{Y_{\tau^*}>0\}}\right) \\
&\leq E^Q\left(Z_\mu \cdot Y_\mu 1_{\{Y_{\tau^*}>0\}}\right) \\
&\leq E^Q(Z_\mu \cdot Y_\mu) = E^P(Y_\mu),
\end{aligned}$$

and the proposition is proved. ◇

We are able now to identify the necessary and sufficient conditions on a pair (τ^*, P^*) to be a *saddle-point* for the stochastic game defined by (1.4) and (1.3).

Theorem 3.10. *A pair (τ^*, P^*) is a saddle point for the stochastic game (i.e. $E^{P^*}(Y_\tau) \leq E^{P^*}(Y_{\tau^*}) \leq E^P(Y_{\tau^*})$, holds for all stopping times $\tau \in \mathcal{S}$, and probability models $P \in \mathcal{P}$), if and only if:*

(i) $Y_{\tau^*} = V_{\tau^*}$, a.e.;

(ii) $\{V_{t\wedge\tau^*}; \mathcal{F}_t\}$ is a P^*-martingale;

(iii) $\{V_t; \mathcal{F}_t\}$ is a P^*-supermartingale.

Proof of Necessity: Suppose that (τ^*, P^*) is a saddle point. (i) follows immediately form Proposition 3.1, and because $E^{P^*}(Y_\tau) \leq E^{P^*}(Y_{\tau^*})$ for all stopping times

$\tau \in \mathcal{S}$; namely,

$$E^{P^*}(Y_{\tau^*}) \;\; \leq \;\; E^{P^*}(V_{\tau^*}) \leq E^{P^*}(V_{\rho_{\tau^*}}) = E^{P^*}(Y_{\rho_{\tau^*}}) \leq E^{P^*}(Y_{\tau^*}).$$

Therefore, we have $E^{P^*}(Y_{\tau^*}) = E^{P^*}(V_{\tau^*})$, and again using the fact that $Y_\tau \leq V_\tau$, holds a.s., for all $\tau \in \mathcal{S}$, we have that $Y_{\tau^*} = V_{\tau^*}$, a.e.

To prove (ii) we need to notice that, because (i) holds, we must have $\rho_\tau \leq \rho_{\tau^*} = \tau^*$ for all τ such that $\tau \leq \tau^*$. Then, because (τ^*, P^*) is saddle-point, and because we can apply Proposition 3.5, we obtain the following inequalities:

$$\inf_{P \in \mathcal{P}} E^P(V_{\tau^*}) \leq \inf_{P \in \mathcal{P}} E^P(V_{\rho_\tau}) \leq E^{P^*}(V_{\rho_\tau}) = E^{P^*}(Y_{\rho_\tau})$$

and

$$E^{P^*}(Y_{\rho_\tau}) \leq E^{P^*}(Y_{\tau^*}) = \inf_{P \in \mathcal{P}} E^P(Y_{\tau^*}) = \inf_{P \in \mathcal{P}} E^P(V_{\tau^*}).$$

Hence, $\inf_{P \in \mathcal{P}} E^P(V_{\rho_\tau}) = E^{P^*}(V_{\rho_\tau}) = E^{P^*}(V_{\tau^*})$, and Proposition 3.7 implies that

$$E^{P^*}(V_\tau) = E^{P^*}(V_{\rho_\tau}) = E^{P^*}(V_{\tau^*}) \tag{3.9}$$

holds for all τ such that $\tau \leq \tau^*$. Therefore, (ii) holds.

To prove (iii) we must combine the results of Propositions 3.5, 3.7, and 3.9. Part(ii) of Proposition 3.9 shows that for all $\tau \in \mathcal{S}_{\tau^*}$, and $P \in \mathcal{P}$ we have

$$E^{P^*}(Y_\tau) \leq E^P(Y_\tau). \tag{3.10}$$

Let us consider μ, σ in $\mathcal{S}_{\tau^*}$, arbitrary stopping times, such that $\mu \leq \sigma$. We can use (3.10), for stopping times ρ_μ, and ρ_σ, respectively, to apply Proposition 3.7. We can do so since we have $Y_{\rho_\mu} = V_{\rho_\mu}$, and $Y_{\rho_\sigma} = V_{\rho_\sigma}$, respectively. Therefore, the value process must be P^*-martingale on the stochastic intervals $[\![\mu, \rho_\mu]\!]$, and $[\![\sigma, \rho_\sigma]\!]$. Combining this observation with Proposition 3.5 we obtain

$$E^{P^*}(V_\mu) = E^{P^*}(V_{\rho_\mu}) = \inf_{P \in \mathcal{P}} E^P(V_{\rho_\mu}) \geq \inf_{P \in \mathcal{P}} E^P(V_{\rho_\sigma}) = E^{P^*}(V_{\rho_\sigma}) = E^{P^*}(V_\sigma),$$

for all stopping times μ, σ in $\mathcal{S}_{\tau^*}$, such that $\mu \leq \sigma$, i.e. we have proved the supermartingale property of the process V from the stoping time $\tau*$ on. At this point we can conclude due to (ii) that, $\{V_t, \mathcal{F}_t\}_{0 \leq t \leq \infty}$ is indeed P^*-supermartingale, hence (iii) is proved.

Proof of Suficiency. Let us assume now that the conditions (i)-(iii) hold. It is easy to see that, since the maximal reward process V is a P^*-supermartingale and a martingale up to time τ^*, due to optional sampling theorem we have

$$E^{P^*}(Y_\tau) \leq E^{P^*}(V_\tau) \leq E^{P^*}(V_{\tau^*}) = E^{P^*}(Y_{\tau^*}),\ \forall\ \tau \in \mathcal{S}. \tag{3.11}$$

To prove the second inequality of the saddle-point property, we observe that (i) implies $\rho_0 \leq \tau*$ a.e., therefore the martingale property of V implies

$$V_0 = E^{P^*}(V_{\rho_0}) = E^{P^*}(V_{\tau^*}). \tag{3.12}$$

We can use Proposition 3.2 for $\tau = 0$, and obtain:

$$V_0 = \inf_{P \in \mathcal{P}} E^P(Y_{\rho_0}) = \inf_{P \in \mathcal{P}} E^P(V_{\rho_0}). \tag{3.13}$$

Hence, (3.12) and (3.13) imply $E^{P^*}(Y_{\rho_0}) \leq E^P(Y_{\rho_0})$ holds for all $P \in \mathcal{P}$; and thus, (ρ_0, P^*) becomes a saddle-point. We may use now the result offered by the part (ii) of Proposition 3.9, for the saddle-point (ρ_0, P^*) to conclude that for all $\tau \in \mathcal{S}_{\rho'}$, thus also for $\tau = \tau^*$, we have

$$E^{P^*}(Y_\tau) \leq E^P(Y_\tau),\ \forall P \in \mathcal{P}. \tag{3.14}$$

Therefore, the pair (τ^*, P^*) is indeed a saddle-point for the stochastic game. $\diamond$

3.4 Existence of a Saddle-Point

From what we have seen in the previous subsection, it seems that if we can find $P^* \in \mathcal{P}$ such that $\{V_t, \mathcal{F}_t\}$ is a P^*-supermartingale, we have already identified a saddle point in the form of the (ρ_0, P^*). The problem that we still face is that such a specific "scenario" may not exist in general, hence we are confronted with the issue of determining appropriate conditions for the existence of such a probability measure. Conditions like "$\mathcal{Z}$ is closed" (as in Remark 2.16), although by no means minimal, seem sufficient for what we need, i.e., they assure us that there is a $Z^* \in \mathcal{Z}$ such that P^* defined as $\frac{dP^*}{dQ}|_{\mathcal{F}_t} \triangleq Z^*_t$ could be interpreted as a "worst case scenario".

Proposition 3.11. *If $\mathcal{Z}$ is closed (as in Remark* 2.16*), then there exit a probability $P^* \in \mathcal{P}$ such that the value process V is a P^*-supermartingale.*

Proof. The Remark 2.16 tells us that $\mathcal{Z}$ is closed, translates to $\mathcal{Z}_\tau$ being closed with respect to the $\mathbf{L^2}$ norm, for all stopping times $\tau \in \mathcal{S}$. This is true then for the stoping time ρ_0. Hence, due to Remark 2.2 we must have that, almost surely,

$$Z^*_{\rho_0} \triangleq \operatorname*{essinf}_{Z \in \mathcal{Z}} Z_{\rho_0} = \lim_{k \to \infty} \downarrow Z^k_{\rho_0} \in \mathcal{Z}_{\rho_0}.$$

In other words, there exist a process $Z^* \in \mathcal{Z}$, such that the above equation is satisfied. Then due to Lemma 3.8, we can say that $Z^*_\tau = \operatorname{essinf}_{Z \in \mathcal{Z}} Z_\tau$, is true a.s. for any stopping time $\tau \in \mathcal{S}_{\rho_0}$.

We observe next that since the value process is non-negative we must have

$E^{P^*}(V_\tau) \leq E^P(V_\tau)$, for all $\tau \in \mathcal{S}_{\rho_0}$, and $P \in \mathcal{P}$. Therefore due to Proposition 3.5 we can conclude that the value process V behaves as a P^*-supermartingale from the time ρ_0 on, since $E^{P^*}(V_\tau) \geq E^{P^*}(V_\mu)$, for any stopping times $\tau, \mu \in \mathcal{S}_{\rho_0}$ such that $\tau \leq \mu$.

We conclude the proof be observing that Proposition 3.2 implies that $V_0 = \inf_{P \in \mathcal{P}} E^P(V_{\rho_0}) = E^{P^*}(V_{\rho_0})$, therefore $\{V_{t\wedge\rho_0}, \mathcal{F}_t\}$ is P^*-martingale.

Hence the $\{V_t, \mathcal{F}_t\}_{0 \leq t \leq \infty}$ is P^*-supermartingale for $P^* \in \mathcal{P}$, chosen such that $Z^*_{\rho_0} \triangleq \operatorname{essinf}_{Z \in \mathcal{Z}} Z_{\rho_0}$. ◇

3.5 Deterministic Approach to the Stochastic Game

We have already seen in the previous subsection that if $\mathcal{Z}$ is closed, then the process $\{V_t, \mathcal{F}_t\}_{0 \leq t \leq \infty}$ is a P^*- supermartingale for some $P^* \in \mathcal{P}$. Therefore, the "value" process V admits the Doob-Meyer decomposition of the form

$$V_t = V_0 + X_t - C_t, \tag{3.15}$$

where X is a P^*-martingale with $X_0 = 0$, and C is an increasing, predictable process with $C_0 = 0$.

The next result will offer an interpretation of the martingale X, just like in the case of the cooperative version of the stochastic game.

Theorem 3.12. *If $\mathcal{Z}$ is closed, there is $P^* \in \mathcal{P}$ such that*

$$V_0 = E^{P^*}\left(\sup_{0 \le t < \infty}(Y_t + \Lambda_t)\right) \tag{3.16}$$

where $\Lambda_t \triangleq X_\infty - X_t$.

The process Λ can be interpreted as the Lagrange multiplier enforcing the constraint that the player should stop at stopping times rather than arbitrary random times.

Proof. Let us denote by $M_t \triangleq V_0 + X_t$, therefore $\Lambda_t \triangleq M_\infty - M_t$. Also, like in the proof of the theorem 2.24 we define the processes $Q_t \triangleq Y_t + \Lambda_t$, and $\nu_t \triangleq \sup_{s \ge t} Q_s$. We shall prove that:

$$E^{P^*}[\nu_t | \mathcal{F}_t] = V_t \quad \text{a.s.} \tag{3.17}$$

holds for every $t \in [0, \infty]$, from which (3.16) will follow immediately by taking $t = 0$. Observe that,

$$\begin{aligned} Q_s &= Y_s + M_\infty - M_s \\ &= Y_s + (V_\infty + C_\infty) - (V_s + C_s) \\ &= V_\infty - (V_s - Y_s) + (C_\infty - C_s). \end{aligned}$$

But $V_s - Y_s \ge 0$, and $C_\infty - C_s \le C_\infty - C_t$ for $s \ge t$, so clearly we have:

$$Q_s \le V_\infty + C_\infty - C_t = M_\infty - C_t.$$

Therefore, if we take the supremum over all $s \geq t$, we obtain:

$$\nu_t \leq M_\infty - C_t, \quad \text{for all } t \in [0, \infty]. \tag{3.18}$$

To prove the reverse inequality, observe that

$$Q_{\rho_t} = Y_{\rho_t} + M_\infty - M_{\rho_t} = M_\infty + V_{\rho_t} - M_{\rho_t} = M_\infty - C_{\rho_t}. \tag{3.19}$$

Also, recall that $E^{P^*}(V_t) = E^{P^*}(V_{\rho_t})$, hence $E^{P^*}(C_t) = E^{P^*}(C_{\rho_t})$, and since the process C is increasing we can conclude that $C_t = C_{\rho_t}$ a.e. Therefore (3.19) becomes

$$Q_{\rho_t} = M_\infty - C_{\rho_t} = M_\infty - C_t \geq \sup_{s \geq t} Q_s. \tag{3.20}$$

Since $\rho_t \in \mathcal{S}_t$, we must have

$$Q_{\rho_t} = M_\infty - C_t = \sup_{s \geq t} Q_s.$$

Therefore we can conclude that

$$E^{P^*}(\nu_t | \mathcal{F}_t) = E^{P^*}(M_\infty - C_t | \mathcal{F}_t) = M_t - C_t = V_t.$$

And this proves our result, for $t = 0$.

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