SMOOTH TRANSFER OF KLOOSTERMAN INTEGRALS

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Abstract
We establish the existence of smooth transfer between absolute Kloosterman integrals and Kloosterman integrals relative to a quadratic extension.

Contents

1. Introduction

Let E/F be a quadratic extension of local non-Archimedean fields, let $\eta : F^\times \to \{\pm 1\}$ be the corresponding quadratic character, and let $\psi : F \to \mathbb{C}^\times$ be a nontrivial character. Let N_m be the subgroup of upper triangular matrices in $\mathrm{GL}(m)$ with unit diagonal, and let A_m be the group of diagonal matrices. We define a character $\theta : N_m(F) \to \mathbb{C}^\times$ by $\theta(n) = \psi\big(\sum n_{i,i+1}\big)$. The group $N_m(F) \times N_m(F)$ operates on $\mathrm{GL}(m, F)$ by $g \mapsto {}^t n_1 g n_2$, and the orbits that intersect $A_m(F)$ form a dense open subset. We define the diagonal orbital integrals of a smooth function of compact support Φ on $\mathrm{GL}(m, F)$:

$$\Omega(\Phi, \psi : a) := \int_{N_m(F) \times N_m(F)} \Phi({}^t n_1 a n_2)\theta(n_1 n_2)\, dn_1\, dn_2. \tag{1}$$

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Likewise, we let $H(m \times m, E/F)$ be the space of Hermitian matrices $m \times m$ and set $S_m(F) = H(m \times m, E/F) \cap \mathrm{GL}(m, E)$. We define a character $n \mapsto \theta(n\overline{n})$ of $N_m(E)$ by $\theta(n\overline{n}) = \psi\big(\sum(n_{i,i+1} + \overline{n_{i,i+1}})\big)$. The group $N_m(E)$ operates on $S_m(F)$ by $s \mapsto {}^t\overline{n}sn$, and the orbits that intersect $A_m(F)$ form a dense open subset of $S_m(F)$. We define the diagonal orbital integrals of a smooth function of compact support Ψ on $S_m(F)$:

$$\Omega(\Psi, E/F, \psi : a) := \int_{N_m(E)} \Phi({}^t\overline{n}an)\theta(n\overline{n})\, dn. \tag{2}$$

We say that Φ *matches* Ψ for ψ if for every $a \in A_m(F)$,

$$\Omega(\Phi, \psi : a) = \gamma(a, \psi)\Omega(\Psi, E/F, \psi : a),$$

where

$$\gamma(a, \psi) := \eta(a_1)\eta(a_1a_2)\cdots\eta(a_1a_2, \dots, a_{m-1}) \quad \text{if } a = \mathrm{diag}(a_1, a_2, \dots, a_m).$$

In this paper we prove that for every Φ there is a matching Ψ, and conversely.

Suppose that E/F is unramified, and suppose that the conductor of ψ is $\mathscr{O}_F$. Let Φ_0 be the characteristic function of the set of matrices with integral entries in $M(m \times m, F)$. Similarly, let Ψ_0 be the characteristic function of the set of matrices with integral entries in $H(m \times m, E/F)$. The *fundamental lemma* asserts that the restriction of Φ_0 to $\mathrm{GL}(m, F)$ matches the restriction of Ψ_0 to $S_m(F)$. This has been established by Ngô [10] in the case of positive characteristic.

In general, if Φ matches Ψ, then there are similar relations between the other orbital integrals (see [3], [4]).

Within the context of the relative trace formula, the problem at hand is the analogue of the transfer problem for the ordinary trace formula. Indeed, when E/F is an extension of function fields, the result presented in this paper, together with the fundamental lemma of Ngô [9], [10], implies the existence of global identities of the form

$$\begin{aligned}\int_{N_m(F)\times N_m(F)\backslash N_m(F_{\mathbb{A}})\times N_m(F_{\mathbb{A}})} \theta(n_1n_2)\, dn_1\, dn_2\Big(\sum_{\xi\in Gl(n,F)} \Phi({}^tn_1\xi n_2)\Big)&\\ = \int_{N_m(E)\backslash N_m(E_{\mathbb{A}})} \theta(n\overline{n})\, dn\Big(\sum_{\xi\in S_m(F)} \Psi({}^t\overline{n}\xi n)\theta(n\overline{n})\, dn\Big).&\end{aligned}$$

In turn, the identities are a crucial step in extending the results of [5] to $\mathrm{GL}(m)$.

Our method is first to linearize the problem by replacing $\mathrm{GL}(m, F)$ and $S_m(F)$ by the vector spaces $M(m \times m, F)$ and $H(m \times m, E/F)$, respectively. We then establish a simple relation between the orbital integrals of a function on either space and its Fourier transform. It follows that if Φ matches Ψ, then the Fourier transform of Φ matches the Fourier transform of Ψ. This proves the existence of many pairs of matching functions and, in turn, is used to prove the existence of the transfer in a very simple way.

Our method is suggested by the work of J. Waldspurger [13]. Waldspurger considers orbital integrals in the context of the Lie algebras and derives the existence of the transfer from the fundamental lemma (assumed to hold) and global identities. He uses the Fourier transform on the Lie algebras in an essential way. It is reasonable to ask whether, conversely, the results of the paper can be used to prove the fundamental lemma in the context of the relative trace formula. We answer this question in the affirmative for the case of $m = 2, 3$ in Section 9. In a forthcoming paper (see [7]), we use the previous results to prove the fundamental lemma for all m.

2. Orbital integrals in $M(m \times m, F)$

We now consider the action of $N_m(F) \times N_m(F)$ on $M(m \times m, F)$ given by $r \mapsto {}^t n_1 r n_2$. We denote by θ_0 the algebraic character $\theta_0 : N(F) \to F$ defined by

$$\theta_0(n) = \sum_{i=1}^{i=m-1} n_{i,i+1}.$$

We say that an element $x \in M(m \times m, F)$ (or its orbit) is *relevant* if the character $(n_1, n_2) \mapsto \theta_0(n_1 n_2)$ is trivial on the stabilizer of x in $N(F) \times N(F)$.

We let $\Delta_i(x)$, $1 \le i \le m$, be the determinant of the matrix obtained by removing the last $m - i$ rows and the last $m - i$ columns of a matrix x. In particular, $\Delta_m(x) = \det x$. The functions Δ_i are invariants of the action of $N_m(F) \times N_m(F)$. For $1 \le i \le m - 1$, we denote by O_i the set defined by $\Delta_i \ne 0$. We denote by B_m the group of upper triangular matrices in $\mathrm{GL}(m)$, by A_m the group of diagonal matrices, and by $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ the simple roots of A_m corresponding to B_m. We denote by N_α the root group corresponding to a root α. We identify the Weyl group of A_m with the group W_m of permutation matrices, and we let $\mathscr{A}_m$ be the set of diagonal matrices in $M(m \times m, F)$. We often drop the index m from the notation.

PROPOSITION 1

In $M(m \times m, F)$ the set defined by $\Delta_m = 0$, $\Delta_{m-1} = 0$ contains no relevant orbit.

Proof

We begin with an elementary lemma.

LEMMA 1

Any $r \in M(m \times m, F)$ can be written (in possibly several ways) in the form

$$r = {}^t n w b$$

with $w \in W_m$, $n \in N_m(F)$, and b an upper triangular matrix.

Proof
We let

$$\mathscr{F}_0 = \{V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n\}$$

be the canonical flag. Thus V_i is the space spanned by the first i vectors of the canonical basis. Let $r \in M(m \times m, F)$ be given. Consider the sequence of subspaces

$$rV_0 \subseteq rV_1 \subseteq rV_2 \subseteq \cdots \subseteq rV_m.$$

It need not be a flag, but, inductively, it is easy to show that there is a flag

$$\mathscr{F} = \{U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_m\}$$

such that

$$rV_i \subseteq U_i, \quad 0 \leq i \leq n.$$

By the standard Bruhat decomposition,

$$\mathscr{F} = {}^t n w \mathscr{F}_0$$

for suitable w and $n \in N_m(F)$. Thus

$$({}^t n w)^{-1} r V_i \subseteq V_i, \quad 0 \leq i \leq n.$$

Since $({}^t n w)^{-1} r$ stabilizes $\mathscr{F}_0$, it is an upper triangular matrix. This proves the lemma. □

To prove our assertion, it suffices then to consider the case of a relevant element of the form

$$r = wb$$

with $w \in W_m$ and b an upper triangular matrix such that $\det b = 0$. Our task is then to show that $\Delta_{m-1}(wb) \neq 0$. We remark that if the column of index i, $i < m$, of the matrix r is zero, then r is irrelevant because then $rn_{\alpha_i} = r$ for all $n_{\alpha_i} \in N_{\alpha_i}(F)$. In particular, the first diagonal entry $b_{1,1}$ of b is nonzero. Then at the cost of multiplying b by a suitable element of N_{α_1} on the right, we may assume that $b_{1,2} = 0$. Again, since the second column of b cannot be zero, we must have $b_{2,2} \neq 0$. Inductively, we see that we may assume b to have the form

$$b = \begin{pmatrix} b' & 0 \\ 0 & 0 \end{pmatrix},$$

where b' is an invertible diagonal matrix of size $(m-1) \times (m-1)$. Thus the last row of b is zero. Since wb is relevant, the first $m-1$ rows of wb cannot be zero. This forces w to have the form

$$w = \begin{pmatrix} w' & 0 \\ 0 & 1 \end{pmatrix},$$

where w' is a permutation matrix in $\mathrm{GL}(m-1)$. Now

$$\Delta_{m-1}(wb) = \det b' \det w'.$$

Thus $\Delta_{m-1}(wb) \neq 0$, as claimed. □

Now it is elementary that every element x such that $\det x = 0$ but $\Delta_{n-1}(x) \neq 0$ is in the orbit of an element of the form

$$\begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, x is relevant if and only if x' is a relevant element in $\mathrm{GL}(m-1, F)$. Also,

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_m) \in \mathscr{A}_m$$

is relevant if and only if $a_1 a_2 \cdots a_{m-1} \neq 0$.

Now we recall the classification of the relevant orbits in $\mathrm{GL}(m, F)$ (cf. [1], [12]). Every orbit has a unique representative of the form wa with $w \in W$ and $a \in A_m(F)$. If this element is relevant, then for every pair of positive roots (α_1, α_2) such that $w\alpha_2 = -\alpha_1$, for $n_{\alpha_i} \in N_{\alpha_i}(F)$ we have

$$^t n_{\alpha_1} w a n_{\alpha_2} = wa \Rightarrow \theta_0(n_{\alpha_1} n_{\alpha_2}) = 0. \tag{3}$$

This condition implies that if α_1 is simple, then α_2 is simple, and conversely. Thus w and its inverse have the property that if they change a simple root to a negative one, then they change it to the opposite of a simple root. Let S be the set of simple roots α such that $w\alpha$ is negative. Then S is also the set of simple roots α such that $w^{-1}\alpha$ is negative and $wS = -S$. Let M be the standard Levi subgroup determined by S. Thus S is the set of simple roots of M for the torus A, w is the longest element of $W \cap M$, and $w^2 = 1$. This being so, if α_2 is simple, then condition (3) implies $\alpha_2(a) = 1$. Thus a is in the center of M.

Let w_m be the $m \times m$ permutation matrix with antidiagonal entries equal to 1. We see that elements of the form

$$g = \begin{pmatrix} w_{m_1} a_1 & 0 & \cdots & 0 \\ 0 & w_{m_2} a_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & w_{m_r} a_r \end{pmatrix},$$

with $m_1 + m_2 + \cdots + m_r = m$, $a_1, a_2, \ldots, a_r \in F^\times$, form a set of representatives for the relevant orbits in $\mathrm{GL}(m, F)$. A set of representatives for the relevant orbits in $M(n \times n, F)$ is formed of the same elements, except that $a_r = 0$ is allowed when $m_r = 1$ (and $w_{m_r} = 1$).

We set $\theta = \psi \circ \theta_0$. We define the orbital integral of a relevant element x:

$$\Omega\,[\Phi, \psi : x] = \int \Phi({}^t n_1 x n_2)\theta(n_1 n_2)\, dn_1\, dn_2;$$

the integral is over the quotient of $N_m(F) \times N_m(F)$ by the stabilizer of x. The normalization of the measure is described below.

It is convenient to introduce the *intermediate orbital integrals*. Let m, n be two integers greater than zero. For $\Phi \in \mathscr{S}(M((m+n) \times (m+n), F))$, $A_n \in \mathrm{GL}(n, F)$, $B_m \in M(m \times m, F)$, we set

$$\Omega_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right] := \int \Phi \left[\begin{pmatrix} 1_n & 0 \\ Y & 1_m \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \begin{pmatrix} 1_n & X \\ 0 & 1_m \end{pmatrix}\right] \times \psi\left[\mathrm{Tr}\, \epsilon_n^m X + \mathrm{Tr}\, Y \tilde{\epsilon}_n^m\right] dX\, dY. \tag{4}$$

Here $\epsilon = \epsilon_n^m$ is the matrix with m rows and n columns whose first row is the row matrix of size n,

$$\epsilon_n = (0, 0, \ldots, 0, 1),$$

and all other rows are zero. Likewise, $\tilde{\epsilon} = \tilde{\epsilon}_n^m$ is the matrix with n rows and m columns whose first column is the column matrix

$$\tilde{\epsilon}_n = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{pmatrix},$$

and all other columns are zero. If $X = (x_{i,j})$, then the measure dX is the tensor product of the self-dual Haar measure $dx_{i,j}$.

The above integral can we written more explicitly as

$$\int \Phi \left[\begin{pmatrix} A_n & A_n X \\ Y A_n & B_m + Y A_n X \end{pmatrix}\right] \psi\left[\mathrm{Tr}(\epsilon X) + \mathrm{Tr}(Y \tilde{\epsilon})\right] dX\, dY,$$

or, after a change of variables,

$$\Omega_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right] = |\det A_n|^{-2m} \int \Phi \left[\begin{pmatrix} A_n & X \\ Y & B_m + Y A_n^{-1} X \end{pmatrix}\right] \times \psi\left[\mathrm{Tr}(\epsilon A_n^{-1} X) + \mathrm{Tr}(Y A_n^{-1} \tilde{\epsilon})\right] dX\, dY.$$

The previous form shows that the integral converges and defines a smooth function on $\mathrm{GL}(n, F) \times M(m \times m, F)$. More precisely, the following result is easily obtained.

LEMMA 2
If Φ is supported on O_n, then $\Omega_m^n[\Phi, \psi : \bullet]$ is a smooth function of compact support on $\mathrm{GL}(n, F) \times M(m \times m, F)$. Conversely, any smooth function of compact support on $\mathrm{GL}(n, F) \times M(m \times m, F)$ is equal to $\Omega_m^n[\Phi, \psi : \bullet]$ for a suitable function Φ supported on O_n.

We also introduce the *normalized* intermediate orbital integrals

$$\tilde{\Omega}_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right] := |\det A_n|^m \times \Omega_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right]. \tag{5}$$

If Ψ is a smooth function of compact support on $\mathrm{GL}(n, F) \times M(m \times m, F)$, we can define its orbital integrals relative to the action $(N_n \times N_n) \times (N_m \times N_m)$ on $\mathrm{GL}(n, F) \times M(m \times m, F)$. They are denoted by $\Omega(\Psi, \psi : x_n, x_m)$, where x_n is relevant in $\mathrm{GL}(n, F)$ and x_m relevant in $M(m \times m, F)$. In particular, consider the case of the function

$$\Psi(A_n, B_m) = \Omega_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right]$$

and a relevant element x of $M((n+m) \times (n+m), F)$ of the form

$$x = \begin{pmatrix} x_n & 0 \\ 0 & x_m \end{pmatrix},$$

where x_n is relevant in $\mathrm{GL}(n, F)$ and x_m is relevant in $M(m \times m, F)$. We then have the following reduction formula:

$$\Omega\,[\Phi, \psi : x] = \Omega\,[\Psi, \psi : x_n, x_m]\,. \tag{6}$$

This formula reduces the computation of the orbital integrals (and the normalization of the measures) to the case of an element of the form $w_n a$, where a is a scalar matrix. Then

$$\Omega[\Phi, \psi : w_n a] = \int \Phi \left[\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a \\ 0 & 0 & 0 & \cdots & a & a x_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a & \cdots & a x_{n-2,n-1} & a x_{n-2,n} \\ 0 & a & a x_{n-1,3} & \cdots & a x_{n-1,n-1} & a x_{n-1,n} \\ a & a x_{n,2} & a x_{n,3} & \cdots & a x_{n,n-1} & a x_{n,n} \end{pmatrix} \right]$$

$$\times\, \psi\Big(\sum_{i=2}^{i=n} x_{i,n-i+2} \Big) \otimes dx_{i,j}\,,$$

where the measures $dx_{i,j}$ are self-dual.

We also denote by $\Omega(\Phi, \psi : a_1, a_2, \dots, a_n)$ the orbital integral $\Omega(\Phi, \psi : a)$, where $a = \operatorname{diag}(a_1, a_2, \dots, a_n)$. We introduce normalized diagonal orbital integrals as follows. For $a \in \mathscr{A}_m(F)$ we set

$$\sigma_n(a) := \Delta_1(a)\Delta_2(a)\cdots\Delta_{n-1}(a).$$

Thus

$$\sigma_n(a) = a_1^{n-1}a_2^{n-2}\cdots a_{n-1}$$

and a is relevant if $\sigma_n(a) \neq 0$. For a relevant, we set

$$\tilde{\Omega}(\Phi, \psi : a) := |\sigma_n(a)|\Omega(\Phi, \psi : a). \tag{7}$$

We have a reduction formula for these normalized orbital integrals. In a precise way, for $\Phi \in \mathscr{S}(M(n+m) \times (n+m), F)$, set

$$\Psi(A_n, B_m) = \tilde{\Omega}_m^n \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right].$$

Consider a diagonal matrix

$$a = \begin{pmatrix} a_n & 0 \\ 0 & a_m \end{pmatrix}$$

with $a_n \in A_n(F)$ and a_m relevant in $\mathscr{A}_m(F)$. The normalized orbital integral of Ψ on the pair (a_n, a_m) is defined to be

$$\tilde{\Omega}[\Psi, \psi : a_n, a_m] = |\sigma_n(a_n)|\,|\sigma_m(a_m)|\Omega[\Psi, \psi : a_n, a_m].$$

It follows from the relation

$$(\det a_n)^m \sigma_n(a_n)\sigma(a_m) = \sigma_{n+m}(a)$$

that it is equal to $\tilde{\Omega}(\Phi, \psi : a)$.

3. Weil formula: The split case

Our goal is to obtain a simple relation between the orbital integrals of a function and the orbital integrals of its Fourier transform. Our main tool is the Weil formula for the Fourier transform of a character of second order. We state the form of the Weil formula that we are using. We consider the direct sum

$$M(n \times m, F) \oplus M(m \times n, F),$$

the first index being the number of rows. A function Φ on that space is written as

$$\Phi\begin{pmatrix} 0_{n,n} & X \\ Y & 0_{m,m} \end{pmatrix}.$$

Thus the matrix written is a square matrix of size $(n+m)\times(n+m)$ with zero diagonal blocks, $X \in M(n \times m, F)$, and $Y \in M(m \times n, F)$. The Fourier transform of such a function is the function defined by

$$\hat{\Phi}\begin{pmatrix} 0_{n,n} & X \\ Y & 0_{m,m} \end{pmatrix} := \int \Phi \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix} \psi \left[-\operatorname{Tr}\left(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \begin{pmatrix} cc0 & U \\ V & 0 \end{pmatrix} \right) \right] dU\, dV$$

$$= \int \Phi \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix} \psi\big[-\operatorname{Tr}(YU) - \operatorname{Tr}(XV) \big]\, dU\, dV. \tag{8}$$

PROPOSITION 2

For $A \in \mathrm{GL}(n, F)$, $B \in \mathrm{GL}(m, F)$, and any Schwartz function Φ on $M(n \times m, F) \oplus M(m \times n, F)$, we have

$$\int \Phi \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \psi\, (\operatorname{Tr} BYAX)\; dX\, dY$$
$$= |\det A|^{-m} |\det B|^{-n} \int \hat{\Phi} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \psi(-\operatorname{Tr} B^{-1} Y A^{-1} X)\, dX\, dY.$$

Proof

Assume first that A and B are unit matrices. Set

$$\Phi_1(U, V) = \int \Phi \begin{pmatrix} 0 & V \\ Y & 0 \end{pmatrix} \psi\big(-\operatorname{Tr}(YU) \big)\, dU.$$

Then the left-hand side of the identity reduces to

$$\int \Phi_1(-X, X)\, dX.$$

On the other hand, using the (partial) Fourier inversion formula, we see that the right-hand side is equal to

$$\int \Phi_1(Y, -Y)\, dY.$$

Our assertion then follows. In general, we remark that $\operatorname{Tr}(BYAX) = \operatorname{Tr} YAXB$. We change X to XB^{-1} and Y to YA^{-1} in the left-hand side, and we change Y to BY and X to AX in the right-hand side. We see that we are reduced to the case of $A = 1$, $B = 1$ applied to the function

$$(X, Y) \mapsto |\det A|^{-m} |B|^{-n} \Phi \begin{pmatrix} 0 & XB^{-1} \\ YA^{-1} & 0 \end{pmatrix},$$

the Fourier transform of which is the function

$$(X, Y) \mapsto \hat{\Phi} \begin{pmatrix} 0 & AX \\ BY & 0 \end{pmatrix}.$$

Our assertion follows. □

We record a simple consequence of this formula. For $P \in M(m \times n, F)$ and $Q \in M(n \times m, F)$,

$$\int \Phi \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \psi\big(\operatorname{Tr}(PX) + \operatorname{Tr}(YQ) + \operatorname{Tr}(BYAX)\big)\, dX\, dY$$
$$= |\det A|^{-m} |\det B|^{-n}$$
$$\times \int \hat{\Phi} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \psi\big(-\operatorname{Tr} B^{-1}(Y+P)A^{-1}(X+Q)\big)\, dX\, dY. \tag{9}$$

4. Inversion formula for the orbital integrals

For $\Phi \in \mathscr{S}(M(n \times n, F))$, we define the Fourier transform of Φ to be

$$\hat{\Phi}(X) = \int \Phi(Y)\psi[-\operatorname{Tr}(XY)]\, dY.$$

The measure is self-dual. Recall that $w_n \in W_n$ is the permutation matrix with unit antidiagonal. We set

$$\check{\Phi}(X) = \hat{\Phi}(w_n X w_n).$$

Our goal in this section is to obtain a relation between the orbital integrals of Φ and the orbital integrals of $\check{\Phi}$. We first establish such a relation for the intermediate orbital integrals.

PROPOSITION 3
Let $n \geq 1$, $m \geq 1$ be two integers. Let $\Phi \in \mathscr{S}(M((m+n) \times (m+n), F))$. Then

$$\int \left(\int \tilde{\Omega}^n_m \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right] \psi\left[-\operatorname{Tr} B_m w_m C_m w_m\right] dB_m \right)$$
$$\times \psi\big[\operatorname{Tr}(w_m C_m^{-1} w_m \epsilon^m_n A_n^{-1} \tilde{\epsilon}^m_n)\big]\psi\big[-\operatorname{Tr}(A_n w_n D_n w_n)\big]\, dA_n$$
$$= \tilde{\Omega}^m_n \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} C_m & 0 \\ 0 & D_n \end{pmatrix} \right].$$

The integrals are for $B_m \in M(m \times m, F)$ and $A_n \in M(n \times n, F)$.

Proof
We consider the partial Fourier transform Θ (with respect to B_m) of the normalized intermediate orbital integral of Φ:

$$\Theta(A_n, C_m) := \int \tilde{\Omega}^n_m \left[\Phi, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right] \psi\left[-\operatorname{Tr} B_m C_m\right] dB_m.$$

If we change C_m to $w_m C_m w_m$ and D_n to $w_n D_n w_n$, we see that the identity of the proposition amounts to

$$\int \Theta(A_n, C_m)\psi[\operatorname{Tr} C_m^{-1}\epsilon_n^m A_n^{-1}\tilde{\epsilon}_n^m]\psi[-\operatorname{Tr} A_n D_n]\, dA_n$$
$$= |\det C_m|^n \int \hat{\Phi}\begin{pmatrix} D_n + w_n Y w_m C_m w_m X w_n & w_n Y w_m C_m \\ C_m w_m X w_n & C_m \end{pmatrix}$$
$$\times \psi[-\operatorname{Tr}\epsilon_m^n X - \operatorname{Tr} Y\tilde{\epsilon}_m^n]\, dX\, dY. \tag{10}$$

Changing X to $w_m Y w_n$ and Y to $w_n X w_m$ and noting the relations

$$w_n \epsilon_m^n w_m = \tilde{\epsilon}_n^m, \qquad w_m \tilde{\epsilon}_m^n w_n = \epsilon_n^m,$$

we see that the right-hand side of the last formula can also be written

$$\int \hat{\Phi}\begin{pmatrix} D_n + X C_m Y & X C_m \\ C_m Y & C_m \end{pmatrix}\psi[-\operatorname{Tr} Y\tilde{\epsilon}_n^m - \operatorname{Tr}\epsilon_n^m X]\, dX\, dY. \tag{11}$$

From now on, let us write ϵ for ϵ_n^m and $\tilde{\epsilon}$ for $\tilde{\epsilon}_n^m$. After a change of variables, we find that Θ is also equal to

$$|\det A_n|^m \int \Phi\left[\begin{pmatrix} A_n & A_n X \\ Y A_n & B_m \end{pmatrix}\right]$$
$$\times \psi[\operatorname{Tr}(\epsilon X) + \operatorname{Tr}(Y\tilde{\epsilon}) + \operatorname{Tr}(C_m Y A_n X)]\, dX\, dY$$
$$\times \psi[-\operatorname{Tr}(C_m B_m)]\, dB_m.$$

After a new change of variables, this becomes

$$|\det A_n|^{-m} \int \Phi\left[\begin{pmatrix} A_n & X \\ Y & B_m \end{pmatrix}\right]$$
$$\times \psi\big[\operatorname{Tr}(\epsilon A_n^{-1} X) + \operatorname{Tr}(Y A_n^{-1}\tilde{\epsilon}) + \operatorname{Tr}(C_m Y A_n^{-1} X)\big]\, dX\, dY$$
$$\times \psi\big[-\operatorname{Tr}(C_m B_m)\big]\, dB_m.$$

We now introduce a partial Fourier transform of Φ:

$$\Phi_1\begin{pmatrix} A_n & U \\ V & C_m \end{pmatrix}$$
$$:= \int \Phi\begin{pmatrix} A_n & X \\ Y & B_m \end{pmatrix}\psi\left[-\operatorname{Tr}\begin{pmatrix} 0 & X \\ Y & B_m \end{pmatrix}\begin{pmatrix} cc0 & U \\ V & C_m \end{pmatrix}\right] dX\, dY\, dB_m.$$

By the Weil formula, the previous expression is then

$$|\det C_m|^{-n} \int \Phi_1\begin{pmatrix} A_n & X \\ Y & C_m \end{pmatrix}\psi\big[-\operatorname{Tr} C_m^{-1}(Y + \epsilon A_n^{-1})A_n(X + A_n^{-1}\tilde{\epsilon})\big]\, dX\, dY.$$

Expanding ψ, we find

$$\begin{aligned}|\det C_m|^{-n}\psi\big[-\operatorname{Tr}(C_m^{-1}\epsilon A_n^{-1}\tilde{\epsilon})\big]\\ \times\int\Phi_1\begin{pmatrix}A_n & X\\ Y & C_m\end{pmatrix}\\ \times\psi\big[-\operatorname{Tr}(C_m^{-1}YA_nX)-\operatorname{Tr}(C_m^{-1}Y\tilde{\epsilon})-\operatorname{Tr}(C_m^{-1}\epsilon X)\big]\,dX\,dY.\end{aligned}$$

Changing variables once more, we obtain, at last,

$$\begin{aligned}\Theta(A_n,C_m)=|\det C_m|^{n}\psi\big[-\operatorname{Tr}(C_m^{-1}\epsilon A_n^{-1}\tilde{\epsilon})\big]\\ \times\int\Phi_1\begin{pmatrix}A_n & XC_m\\ C_mY & C_m\end{pmatrix}\\ \times\psi\big[-\operatorname{Tr}C_mYA_nX-\operatorname{Tr}(Y\tilde{\epsilon})-\operatorname{Tr}(\epsilon X)\big]\,dX\,dY\end{aligned}$$

To obtain (10), we must multiply by

$$\psi\big[\operatorname{Tr}(C_n^{-1}\epsilon A_n^{-1}\tilde{\epsilon})\big]$$

and take the Fourier transform of the result with respect to A_n. The Fourier transform is evaluated at D_n. We indeed obtain (10). The proof shows that for $C_m\in \mathrm{GL}(m,F)$, the integral converges as an iterated integral. □

Our main result in this section is the following inversion formula for diagonal orbital integrals.

THEOREM 1
For $\Phi\in\mathscr{S}(M(n\times n,F))$,

$$\begin{aligned}\tilde{\Omega}(\check{\Phi},\overline{\psi}:a_1,a_2,\ldots,a_n)\\ =\int\tilde{\Omega}(\Phi,\psi:p_1,p_2,\cdots,p_n)\\ \times\psi\Big(-\sum_{i=1}^{i=n}p_ia_{n+1-i}+\sum_{i=1}^{i=n-1}\frac{1}{p_ia_{n-i}}\Big)\,dp_n\,dp_{n-1}\cdots dp_1,\end{aligned}$$

where the multiple integral is only an iterated integral.

Proof
For $n=1$ the formula is just the definition of the Fourier transform. Thus we may assume that $n>1$ and that our assertion is true for $n-1$. We prove it for n. We apply the formula of Proposition 3 to the pair of integers $(n-1,1)$. We obtain, for fixed

$a_1 \in F^\times$,

$$\int \left(\int \tilde{\Omega}_1^{n-1} \left[\Phi, \psi : \begin{pmatrix} A_{n-1} & 0 \\ 0 & p_n \end{pmatrix} \right] \psi \left[-p_n a_1 \right] dp_n \right)$$
$$\times \psi\big[\operatorname{Tr}(a_1^{-1} \epsilon_{n-1} A_{n-1}^{-1} \tilde{\epsilon}_{n-1}) \big] \psi \big[-\operatorname{Tr}(A_{n-1} w_{n-1} D_{n-1} w_{n-1}) \big] \, dA_{n-1}$$
$$= \tilde{\Omega}_{n-1}^1 \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} a_1 & 0 \\ 0 & D_{n-1} \end{pmatrix} \right].$$

This formula shows that the function $\Phi_1(A_{n-1})$ defined by

$$\Phi_1(A_{n-1}) := \psi\big[\operatorname{Tr}(a_1^{-1} \epsilon_{n-1} A_{n-1}^{-1} \tilde{\epsilon}_{n-1}) \big]$$
$$\times \int \tilde{\Omega}_1^{n-1} \left[\Phi, \psi : \begin{pmatrix} A_{n-1} & 0 \\ 0 & p_n \end{pmatrix} \right] \psi \left[-p_n a_1 \right] dp_n,$$

which a priori is defined only on $\mathrm{GL}(n-1, F)$, extends in fact to a Schwartz-Bruhat function on $M((n-1) \times (n-1), F)$, still denoted by Φ_1 such that $\check{\Phi}_1 = \Theta$, where Θ is the Schwartz function on $M((n-1) \times (n-1), F)$ defined by

$$\Theta(D_{n-1}) = \tilde{\Omega}_{n-1}^1 \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} a_1 & 0 \\ 0 & D_{n-1} \end{pmatrix} \right].$$

The induction hypothesis applied to Φ_1 gives

$$\tilde{\Omega}(\Theta, \overline{\psi} : a_2, a_3, \ldots, a_n)$$
$$= \int \tilde{\Omega}(\Phi_1, \psi : p_1, p_2, \ldots, p_{n-1})$$
$$\times \psi \left(- \sum_{i=1}^{i=n-1} p_i a_{n+1-i} + \sum_{i=1}^{i=n-2} \frac{1}{p_i a_{n-i}} \right) dp_{n-1} \cdots dp_1. \tag{12}$$

On the other hand, with

$$a = \operatorname{diag}(a_1, a_2, \ldots, a_m), \qquad a' = \operatorname{diag}(a_2, a_3, \ldots, a_n),$$

we have

$$\tilde{\Omega}(\Theta, \overline{\psi} : a_2, a_3, \ldots, a_n)$$
$$= |\sigma_{n-1}(a')| \Omega(\Theta, \overline{\psi} : a')$$
$$= |a_1|^{n-1} |\sigma_{n-1}(a')|$$
$$\times \int_{N_{n-1}(F) \times N_{n-1}(F)} \Omega_{n-1}^1 \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} a_1 & 0 \\ 0 & {}^t u_2 a' u_1 \end{pmatrix} \right] \overline{\theta}(u_1 u_2) \, du_1 \, du_2$$
$$= |\sigma_n(a)| \Omega(\check{\Phi}, \overline{\psi} : a)$$
$$= \tilde{\Omega}(\check{\Phi}, \overline{\psi} : a_1, a_2, \ldots, a_n).$$

Hence we get

$$\begin{aligned}\tilde{\Omega}(\check{\Phi},\overline{\psi} : a_1, a_2, \dots, a_n) \\ = \int \tilde{\Omega}(\Phi_1, \psi : p_1, p_2, \dots, p_{n-1}) \\ \times \psi\Big(-\sum_{i=1}^{i=n-1} p_i a_{n+1-i} + \sum_{i=1}^{i=n-2} \frac{1}{p_i a_{n-i}}\Big)\, dp_{n-1} \cdots dp_1. \qquad (13)\end{aligned}$$

To complete the proof, we have to compute the orbital integral of the function $A_{n-1} \mapsto \Phi_1(A_{n-1})$ on the matrix $p' = \operatorname{diag}(p_1, p_2, \dots, p_{n-1})$. Note that the factor

$$\psi\big[\operatorname{Tr}(a_1^{-1}\epsilon_{n-1}A_{n-1}^{-1}\tilde{\epsilon}_{n-1})\big],$$

which appears in the definition of Φ_1, is constant on the orbits of $N_{n-1}(F) \times N_{n-1}(F)$ and, in particular, takes the value $\psi[1/(a_1 p_{n-1})]$ on the orbit of p'. Observe, furthermore, that if ϕ is the characteristic function of a compact open set in $F^\times$, then the function

$$(A_{n-1}, p_n) \mapsto \phi(\det A_{n-1})\tilde{\Omega}_1^{n-1}\left[\Phi, \psi : \begin{pmatrix} A_{n-1} & 0 \\ 0 & p_n \end{pmatrix}\right]$$

is a smooth function of compact support on $\mathrm{GL}(n-1, F) \times F$. In particular,

$$\int\int \tilde{\Omega}_1^{n-1}\left[\Phi, \psi : \begin{pmatrix} {}^t u_1 x u_2 & 0 \\ 0 & p_n \end{pmatrix}\right] \theta(u_1u_2)\psi[-p_n a_1]\, dp_n\, du_1\, du_2$$

is an absolutely convergent integral where the order of integration can be reversed. Moreover, for a given p_n,

$$\begin{aligned}|\sigma_{n-1}(p')| \int \tilde{\Omega}_1^{n-1}\left[\Phi, \psi : \begin{pmatrix} {}^t u_1 p' u_2 & 0 \\ 0 & p_n \end{pmatrix}\right] \theta(u_1u_2)\, du_1\, du_2 \\ = \tilde{\Omega}\,[\Phi, \psi : p_1, p_2, \dots, p_n].\end{aligned}$$

It follows that

$$\begin{aligned}
&\tilde{\Omega}[\Phi_1, \psi : p'] \\
&\qquad = \psi\left[\frac{1}{a_1 p_{n-1}}\right] |\sigma_{n-1}(p')| \\
&\qquad\quad \times \int\int \tilde{\Omega}_1^{n-1}\left[\Phi, \psi : \begin{pmatrix} {}^t u_1 p' u_2 & 0 \\ 0 & p_n \end{pmatrix}\right] \theta(u_1 u_2)\psi[-p_n a_1]\, du_1\, du_2\, dp_n \\
&\qquad = \psi\left[\frac{1}{a_1 p_{n-1}}\right] |\sigma_{n-1}(p')| |\det p'| \\
&\qquad\quad \times \int \left(\Omega_1^{n-1}\left[\Phi, \psi : \begin{pmatrix} {}^t u_1 p' u_2 & 0 \\ 0 & p_n \end{pmatrix}\right] \theta(u_1 u_2)\psi[-p_n a_1]\, du_1\, du_2\right) dp_n \\
&\qquad = \psi\left[\frac{1}{a_1 p_{n-1}}\right] |\sigma_n(p)| \int \Omega[\Phi, \psi : p_1, p_2, \ldots, p_n]\psi[-p_n a_1]\, dp_n \\
&\qquad = \psi\left[\frac{1}{a_1 p_{n-1}}\right] \int \tilde{\Omega}[\Phi, \psi : p_1, p_2, \ldots, p_n]\psi[-p_n a_1]\, dp_n.
\end{aligned}$$

Upon inserting this result in the right-hand side of (13), we find the identity of the theorem. □

There is another more elementary formula that comes directly from the Fourier inversion formula.

PROPOSITION 4
For $\Phi \in \mathscr{S}(M(n \times n, F))$, $n > 1$, the function

$$\phi(a) = \Omega\,[\Phi, \psi : w_n a]$$

is a smooth function of compact support on $F^\times$. Furthermore,

$$\phi(a) = |a|^{-n^2+1} \int \Omega\left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} -w_{n-1} a^{-1} & 0 \\ 0 & b \end{pmatrix}\right] db.$$

Proof
After a change of variables, we find

$$\phi(a) = |a|^{(n-n^2)/2} \int \Phi \left[\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a \\ 0 & 0 & 0 & \cdots & a & x_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a & \cdots & x_{n-2,n-1} & x_{n-2,n} \\ 0 & a & x_{n-1,3} & \cdots & x_{n-1,n-1} & x_{n-1,n} \\ a & x_{n,2} & x_{n,3} & \cdots & x_{n,n-1} & x_{n,n} \end{pmatrix} \right]$$
$$\times \psi\left(a^{-1} \sum_{i=2}^{i=n} x_{i,n-i+2}\right) \otimes dx_{i,j}.$$

Since $|\Delta_n|$ is bounded above on the support of Φ, the above integral vanishes for $|a|$ sufficiently large. On the other hand, since the integrand is a smooth function of the $x_{i,j}$, the integral vanishes for $|a|$ small enough.

We pass to the second assertion. In terms of $\check{\Phi}$ we find that ϕ is equal to

$$\int \check{\Phi} \left[\begin{pmatrix} 0 & 0 & 0 & \cdots & -a^{-1} & x_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -a^{-1} & \cdots & x_{n-3,n-1} & x_{n-3,n} \\ 0 & -a^{-1} & x_{n-2,3} & \cdots & x_{n-2,n-1} & x_{n-2,n} \\ -a^{-1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n-1} & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n-1} & x_{n,n} \end{pmatrix} \right]$$
$$\times |a|^{(n-n^2)/2} \psi\left(a \sum_{i=1}^{i=n} x_{i,n-i+1}\right) \otimes dx_{i,j}.$$

We define a function Φ_1 on $\mathrm{GL}(n-1, F)$ by

$$\Phi_1(A_{n-1}) = \int \Omega_1^{n-1} \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} A_{n-1} & 0 \\ 0 & b \end{pmatrix} \right] db.$$

We remark that if f is a smooth function of compact support on $F^\times$, then the product $f(\det A_{n-1})\Phi_1(A_{n-1})$ is a smooth function of compact support on $\mathrm{GL}(n-1, F)$. It easily follows that

$$\int \Omega \left[\check{\Phi}, \overline{\psi} : \begin{pmatrix} -w_{n-1}a^{-1} & 0 \\ 0 & b \end{pmatrix} \right] db = \Omega(\Phi_1; \overline{\psi} : -w_{n-1}a^{-1}).$$

Now

$$\Phi_1(A_{n-1}) = \int \check{\Phi}\left[\begin{pmatrix} A_{n-1} & A_{n-1}X \\ YA_{n-1} & b+YA_{n-1}X \end{pmatrix}\right]\psi(-\epsilon_{n-1}X - Y\tilde{\epsilon}_{n-1})\,dX\,dY\,db$$
$$= |\det A_{n-1}|^{-2}$$
$$\times \int \check{\Phi}\left[\begin{pmatrix} A_{n-1} & X \\ Y & b \end{pmatrix}\right]\psi(-\epsilon_{n-1}A_{n-1}^{-1}X - YA_{n-1}^{-1}\tilde{\epsilon}_{n-1})\,dX\,dY\,db.$$

Now

$$\Omega[\Phi_1, \overline{\psi} : -w_{n-1}a^{-1}] = \int \Phi_1(A_{n-1})\psi\Big(-\sum_{i=2}^{i=n-1} x_{i,n-i+1}\Big)\,dx_{i,j},$$

where A_{n-1} is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & -a^{-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -a^{-1} & \cdots & -a^{-1}x_{n-3,n-1} \\ 0 & -a^{-1} & -a^{-1}x_{n-2,3} & \cdots & -a^{-1}x_{n-2,n-1} \\ -a^{-1} & -a^{-1}x_{n-1,2} & -a^{-1}x_{n-1,3} & \cdots & -a^{-1}x_{n-1,n-1} \end{pmatrix}.$$

After a change of variables, we find

$$|a|^{((n-1)^2-(n-1))/2}\int \Phi_1(A_{n-1})\psi\Big(a\sum_{i=2}^{i=n-1} x_{i,n-i+1}\Big)\,dx_{i,j},$$

where now A_{n-1} is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & -a^{-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -a^{-1} & \cdots & x_{n-3,n-1} \\ 0 & -a^{-1} & x_{n-2,3} & \cdots & x_{n-2,n-1} \\ -a^{-1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n-1} \end{pmatrix}.$$

Combining with the previous formula for Φ_1, we find that its orbital integral on $-w_{n-1}a^{-1}$ is

$$|a|^{((n-1)^2-(n-1))/2+2(n-1)}\int \check{\Phi}(A_n)\psi\Big(a\sum_{i=1}^{i=n-1} x_{i,n-i+1}\Big)\,dx_{i,j},$$

where A_n is now the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & -a^{-1} & x_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -a^{-1} & \cdots & x_{n-3,n-1} & x_{n-3,n} \\ 0 & -a^{-1} & x_{n-2,3} & \cdots & x_{n-2,n-1} & x_{n-2,n} \\ -a^{-1} & x_{n-1,2} & x_{n-1,3} & \cdots & x_{n-1,n-1} & x_{n-1,n} \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n-1} & x_{n,n} \end{pmatrix}.$$

Comparing with the last expression for $\phi(a)$, we obtain our assertion. □

5. Orbital integrals in $H(n \times n, E/F)$

We let E/F be a quadratic extension of F and denote by $H(n \times n, E/F)$ the space of $n \times n$ Hermitian matrices, that is, the matrices m such that ${}^t m = \overline{m}$. The group $N_n(E)$ operates on $H(n \times n, E/F)$ by $m \mapsto {}^t\overline{u}mu$. Viewed as functions on $H(n \times n, E/F)$, the functions Δ_i are invariants of this action. For $1 \leq i < n$, we let O'_i be the subset of $H(n \times n, E/F)$ defined by $\Delta_i \neq 0$. We define an algebraic character $\theta_1 : N(E) \to F$ by $\theta_1(u) = \sum_{i=1}^{i=n-1} u_{i,i+1}$. Often we write $\theta_1(u) = \theta_0(u\overline{u})$. We say that an element x (or its orbit) is relevant if θ_1 is trivial on the stabilizer of x in $N_n(E)$.

PROPOSITION 5

In $H(n \times n, E/F)$ the set defined by $\Delta_{n-1} = 0$, $\Delta_n = 0$ contains no relevant orbit.

Proof

Every $s \in H(n \times n, E/F)$ has the form $s = {}^t nwb$ with $n \in N_n(E)$, $w \in W_n$, and b upper triangular. Suppose that $\det s = 0$, and suppose that s is relevant. We have to show that $\Delta_{n-1}(s) \neq 0$. Now, if a column of s with index $i < n$ is zero, then the row with index i is also zero because s is Hermitian, and then s is irrelevant since it is fixed by the group $N_{\alpha_i}(E)$. In particular, $b_{1,1} \neq 0$. As before, at the cost of replacing s by ${}^t\overline{n}_1 s n_1$, we may assume that $s = {}^t nwb$, where b has the form

$$b = \begin{pmatrix} b' & 0 \\ 0 & 0 \end{pmatrix},$$

where b' is an invertible diagonal matrix in $\mathrm{GL}(n-1, E)$. In particular, the last row of b is zero. We may replace s by the element $wb\overline{n}^{-1}$, and the last row of $b\overline{n}^{-1}$ is again zero. Since the rows of $wb\overline{n}^{-1}$ with index less than n cannot be zero, w must have the form

$$w = \begin{pmatrix} w' & 0 \\ 0 & 0 \end{pmatrix}$$

with $w' \in W_{n-1}$. Then $\Delta_{n-1}(wb\overline{n}^{-1}) = \det b \det w' \neq 0$, as claimed. □

As before, an element x such that $\det x = 0$ but $\Delta_{n-1}(x) \neq 0$ is in the same orbit as an element of the form

$$\begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, x is relevant if and only if x' is a relevant element in $\mathrm{GL}(m-1, E) \cap H((m-1) \times (m-1), E/F)$.

Now we recall the classification of the relevant orbits in $\mathrm{GL}(n-1, E) \cap H((n-1) \times (n-1), E/F)$. Every orbit has a unique representative of the form wa with $w \in W$, $w^2 = 1$, $a \in A_m(E)$, and $waw = \overline{a}$ (cf. [11]). Suppose that α is a simple root such that $w\alpha = -\beta$ where β is positive. For $n_\alpha \in N_\alpha$, define $n_\beta \in N_\beta$ by

$${}^t\overline{n}_\beta w a n_\alpha = wa.$$

Then

$${}^t\overline{n}_\alpha w a n_\beta = wa.$$

There exists an element $n_{\alpha+\beta} \in N_{\alpha+\beta}$ (i.e., $n_{\alpha+\beta} = 1$ if $\alpha + \beta$ is not a root) such that $n = n_\alpha n_\beta n_{\alpha+\beta}$ satisfies

$${}^t\overline{n} w a n = wa.$$

If wa is relevant, this relation implies

$$\theta_0(n_\alpha \overline{n}_\alpha n_\beta \overline{n}_\beta) = 0.$$

If β is not simple, this leads to a contradiction. Thus β is simple. Since $w^2 = 1$, we see that, as before, there is a standard Levi subgroup M such that w is the longest element in $W_m \cap M$. The above relation also implies that a is in the center of M and in $A(F)$.

We conclude that the set of representatives for the relevant orbits of $N_m(F) \times N_m(F)$ in $M(m \times m, F)$ given in Section 2 is also a set of representatives for the relevant orbits of $N_m(E)$ in $H(m \times m, E/F)$.

We set $\theta(u\overline{u}) = \psi(\theta_0(u\overline{u}))$. The orbital integrals

$$\Omega[\Phi, \psi, E/F : x] = \int \Phi({}^t\overline{u} x u)\theta(u\overline{u})\, du$$

of a relevant element x are defined as before: the integral is over the quotient of $N_m(E) \cap M$ by the stabilizer of x. Our goal is to study these orbital integrals in complete analogy with the previous discussion.

It is convenient to introduce the *intermediate orbital integrals*. Let m, n be two integers greater than zero. For $\Phi \in \mathscr{S}(H((m+n) \times (m+n), E/F))$, $A_n \in S_n(F)$,

$B_m \in H(m \times m, E/F)$, we set

$$\begin{aligned}\Omega_m^n &\left[\Phi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right] \\ &:= \int \Phi\left[\begin{pmatrix} 1_n & 0 \\ {}^t\overline{X} & 1_m \end{pmatrix}\begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\begin{pmatrix} 1_n & X \\ 0 & 1_m \end{pmatrix}\right] \\ &\quad \times \psi\big[\operatorname{Tr}(\epsilon X) + \operatorname{Tr}({}^t\overline{X}\tilde{\epsilon})\big]\, dX. \end{aligned} \tag{14}$$

Here ϵ and $\tilde{\epsilon}$ are as above. Note that $\tilde{\epsilon} = {}^t\epsilon$. If $X = (x_{i,j})$, then the measure dX is the tensor product of the self-dual Haar measures $dx_{i,j}$. The integral converges and defines a smooth function on $S_n(F) \times M(m \times m, F)$. In particular, we have the following easy result.

LEMMA 3
If Φ is supported on O'_n, then the function $\Omega_m^n(\Phi, \psi : \bullet)$ is a smooth function of compact support on $S_n(F) \times M(m \times m, F)$. Conversely, every smooth function of compact support on that space is of this form for a suitable Φ supported on O'_n.

After a change of variables, we find, keeping in mind that $\det A_n \in F^\times$,

$$\begin{aligned}\Omega_m^n &\left[\Phi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right] \\ &= |\det A_n|_F^{-2m} \int \Phi\left[\begin{pmatrix} A_n & X \\ {}^t\overline{X} & B_m + {}^t\overline{X} A_n^{-1} X \end{pmatrix}\right] \\ &\quad \times \psi\big[\operatorname{Tr}(\epsilon A_n^{-1} X) + \operatorname{Tr}({}^t\overline{X} A_n^{-1}\tilde{\epsilon})\big]\, dX. \end{aligned}$$

We also introduce the *normalized* intermediate orbital integrals

$$\begin{aligned}\tilde{\Omega}_m^n &\left[\Phi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right] \\ &:= \eta(\det A_n)^m |\det A_n|_F^m \times \Phi_m^n\left[\Phi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right]. \end{aligned} \tag{15}$$

One can define the orbital integrals of a smooth function of compact support Ψ on $S_n(F) \times H(m \times m, E/F)$ relative to the action of $(N_n(E)) \times (N_m(E))$ on $S_n(F) \times H(m \times m, E/F)$. They are denoted by $\Omega(\Psi, \psi : x_n, x_m)$, where x_n is relevant in $S_n(F)$ and x_m is relevant in $H(m \times m, E/F)$. In particular, consider the case of the function

$$\Psi(A_n, B_m) = \Omega_m^n\left[\Phi, \psi, E/F : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix}\right]$$

and a relevant element x of $H(m \times m, E/F)$ of the form

$$x = \begin{pmatrix} x_n & 0 \\ 0 & x_m \end{pmatrix},$$

where x_n is relevant in $S_n(F)$ and x_m is relevant in $H(m \times m, E/F)$. We then have the reduction formula

$$\Omega\,[\Phi, E/F, \psi : x] = \Omega\,[\Psi, E/F, \psi : x_n, x_m]\,. \tag{16}$$

This formula reduces the computation of the orbital integrals (and the normalization of the measures) to the case of an element of the form $w_n a$, where $a \in F^\times$ is a scalar matrix. Then

$$\begin{aligned}\Omega[\Psi, E/F, \psi : w_n a] \\ = \int \Phi \left[\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a \\ 0 & 0 & 0 & \cdots & a & ax_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a & \cdots & ax_{n-2,n-1} & ax_{n-2,n} \\ 0 & a & ax_{n-1,3} & \cdots & ax_{n-1,n-1} & ax_{n-1,n} \\ a & ax_{n,2} & ax_{n,3} & \cdots & ax_{n,n-1} & ax_{n,n} \end{pmatrix}\right] \\ \times \psi\Big(\sum_{i=2}^{i=n} x_{i,n-i+2}\Big) \otimes dx_{i,j},\end{aligned}$$

where $x_{i,i} \in F$, $x_{i,j} \in E$, $x_{i,j} = \overline{x}_{j,i}$, where the measures $dx_{i,j}$, $i < j$, on E and the measures $dx_{i,i}$ on F are self-dual.

We also denote by $\Omega(\Phi, E/F, \psi : a_1, a_2, \ldots, a_n)$ the orbital integral $\Omega(\Phi, E/F, \psi : a)$, where $a = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ is relevant. We introduce normalized diagonal orbital integrals as follows:

$$\begin{aligned}&\tilde{\Omega}(\Phi, E/F, \psi : a_1, a_2, \ldots, a_n) \\ &\quad := \eta\big(\sigma_n(a)\big)|\sigma_n(a)|\Omega(\Phi, E/F, \psi : a_1, a_2, \ldots, a_n).\end{aligned} \tag{17}$$

As before, there is a reduction formula for the normalized diagonal orbital integrals.

6. Weil formula: The twisted case

As before, our goal is to obtain a simple relation between the orbital integrals of a function and the orbital integrals of its Fourier transform. Again, our main tool is the Weil formula for the Fourier transform of a character of second order. We first recall the one-variable case. Define the Fourier transform of $\Phi \in \mathscr{S}(E)$ by

$$\hat{\Phi}(z) = \int_E \Phi(u)\psi(-uz - \overline{uz})\,du.$$

Then, for $a \in F^\times$,

$$\int \hat{\Phi}(z)\psi(az\bar{z})\,dz = |a|_F^{-1}\eta(a)c(E/F,\psi)\int \Phi(z)\psi\Big(-\frac{z\bar{z}}{a}\Big)\,dz.$$

We may take this formula as a definition of the constant $c(E/F,\psi)$. We also have

$$\int \Phi(z)\psi(az\bar{z})\,dz = |a|_F^{-1}\eta(a)c(E/F,\psi)\int \hat{\Phi}(z)\psi\Big(-\frac{z\bar{z}}{a}\Big)\,dz.$$

Applying the formula twice, we get the relation

$$c(E/F,\psi)c(E/F,\overline{\psi}) = 1.$$

More generally, if Φ is a Schwartz function on the space of column vectors of size n with entries in E, we define its Fourier transform, a function on the space of row vectors of size n, by

$$\hat{\Phi}(X) = \int \Phi(U)\psi[-ZU - \overline{ZU}]\,dU.$$

PROPOSITION 6
For every matrix $A \in S_n(F)$, we have

$$\int_{E^n} \hat{\Phi}(Z)\psi[ZA\,{}^t\overline{Z}]\,dZ$$
$$= |\det A|_F^{-1}\eta(\det A)c(E/F,\psi)^n \int_{E^n} \Phi(X)\psi[-{}^t\overline{X}A^{-1}X]\,dX.$$

Proof
If A is a diagonal matrix (with diagonal entries in $F^\times$), the formula follows at once from the case of $n = 1$. In general, we may write

$$A = Ma\,{}^t\overline{M},$$

with $M \in \mathrm{GL}(n,E)$, a diagonal. Then the left-hand side is

$$|\det M|_E^{-1}\int \hat{\Phi}(ZM^{-1})\psi[Za\,{}^t\overline{Z}]\,dZ.$$

Since $Z \mapsto |\det M|_E^{-1}\hat{\Phi}(ZM^{-1})$ is the Fourier transform of $X \mapsto \Phi(MX)$, this is equal to

$$\eta(\det a)|\det a|^{-1}c(E/F,\psi)^n \int \Phi(MX)\psi(-{}^t\overline{X}a^{-1}X)\,dX.$$

After a change of variables, this becomes

$$\eta(\det a)|\det M|_E^{-1}|\det a|^{-1}c(E/F,\psi)^n\int \Phi(X)\psi(-{}^t\overline{X}A^{-1}X)\,dX.$$

Since $|\det A|_F = |\det a|_F|\det M|_E$, our assertion follows. □

On the space $H(n\times n, E/F)$, we define the Fourier transform by

$$\hat{\Psi}(X)=\int \Psi(Y)\psi\big[-\operatorname{Tr}(XY)\big]\,dY,$$

where the measure is self-dual. Thus if $Y=(y_{i,j})$, then $y_{i,i}\in F$, $y_{j,i}=\overline{y}_{i,j}$ for $i<j$, and the measure dY is the tensor product of the self-dual Haar measures $dy_{i,i}$ and $dy_{i,j}$, $i<j$. Note that $\operatorname{Tr}(XY)$ is indeed in F. In $H((n+m)\times(n+m), E/F)$ we consider the subspace of matrices with zero diagonal $n\times n$ and $m\times m$ blocks. The Fourier transform of a function in that space is then defined by

$$\hat{\Phi}\begin{pmatrix}0_{n,n} & {}^t\overline{X}\\ X & 0_{m,m}\end{pmatrix}=\int \Phi\begin{pmatrix}0_{n,n} & V\\ {}^t\overline{V} & 0_{m,m}\end{pmatrix}\psi\big[-\operatorname{Tr}(XV+\overline{XV})\big]\,dV.$$

This being so, the form of Weil formula that we are using is as given in the next proposition.

PROPOSITION 7

Let A, B be Hermitian matrices in $\mathrm{GL}(n,E)$ and $\mathrm{GL}(m,E)$, respectively. Then

$$\begin{aligned}\int \hat{\Phi}\begin{pmatrix}0_{n,n} & {}^t\overline{Z}\\ Z & 0_{m,m}\end{pmatrix}&\psi\big[\operatorname{Tr}(BZA\,{}^t\overline{Z})\big]\,dZ\\ &=\eta(\det A)^m|\det A|_F^{-m}\eta(\det B)^n|\det B|_F^{-n}c(E/F,\psi)^{mn}\\ &\qquad\times\int \Phi\begin{pmatrix}0_{n,n} & X\\ {}^t\overline{X} & 0_{m,m}\end{pmatrix}\psi\big[-\operatorname{Tr}(B^{-1}\,{}^t\overline{X}A^{-1}X)\big]\,dX.\end{aligned}$$

Proof

Let us observe that $\operatorname{Tr}(BZA\,{}^t\overline{Z})$ is indeed in F. As before, we may reduce the computation to the case where B is the diagonal matrix $\operatorname{diag}(b_1,b_2,\dots,b_m)$. We may regard Z as a matrix

$$Z=\begin{pmatrix}Z_1\\ Z_2\\ \dots\\ Z_m\end{pmatrix},$$

where each Z_i is a row of size n. Then the formula follows from the previous formula applied to the matrices $b_1A, b_2A,\dots,b_mA$. □

7. Inversion formula in the Hermitian case

As before, we set

$$\check{\Psi}(X) = \hat{\Psi}(w_m X w_m).$$

We then have the following result.

PROPOSITION 8

Let $n \geq 1$, $m \geq 1$ be two integers. Let $\Psi \in \mathscr{S}\big(H((m+n)\times(m+n), E/F)\big)$. Then we have

$$\begin{aligned}\int \left(\int \tilde{\Omega}_m^n \left[\Psi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right] \psi[-\operatorname{Tr} B_m w_m C_m w_m]\, dB_m \right) \\ \times \psi\big[\operatorname{Tr}(w_m C_m^{-1} w_m \epsilon A_n^{-1} \tilde{\epsilon})\big] \psi\big[-\operatorname{Tr}(A_n w_n D_n w_n)\big]\, dA_n \\ = c(E/F, \psi)^{mn} \tilde{\Omega}_n^m \left[\check{\Psi}, E/F, \overline{\psi} : \begin{pmatrix} C_m & 0 \\ 0 & D_n \end{pmatrix} \right].\end{aligned}$$

Proof

The proof follows step by step the proof of the corresponding result for $M((n+m)\times(n+m), F)$. We consider the partial Fourier transform Θ (with respect to B_n) of the normalized intermediate orbital integral of Ψ:

$$\Theta\,(A_n, C_m) := \int \tilde{\Omega}_m^n \left[\Psi, E/F, \psi : \begin{pmatrix} A_n & 0 \\ 0 & B_m \end{pmatrix} \right] \psi\,[-\operatorname{Tr} B_m C_m]\; dB_m.$$

The critical step is as follows. We get for Θ:

$$\begin{aligned}\eta(\det A_n)^m |\det A_n|^{-m} \int \Psi \left[\begin{pmatrix} A_n & X \\ {}^t\overline{X} & B_m \end{pmatrix} \right] \\ \times \psi\big[\operatorname{Tr}(\epsilon A_n^{-1} X) + \operatorname{Tr}({}^t\overline{X} A_n^{-1} \tilde{\epsilon}) + \operatorname{Tr}(C_m\, {}^t\overline{X} A_n^{-1} X)\big]\, dX \\ \times \psi\big[-\operatorname{Tr}(C_m B_m)\big]\, dB_m.\end{aligned}$$

We now introduce a partial Fourier transform of Ψ:

$$\Psi_1 \begin{pmatrix} A_n & U \\ {}^t\overline{U} & C_m \end{pmatrix} := \int \Psi \begin{pmatrix} A_n & X \\ {}^t\overline{X} & B_m \end{pmatrix} \psi \left[-\operatorname{Tr} \begin{pmatrix} 0 & X \\ {}^t\overline{X} & B_m \end{pmatrix} \begin{pmatrix} 0 & U \\ {}^t\overline{U} & C_m \end{pmatrix} \right] dX\, dB_m.$$

By the Weil formula, the previous expression is then

$$\begin{aligned}|\det C_m|^{-n} \eta(\det C_m)^n c(E/F, \psi)^{mn} \\ \times \int \Psi_1 \begin{pmatrix} A_n & X \\ {}^t\overline{X} & C_m \end{pmatrix} \psi\big[-\operatorname{Tr} C_m^{-1}({}^t\overline{X} + \epsilon A_n^{-1}) A_n (X + A_n^{-1}\tilde{\epsilon})\big]\, dX.\end{aligned}$$

The rest of the proof is identical to the proof of Proposition 3. □

Our main result in this section is the following inversion formula.

THEOREM 2

For $\Phi \in \mathscr{S}(H(n \times n, E/F))$,

$$c(E/F,\psi)^{(n(n-1))/2}\tilde{\Omega}(\check{\Phi}, \overline{\psi}, E/F : a_1, a_2, \ldots, a_n)$$
$$= \int \tilde{\Omega}(\Phi, \psi, E/F : p_1, p_2, \ldots, p_n)$$
$$\times \psi\Big(-\sum_{i=1}^{i=n} p_i a_{n+1-i} + \sum_{i=1}^{i=n-1} \frac{1}{p_i a_{n-i}}\Big)\, dp_n\, dp_{n-1} \cdots dp_1,$$

where the integral is only an iterated integral.

Proof

Again the formula is trivial for $n = 1$, so we may assume that $n > 1$ and that our assertion is true for $n-1$. We prove it for n. We apply the formula of Proposition 8 to the pair of integers $(n-1, 1)$. We obtain

$$\int \left(\int \tilde{\Omega}_1^{n-1}\left[\Psi, E/F, \psi : \begin{pmatrix} A_{n-1} & 0 \\ 0 & p_n \end{pmatrix}\right] \psi[-p_n a_1]\, dp_n\right)$$
$$\times \psi\big[\operatorname{Tr}(a_1^{-1}\epsilon_{n-1} A_{n-1}^{-1}\tilde{\epsilon}_{n-1})\big]\psi\big[-\operatorname{Tr}(A_{n-1} w_{n-1} D_{n-1} w_{n-1})\big]\, dA_{n-1}$$
$$= c(E/F, \psi)^{n-1}\tilde{\Omega}_{n-1}^1\left[\check{\Psi}, E/F, \overline{\psi} : \begin{pmatrix} a_1 & 0 \\ 0 & D_{n-1} \end{pmatrix}\right].$$

We set

$$\Psi_1(A_{n-1}) := \psi\big[\operatorname{Tr}(a_1^{-1}\epsilon_{n-1} A_{n-1}^{-1}\tilde{\epsilon}_{n-1})\big]$$
$$\times \int \tilde{\Omega}_1^{n-1}\left[\Psi, E/F, \psi : \begin{pmatrix} A_{n-1} & 0 \\ 0 & p_n \end{pmatrix}\right][-p_n a_1]\, dp_n.$$

Then $\check{\Psi}_1(D_{n-1}) = \Theta(D_{n-1})$, where

$$\Theta(D_{n-1}) = c(E/F, \psi)^{n-1}\tilde{\Omega}_{n-1}^1\left[\check{\Psi}, E/F, \overline{\psi} : \begin{pmatrix} a_1 & 0 \\ 0 & D_{n-1} \end{pmatrix}\right].$$

The induction hypothesis applied to Ψ_1 gives

$$c(E/F,\psi)^{(n-1)(n-2)/2}\tilde{\Omega}(\Theta, E/F, \overline{\psi} : a_2, a_3, \ldots, a_n)$$
$$= \int \tilde{\Omega}(\Psi_1, E/F, \psi : p_1, p_2, \ldots, p_{n-1})$$
$$\times \psi\Big(-\sum_{i=1}^{i=n-1} p_i a_{n+1-i} + \sum_{i=1}^{i=n-2} \frac{1}{p_i a_{n-i}}\Big)\, dp_{n-1} \cdots dp_1. \qquad (18)$$

The rest of the proof is the same as before once we observe that

$$c(E/F,\psi)^{n-1}c(E/F,\psi)^{(n-1)(n-2)/2}=c(E/F,\psi)^{n(n-1)/2}. \qquad \square$$

We leave it to the reader to formulate and prove the analogue of Proposition 4 in the present situation.

8. Smooth matching

Definition 1

We say that a function $\Phi\in\mathscr{S}(M(n\times n,F))$ and a function $\Psi\in\mathscr{S}(H(n\times n,E/F))$ have matching orbital integrals for ψ, and we write $\Phi\overset{\psi}{\longleftrightarrow}\Psi$ if for every diagonal matrix $a\in\mathscr{A}_n$ with $\Delta_{n-1}(a)\neq 0$,

$$\tilde{\Omega}(\Phi,\psi;a)=\tilde{\Omega}(\Psi,E/F,\psi:a).$$

From the inversion formula for the orbital integrals, we have at once the following result.

PROPOSITION 9

If $\Phi\overset{\psi}{\longleftrightarrow}\Psi$, *then* $\check{\Phi}\overset{\overline{\psi}}{\longleftrightarrow}c(E/F,\psi)^{n(n-1)/2}\check{\Psi}$, *and conversely.*

Now we can formulate our main result.

THEOREM 3

Given $\Phi\in\mathscr{S}(M(n\times n,F))$, *there is* $\Psi\in\mathscr{S}(H(n\times n,E/F))$ *with matching orbital integrals for* ψ, *and conversely.*

Proof

We treat the case of Φ. The case of Ψ is similar. The case of $n=1$ being trivial, we may assume that $n>1$ and that our assertion is proved for $n'<n$. For $i<n$ we can consider smooth functions of compact support on $\mathrm{GL}(i,F)\times M((n-i)\times(n-i),F)$ and $S_i(F)\times H((n-i)\times(n-i),E/F)$, respectively, and their orbital integrals for the action of the groups $N_i(F)\times N_i(F)\times N_{n-i}(F)\times N_{n-i}(F)$ and $N_i(E)\times N_{n-i}(E)$, respectively. It follows from the induction hypothesis that any function on the first space matches a function on the second space for ψ. Now let O_i (resp., O_i') be the open set of $M(n\times n,F)$ (resp., $H(n\times n,E/F)$) defined by $\Delta_i\neq 0$. If Φ is supported on O_i, then its normalized intermediate orbital integral

$$\tilde{\Omega}^i_{n-i}[\Phi,\psi:g_i,m_{n-i}]$$

is a smooth function of compact support $\Phi_{i,n-i}$ on $\mathrm{GL}(i, F)\times M((n-i)\times(n-i), F)$; it matches a function of compact support $\Psi_{i,n-i}$ on $S_i(F)\times H((n-i)\times(n-i), E/F)$, which in turn is the normalized intermediate orbital integral of a function Ψ supported on O'_i. Now consider diagonal matrices $a_i \in A_i$, $a_{n-i} \in M((n-i)\times(n-i), F)$, with $\Delta_{n-i-1}(a_{n-i}) \neq 0$. Then we have

$$\tilde{\Omega}\left[\Phi, \psi : \begin{pmatrix} a_i & 0 \\ 0 & a_{n-i} \end{pmatrix}\right] = \tilde{\Omega}\left[\Phi_{i,n-i}, \psi : a_i, a_{n-i}\right].$$

There is a similar formula for Ψ. Thus, in fact, $\Phi \overset{\psi}{\longleftrightarrow} \Psi$. Hence our assertion is true for Φ supported on O_i, $i < n$.

Next, suppose that Φ is such that

$$\Omega(\Phi, \psi : w_n a) = 0$$

for all $a \in F^\times$. Our classification of relevant orbits shows then that all the integrals of Φ over relevant orbits contained in the closed set Q defined by

$$\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1} = 0$$

vanish. Hence Φ has the same orbital integrals as a function supported on the complement of Q. We may assume that the support of Φ is contained in the complement of Q. Using a partition of unity, we see that we can write

$$\Phi = \sum_{i=1}^{i=n-1} \Phi_i,$$

where Φ is supported on O_i. Then $\Phi_i \overset{\psi}{\longleftrightarrow} \Psi_i$, where the function Ψ_i is supported on O'_i and

$$\Phi \overset{\psi}{\longleftrightarrow} \sum_{i=1}^{i=n-1} \Psi_i.$$

Now let Φ be an arbitrary function. Then

$$\phi(a) := |a|^{(n+1)(n-1)}\Omega[\Phi, \psi : w_n a]$$

is a smooth function of compact support on $F^\times$. Let Φ_1 be a function of compact support contained in O_{n-1}. Then

$$\Omega^1_{n-1}\left[\Phi_1, \overline{\psi} : \begin{pmatrix} g_{n-1} & 0 \\ 0 & b \end{pmatrix}\right]$$

is an arbitrary smooth function of compact support on $\mathrm{GL}(n-1, F)\times F$. In particular, we can choose Φ_1 so that

$$\phi(a) = \int \Omega\left[\Phi_1, \overline{\psi} : \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & p \end{pmatrix}\right] dp.$$

Let Φ_2 be the function such that $\check{\Phi}_2 = \Phi_1$. Since Φ_1 matches a function for $\overline{\psi}$, it follows from Proposition 9 that $\Phi_2 \xleftrightarrow{\psi} \Psi_2$ for a suitable function Ψ_2. On the other hand, by Proposition 4 applied to Φ_2,

$$\Omega(\Phi, \psi : w_n a) = \Omega(\Phi_2, \psi : w_n a),$$

so that

$$\Omega(\Phi - \Phi_2, \psi : w_n a) = 0.$$

By the previous case, it follows that $\Phi - \Phi_2 \xleftrightarrow{\psi} \Psi_3$ for a suitable function Ψ_3. Then $\Phi \xleftrightarrow{\psi} \Psi_2 + \Phi_3$ and our assertion follows. □

9. Application to the fundamental lemma

Suppose that E/F is unramified, and suppose that the conductor of ψ is $\mathscr{O}_F$. Let Φ_0 be the characteristic function of the set of matrices with integral entries in $M(m \times m, F)$. Similarly, let Ψ_0 be the characteristic function of the set of matrices with integral entries in $H(m \times m, E/F)$. The *fundamental lemma* asserts that Φ_0 matches Ψ_0. We prove this for $m = 2$ and $m = 3$. The case where $|\det a| = 1$ is already known, but the proof presented below is much simpler. First $\tilde{\Omega}(\Phi_0, \psi : a) = \tilde{\Omega}(\Phi_0, \overline{\psi} : a)$, and likewise for Ψ_0. It follows that the difference

$$\omega(a) := \tilde{\Omega}(\Phi_0, \psi : a) - \tilde{\Omega}(\Psi_0, E/F, \psi : a)$$

satisfies the functional equation

$$\omega(a_1, a_2, \dots, a_m) = \int \omega(p_1, p_2, \dots, p_m) \times \psi\Big(-\sum_{i=1}^{i=m} p_i a_{m+1-i} + \sum_{i=1}^{i=m-1} \frac{1}{p_i a_{m-i}}\Big)\, dp_m\, dp_{m-1} \cdots dp_1. \tag{19}$$

Moreover, it is supported on the set $|a_1 a_2 \cdots a_m| \le 1$. The fundamental lemma asserts that $\omega = 0$.

For $m = 2$ it is easily checked that

$$\int \Omega(\Phi_0, \psi : a_1, a_2)\, da_2$$

is 1 if $|a_1| = 1$ and 0 otherwise, and likewise for

$$\int \Omega(\Phi_0, E/F, \psi : a_1, a_2)\, da_2.$$

Thus

$$\int \omega(a_1, a_2) da_2 = 0.$$

If we set

$$\mu(a_1, b_1) := \int \omega(a_1, p_2)\psi(-p_2 b_1)\, dp_2,$$

then (19) is equivalent to the relation

$$\mu(a_1, -b_1) = \mu(b_1, a_1)\psi\Big(\frac{1}{a_1 b_1}\Big). \tag{20}$$

Moreover, $\mu(a_1, b_1 + t) = \mu(a_1, b_1)$ for $|t| \le |a_1|$. Since $\mu(a_1, 0) = 0$, we see that $\mu(a_1, b_1) \neq 0$ implies $|b_1| > |a_1|$. By (20), $\mu(a_1, b_1) \neq 0$ also implies $|b_1| > |a_1|$. Thus $\mu = 0$ or, equivalently, $\omega = 0$, and we are done.

For $m = 3$ we first establish the following lemma.

LEMMA 4

If

$$\int \omega(a_1, a_2, a_3)\, da_3 = 0,$$

then $\omega = 0$.

Proof

Indeed, if we set

$$\mu(a_1, a_2, b_1) := \int \omega(a_1, a_2, a_3)\psi(-a_3 b_1)\, da_3$$

and

$$\sigma(a_1, a_2, b_1) := \mu(a_1, a_2, b_1)\psi\Big(\frac{1}{a_2 b_1}\Big),$$

then (19) is equivalent to

$$\sigma(a_1, a_2, -b_1) = \int \sigma(b_1, p_2, a_1)\psi(p_2 a_2)\, dp_2. \tag{21}$$

The condition on the support of ω implies that $\mu(a_1, a_2, b_1 + t) = \mu(a_1, a_2, b_1)$ for $|t| \le |a_1 a_2|$. Under the assumption of the lemma, we thus have $\mu(a_1, a_2, b_1) = 0$ for $|b_1| \le |a_1 a_2|$. Equivalently, $\sigma(a_1, a_2, b_1) \neq 0$ implies $|a_2| \le |\varpi b_1 a_1^{-1}|$. From (21) we have $\sigma(a_1, a_2 + t, b_1) = \sigma(a_1, a_2, b_1)$ for $|t| \le |\varpi^{-1} b_1 a_1^{-1}|$. Thus $\sigma = 0$ and $\omega = 0$. □

Thus it suffices to prove the relation

$$\int \Omega(\Phi_0, \psi : a_1, a_2, a_3)\, da_3 = \eta(a_2) \int \Omega(\Psi_0, E/F, \psi : a_1, a_2, a_3)\, da_3.$$

Now $\int \Omega_1^2(\Phi_0, \psi : g, a_3)\, da_3$ is $|\det g|_F^{-2}$ times the characteristic function Φ_1 of the set of $g \in M(2 \times 2, F)$ such that

$$\|g\| \le 1, \qquad \|(0,1)g^{-1}\| \le 1, \qquad \|g^{-1\,t}(0,1)\| \le 1.$$

Likewise, $\int \Omega_1^2(\Psi_0, E/F, \psi : g, a_3)\, da_3$ is $|\det g|_F^{-2}$ times the characteristic function Ψ_1 of the set of $g \in H(2 \times 2, E/F)$ such that

$$\|g\| \le 1, \qquad \|(0,1)g^{-1}\| \le 1.$$

Thus it suffices to show that

$$\Omega(\Phi_1, \psi : a_1, a_2) = \eta(a_2)\Omega(\Psi_1, E/F, \psi : a_1 a_2). \tag{22}$$

If $|a_1 a_2| = 1$, then this relation is equivalent to

$$\Omega(\Phi', \psi : a_1, a_2) = \eta(a_1)\Omega(\Psi', E/F, \psi : a_1 a_2),$$

where Φ' is the characteristic function of $\mathrm{GL}(2, \mathcal{O}_F)$ and Ψ' is the characteristic function of $\mathrm{GL}(2, \mathcal{O}_E) \cap H(2 \times 2, E/F)$; in turn, this relation is a special case of the fundamental lemma for $m = 2$.

Now

$$\Omega(\Phi_1, \psi : a_1, a_2) = \int \psi(x_1 + x_2)\, dx_1\, dx_2$$

over the set

$$|a_2| \ge 1, \qquad |a_1 x_i| \le 1, \qquad |x_i| \le |a_1|^{-1}, \qquad |a_2 + a_1 x_1 x_2| \le 1.$$

If $|a_2| = 1$, then this is 1. If $|a_2| > 1$, then this is zero unless $|a_1 a_2| = 1$. A similar remark applies to $\Omega(\Psi_0, E/F, \psi : a_1, a_2)$. The relation (22) follows. We are done.

10. Concluding remarks

More generally, one can use Propositions 3 and 4 to prove inductively the theorem of density established in [3]; in a precise way, if Φ is such that $\Omega(\Phi, \psi : a) = 0$ for all diagonal matrices a, then $\Omega(\check{\Phi}, \overline{\psi} : a) = 0$, and $\Omega(\Phi, \psi : g) = 0$, $\Omega(\check{\Phi}, \overline{\psi} : g) = 0$ for all relevant g. Similarly, one can prove that if $\Phi \overset{\psi}{\longleftrightarrow} \Psi$, then for every g in the common set of representatives for the two sets of orbits,

$$\Omega(\Phi, \psi : g) = \gamma(g, \psi)\Omega(\Psi, E/F, \psi : g),$$

where the constant $\gamma(g,\psi)$ (the *transfer factor*) does not depend on the functions, a result that also follows from [3]. This gives another way to compute the transfer factors.

Since the determinant is an invariant of the two actions, it follows from the theorem that for any smooth function of compact support on $\mathrm{GL}(m, F)$ there is a function of compact support Ψ on $S_m(F)$ such that $\Phi \overset{\psi}{\longleftrightarrow} \Psi$.

Except for the last section, the previous discussion applies directly to the extension $\mathbb{C}/\mathbb{R}$.

Suppose that E/F is unramified, and suppose that the conductor of ψ is $\mathscr{O}_F$. Then $c(E/F,\psi) = 1$. If E/F is a quadratic extension of global fields and ψ is a nontrivial character of $F_{\mathbb{A}}/F$, then the product of the constants $c(E_v/F_v, \psi_v)$ over all places v of F inert in E is 1. Likewise, for g relevant in $\mathrm{GL}(n, F) \cap S_n(F)$ the product of the local transfer factors is 1.

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