RANKIN-SELBERG CONVOLUTIONS

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Dedicated to André Weil

0. Introduction and Notations.

(0.1) The L-functions attached to pairs of automorphic forms had their origin in the papers of R. Rankin [R.R.] and A. Selberg [A.S.]. The existence of poles for these "convolutions" led to two results. For one a derivation of asymptotic estimates for the growth of Fourier coefficients of classical automorphic forms; and, in the second place, a result of Rankin on the non-vanishing of Hecke's (actually Ramanujan's) L-function on the edge of the critical strip. In this context Rankin's method is completely analogous to that of De la Vallée Poussin—the square of a Dirichlet character being replaced by what is now recognized as the square of an automorphic representation. C. J. Moreno, again following classical methods, was able to refine Rankin's result to obtain zero-free regions for Hecke's L-functions [C.J.M.].

The conceptual leap that the power operation on Dirichlet characters should generalize to automorphic forms was made by R. P. Langlands [R.P.L.I.]. Today this idea is perhaps best expressed in terms of the language of the group $G_{\Pi(F)}$, a group conjecturally attached by Langlands to any global field F [R.P.L.III]. The reader familiar with this notion will realize that the existence of $G_{\Pi(F)}$ implies the existence of an algebra structure on the set of "automorphic representations" (actually isobaric representations) of the linear groups over F—an algebra in fact isomorphic to the Grothendieck ring of (completely reducible) representations of $G_{\Pi(F)}$.

Suppose then that π and σ are automorphic cuspidal representations (say of G_r and G_t respectively). Granting the above, we could define the L-functions we want to study by simply setting

$$L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma), \tag{1}$$

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L-function is initially defined for cusp forms but extends naturally to all $\pi \boxtimes \sigma$ being the "product" of π and σ, and the L-function on the right being the principal L-function of Godement-Jacquet [R.G.-J]. (The latter isobaric forms.) The analytic properties of $L(s, \pi \times \sigma)$, including the exact location of the poles, would follow automatically from (1). We shall, in fact, however, establish these properties by expressing our L-functions as integrals, integrals completely analogous to those of Rankin and Selberg (at least for $r = t$). Our principal global result, whose proof will appear elsewhere, states that $L(s, \pi \times \sigma)$ is then meromorphic and satisfies the appropriate functional equation. Moreover, if π and σ are unitary, then $s = 1$ is at most a simple pole of $L(s, \pi \times \sigma)$ and is in fact a pole if and only if $\pi \simeq \tilde{\sigma}$. Here $\tilde{\sigma}$ is the representation contragredient to σ. (The location of the other poles may be trivially deduced from this). Given the results of the present paper and granting those of [J-P-S II], this fact is an easy consequence of [J-S II] (cf. (0.2) below for more discussion).

The function $L(s, \pi \times \sigma)$ may also be defined as an Euler-product. Thus if

$$\pi = \bigotimes_v \pi_v, \qquad \sigma = \bigotimes_v \sigma_v,$$

are the respective factorizations of π and σ into local components, then

$$L(s, \pi \times \sigma) = \prod_v L_v(s, \pi_v \times \sigma_v). \tag{2}$$

Explicit formulas for the local factors $L_v(s, \pi_v \times \sigma_v)$ are given in Section 8, Section 9. These formulas could also be heuristically derived from the local version of Langlands' conjecture. We include here a review of certain aspects of this conjecture—for the linear groups.

Let v then be a place of F and F_v the corresponding completion. Let us denote by W_v the local Weil group, and by W_v' the direct product $W_v \times SL(2, \mathbf{C})$. We denote by $\mathcal{G}_v$ the set (of complex) representations ρ of W_v' of the form

$$\rho = \xi \otimes a, \tag{3}$$

where ξ is a completely reducible finite-dimensional representation of W_v and a an algebraic representation of $SL(2, \mathbf{C})$. Thus ρ is uniquely a direct sum of representations of the form

$$\rho = \xi \otimes a_m, \tag{4}$$

where ξ is irreducible and a_m is the unique irreducible representation of $SL(2, \mathbf{C})$ of degree m. We will also denote by $\mathcal{G}_{v,r}$ the set of such representations of degree r.

We associate to each $\rho \in \mathcal{G}_v$ a local Euler factor $L(s, \rho)$ as follows. If ρ is as in (4) above, we set

$$(5) \qquad L(s, \rho) = L(s, \xi \otimes \| \ \|^{(m-1)/2}),$$

where $\| \ \|$ is the unramified quasi-character of W_v corresponding to the normalized absolute value α_v of F_v, and where the L-factor on the right is the local Artin L-function. Thus $L(s, \rho) = 1$ unless ξ is one-dimensional of the form $\xi = \| \ \|^p$ in which case

$$(6) \qquad L(s, \rho) = (1 - q_v^{-(s+p+(m-1)/2)})^{-1}.$$

Here as usual q_v is the cardinality of the residue class field of F_v. In general, if $\rho = \oplus \rho_i$, $1 \le i \le t$, is a sum of irreducible representations, we set

$$(7) \qquad L(s, \rho) = \prod_{1 \le i \le t} L(s, \rho_i).$$

Next let $\mathcal{A}_{r,v}$ denote the set of (classes of) irreducible admissible representations of $GL_r(F_v) = G_{r,v}$. This set may be described as follows. We begin with the generalized special representations of Berstein-Zelevinsky [B-Z]. These consist precisely of the (quasi-) square-integrable representations of $G_{r,v}$; they are parametrized by a pair consisting of an integer m dividing r, and, if $r = ma$, a supercuspidal representation ξ of $G_{a,v}$. We will denote the corresponding representation by $\sigma_m(\xi)$. (For an exact description see Section 8 below.) In terms of the sum operation, denoted by $\boxplus$, introduced by Langlands in [R.P.L. III], every element of $\mathcal{A}_{r,v}$ is uniquely expressible as a sum of representations of the form $\sigma_m(\xi)$ (c.f. (9.5) below).

As in the case of the group W_v', one can attach an Euler factor to any representation ρ of $GL_r(F_v)$:

if ρ is square-integrable of the form $\rho = \sigma_m(\xi)$, we set

$$(8) \qquad L(s, \rho) = L(s, \xi \otimes \alpha_v^{(m-1)/2});$$

if $\rho = \boxplus \rho_i$, $1 \le i \le t$, where the ρ_i are square-integrable we set

$$(9) \qquad L(s, \rho) = \prod_{1 \leq i \leq t} L(s, \rho_i).$$

(see also [H.J. III].

As is well-known, the group of one-dimensional representations (quasi-characters) of the local Weil group W_v is naturally isomorphic to the group of quasi-characters of $GL_1(F_v) \simeq F^\times$. This fact, together with functoriality with respect to extension of the ground field, is a simple reformulation of the local reciprocity isomorphism [J.T.].

One expects in general a "natural" bijection

$$l_r : \mathcal{G}_{v,r} \to \mathcal{A}_{v,r}$$

reducing as above to local class field theory for $r = 1$. For our purpose it is enough to require the following:

i) $l_r(\rho \otimes \| \ \|^s) = l_r(\rho) \otimes \alpha_v^s$, for all $\rho \in \mathcal{G}_{v,r}$ and all $s \in \mathbf{C}$;

ii) l_r commutes with taking contragredients;

iii) l_r maps the irreducible representations of W_v (of degree r) precisely onto the set of supercuspidal representations of $G_{r,v}$;

iv) If ξ is an irreducible representation of W_v of degree a, then, with $r = ma$,

$$l_r(\xi \otimes a_m) = \sigma_m(l_a(\xi)),$$

and finally;

v) l_r is additive: if $\rho_i \in \mathcal{G}_{v,r_i}$ $(i = 1, 2)$ then, with $r = r_1 + r_2$,

$$l_r(\rho_1 \oplus \rho_2) = l_{r_1}(\rho_1) \boxplus l_{r_2}(\rho_2).$$

Any sequence of bijections satisfying these requirements automatically preserves Euler-factors.

Next let π and σ be a pair of irreducible (admissible) representations of $GL_r(F_v)$ and $GL_t(F_v)$ respectively. The associated local factor $L(s, \pi \times \sigma)$ is defined in this paper in terms of the poles of certain integrals which are the local analogues of the Rankin-Selberg integrals. Our results on the local factors, may be derived formally from the above axioms by requiring that the identity

$$(10) \qquad L(s, \rho \otimes \eta) = L(s, l_r(\rho) \times l_t(\eta))$$

hold for all $\rho \in \mathcal{G}_{v,r}$, $\eta \in \mathcal{G}_{v,t}$. For example, suppose $\rho = \rho_0 \otimes a_m$, $\eta = \eta_0 \otimes a_n$, where ρ_0 and η_0 are irreducible representations of W_v, respectively of degree a and b. Then

$$\rho \otimes \eta \simeq (\rho_0 \otimes \eta_0) \otimes (a_m \otimes a_n). \tag{11}$$

On the other hand, if $m \geq n$

$$a_m \otimes a_n = \bigoplus_{\nu} a_{m+n-1-2\nu}, \tag{12}$$

the sum for $0 \leq \nu \leq n - 1$.

It follows form (7), (10) and our assumptions on l_r that if $m \geq n$ then

$$L(s, \sigma_m(\pi_1) \times \sigma_n(\pi_2)) = \prod_{i=1}^{n} L(s, \pi_1 \times (\pi_2 \otimes \alpha^{m+n-2\nu-1})) \tag{13}$$

for any pair of supercuspidal representations π_1, π_2. This identity except for notation is equivalent to Theorem (8.2). Theorem (9.5) is also consistent with the identity (10). We remark also that, if π and σ are both unramified, and if A (resp. B) is the associated Langlands' class, then

$$L(s, \pi \times \sigma) = \det(1 - q^{-s} A \otimes B)^{-1} \tag{14}$$

(c.f. [J-S II] part I, Section 2.).

As is well known, the irreducible admissible representations of $GL_n(\mathbf{R})$ (resp. $GL_n(\mathbf{C})$) are parametrized by the semi-simple representations of $W_{\mathbf{C}/\mathbf{R}}$ (resp. $\mathbf{C}^\times$) of degree r. The analogue of (10) in these cases is then a theorem whose proof we have indicated elsewhere (c.f. [J-P-S II]).

Similarly, if $\epsilon(s, \pi \times \sigma, \psi)$ is the "Gauss sum" appearing in the functional equation for the global L-function $L(s, \pi \times \sigma)$, then

$$\epsilon(s, \pi \times \sigma, \psi) = \prod_{v} \epsilon_v(s, \pi_v \times \sigma_v, \psi_v) \tag{15}$$

is a product of local Gauss sums (c.f. (0.2) below). (Here $\psi = \Pi_v \psi_v$ is a basic (additive) character of $F_{\mathbf{A}}$.) As has been suggested by Gelfand and Kazhdan [G-K], the analogue of (10) for these local factors may be added to our list of axioms for l_r:

(16) $$\epsilon_v(s, \rho \otimes \eta, \psi) = \epsilon_v(s, l_r(\rho) \times l_t(\eta)),$$

again for all $\rho \in \mathcal{G}_{v,r}$, $\eta \in \mathcal{G}_{v,t}$. (For the definition of the factor on the left see for example [J.T.].)

It is expected that these six axioms uniquely specify the map l_r—provided its exists (loc. cit.). We remark that (16) is consistent with a number of results in the literature, see [S.G.-J.] for example when $\deg \rho = \deg \eta = 2$ and $\rho = \tilde{\eta}$.

It is also consistent with Theorem (3.1)(i) below.

Although much weaker than Langlands' conjecture, our result has several applications. Perhaps the more interesting application is to the theory of Eisenstein series. For GL_n, the constant terms of the Eisenstein series can be expressed in terms of the convolutions and finite products of normalized (local) intertwining operators. The "normalizing constant" can be computed in terms of our local factors. For us this is based on the fact that the intertwining integral is a residue of the integrals we consider. Thus the analytic properties of the constant term of the Eisenstein series can, in principle, be derived from our precise theory. In particular an explicit description of the "discrete spectrum", in terms of cusp forms, should follow from properties of our L-functions. (We understand that F. Shahidi [F.S. II, III] has also obtained definitive results on the normalizing constant.)

Another application is to the characterization of automorphic representations: if π is an irreducible representation of $G_r(F_{\mathbf{A}})$ satisfying some auxillary conditions, then π is cuspidal and automorphic if and only if for every automorphic representation σ of $G_{r-2}(F_{\mathbf{A}})$, the corresponding L-function has the appropriate analytic behavior (if $r = 2$ one must replace G_{r-2} by $F_{\mathbf{A}}^{\times}$). The authors must confess that they have not understood the final meaning of this result.

(0.2). To motivate our principal results, Theorems (3.1), (8.2) and (9.5) below, we turn again to a global setting, recalling certain results proved in [J-S II]. Suppose for example that $r = t$ and that π and σ are automorphic, cuspidal representations of $GL_r(\mathbf{A})$. The analytic continuation of the global L-function $L(s, \pi \times \sigma)$ is based on the study of integrals of the form

(1) $$\int_{G_r(F)\backslash G_r(\mathbf{A})} \phi'\phi(g) \,|\det g\,|^s \sum_{\alpha \in F^r - \{0\}} \phi(\alpha g) dg,$$

where ϕ and ϕ' are cusp forms belonging to π and σ respectively and Φ is a Schwartz-Bruhat function on $\mathbf{A}^r$. Note, that if $r = 1$, then π (resp. π') is an idele-class character and we may take $\phi = \pi$ (resp. $\phi' = \pi'$). Then with $\chi = \pi\pi'$ the integral (1) is equal to the integral

$$\int_{\mathbf{A}^\times} \chi(t)\Phi(t)\,|t|^s d^\times t, \tag{2}$$

an integral which, in a well-known sense, has $L(s, \chi)$ as its greatest common divisor.

Granting the results of [J-P-S II], it is also true in general that $L(s, \pi \times \sigma)$ is the "*g.c.d.*" of the integrals (1) (for $r = t$) and of similar integrals for $r > t$ (c.f. [J-S II]).

Again, for $r = t$, the integral (1) may be transformed into an integral of the form

$$\int_{N_r(\mathbf{A})\backslash G_r(\mathbf{A})} W(g)W'(g)\Phi[\eta g]\,|\det g|^s dg, \tag{3}$$

with η as (0.3) and where W (resp. W') is the Whittaker function associated to ϕ (resp. ϕ') (loc. cit. (4.5) part I). In turn the integral (3) is a finite sum of products of local integrals of a similar form:

$$\int_{N_r(F_v)\backslash G_r(F_v)} W_v(g)W_v'(g)\Phi_v(g)\,|\det g|^s dg, \tag{4}$$

where W_v (resp. W_v') is a Whittaker function associated to the local component π_v (resp. σ_v) of π (resp. σ) (loc. cit. (4.6) part I). (For $r > t$ the analogous factorization is given in section (3.4) especially (3.4.1) of loc. cit. part II.)

The integrals (4) (for v non-archimedean) and their analogues for a general pair of integers (r, t) form the principal object of study of this paper (c.f. (2.4) below). Our first principal result is Theorem (2.7). It contains the definition of the local L and ϵ-factors. For example if $r = t$ then $L(s, \pi_v \times \sigma_v)$ is the "*g.c.d.*" of the integrals (4). For technical reasons we need to consider certain non-irreducible representations, those of "Whittaker type" (c.f. (2.1) below)—which include the more natural class of induced representations of "Langlands' type" (c.f. sections (9.4), (9.5)). The proof of Theorem (2.7) is an easy consequence of a result of [B-Z]: namely, the restriction of an irreducible representation of $G_r(F)$ to $P_r(F)$ has a finite composition series (c.f. (0.3)).

The remainder of the paper is devoted to proving the properties of the local L and ϵ-factor which one would expect from (0.1.10) and (0.1.16). The first result, which is to some extent of an intermediate nature, is Theorem (3.1) which we now state.

Suppose then that F is p-adic field, that r and t are integers and that π (resp. σ) is a representation of $G_r(F)$ (resp. $G_t(F)$) of Whittaker type. Moreover, suppose that, in the notations of (0.3) and (3.1),

$$\sigma = \mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2, \ldots, \sigma_p), \tag{5}$$

where Q is an appropriate parabolic subgroup of G_t and the σ_i are again of Whittaker type. Set as in (3.1)

$$\gamma(s, \pi \times \sigma, \psi) = \epsilon(s, \pi \times \sigma, \psi)L(1 - s, \pi^\iota \times \sigma^\iota)/L(s, \pi \times \sigma). \tag{6}$$

Then Theorem (3.1) states that

$$\gamma(s, \pi \times \sigma, \psi) = \prod_{1 \leq i \leq p} \gamma(s, \pi \times \sigma_i, \psi), \tag{7}$$

$$L(s, \pi \times \sigma) \in \prod_{1 \leq i \leq p} L(s, \pi \times \sigma_i)\mathbf{C}[q^{-s}, q^s]. \tag{8}$$

Both (7) and (8) are proved simultaneously in Section 3 to Section 7.

In Section 3 the proof of Theorem (3.1) is reduced, in a straightforward fashion, to the case $p = 2$. Next in Section 4 (with $p = 2$) we treat the case $r \geq t$. Here the principal point is to prove the identities (4.4.1) (for $r = t$) and the analogous identity (4.10.1) (for $r > t$). It is important to note that these identities are identities of *formal* Laurent series. The case $r = 1$ is treated in Section 5. As in Section 4 of [J-P-S I] the proof is reduced to known assertions on the Zeta-integrals of [R.G.-J]. In Section 6 we treat the case $t = r + 1$, $t_1 = r$, $t_2 = 1$. Essential use is made of an explicit integral representation for the elements of $\mathcal{W}(\sigma; \psi)$ (Proposition (6.1)). Finally in Section 7, the proof of Theorem (3.1) is reduced to the cases treated in Section 4 to Section 6.

Sections 8 and 9 contain the principal results concerning the local factor $L(s, \pi \times \sigma)$. Namely, suppose π (resp. σ) is represented, according to

Langlands' classification, as a quotient of an induced representation ξ (resp. η) (c.f. (9.4), (9.5)). We then have, essentially by definition,

$$L(s, \pi \times \sigma) = L(s, \xi \times \eta). \tag{9}$$

We could also set

$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \xi \times \eta, \psi). \tag{10}$$

We then give an explicit formula for the L-factor in Theorem (8.2) and Theorem (9.5) (see also Proposition (8.1)). Here in the proofs essential use is made of Theorem (3.1). Finally the ϵ-factor for an arbitrary pair of irreducible representations may be computed, using Theorem (3.1), in terms of the corresponding factor for a pair of supercuspidal representations.

It is our pleasure to acknowledge our debt to D. Kazhdan and I. M. Gelfand. Their paper in [G-K] gave the initial impetus to much of this research. Their unpublished work on the gamma-functions for representations over finite fields had a great influence on one of us.

(0.3). We now list our principal notations. The ground field F is local, non-archimedean. The group G_r is the group $GL(r)$, regarded as an F-group. Hence $G_r(F) = GL(r, F)$. If $t < r$ we will regard G_t as a subgroup of G_r via the map

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix}. \tag{1}$$

The standard parabolic subgroup of type $(r_1, r_2, \ldots, r_m)$ is the subgroup Q of matrices of the form:

$$g = \begin{pmatrix} g_1 & & & & * \\ & g_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & g_{r_m} \end{pmatrix}, \qquad g_i \in G_{r_i}. \tag{2}$$

The transpose of Q (noted ${}^tQ$ or $\bar{Q}$) is the *lower* standard parabolic subgroup of type $(r_1, r_2, \ldots, r_m)$. We denote by B_r (or B) the standard parabolic subgroup of type $(1, 1, 1, \ldots, 1)$ in G_r and by N_r (or N) its unipotent radical. The subgroup of diagonal matrices in G_r is noted A_r. If $a \in A_r$ has diagonal entries $a_1, a_2, \ldots, a_r$, we write

$$a = \operatorname{diag}(a_1, a_2, \ldots, a_r). \tag{3}$$

We let P_r (or P) be the subgroup of matrices of the form

$$p = \begin{pmatrix} m & u \\ 0 & 1 \end{pmatrix}, \qquad m \in G_{r-1}. \tag{4}$$

Then, if Z_r is the center of G_r, the group $P_r Z_r$ is a parabolic subgroup of type $(r-1, 1)$ in G_r. Its unipotent radical U_r (or U) is the subgroup of $p \in P_r$ with $m = 1$.

We set, for $t < r$,

$$w_r = \overbrace{\begin{pmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}}^{r} \Bigg\} r, \qquad w_{r,t} = \begin{pmatrix} 1 & 0 \\ 0 & w_{r-t} \end{pmatrix}, \tag{5}$$

$\eta_r = \overbrace{(0, 0, \ldots, 0, 1)}$. By convention w_1 is the 1×1 identity matrix. If g is in G_r, we denote by g^ι the inverse transpose of g:

$$g^\iota = {}^t g^{-1}. \tag{6}$$

We let $\underline{R}$, $\underline{P}$, $\tilde{\omega}$, q, v be the ring of integers, the maximal ideal in $\underline{R}$, a generator of $\underline{P}$, the cardinality of $\underline{R}/\underline{P}$, and the normalized valuation, respectively. We denote by $|x|$, $\alpha_F(x)$ or $\alpha(x)$ the absolute value of an element x of F. We usually write $d^\times a$ for the Haar measure on the multiplicative group $F^\times$. We set

$$K_r = GL(r, \underline{R}). \tag{7}$$

If G is a locally compact group, then δ_G is the module of G. It is defined by the condition that if $d_r(g)$ is a right Haar-measure then

$\delta^{-1}(g)d_r(g)$ is a left Haar-measure. If H is a closed subgroup of G and σ a representation of H then we denote by

$$\text{Ind}(G, H; \sigma) \tag{8}$$

the representation ξ of G induced by σ. The precise definition depends on the context, but, in any case, the space of ξ consists of functions f on G with values in the space of σ, which, on the left, transform under H as follows:

$$f(hg) = \delta_H^{1/2}(h)\delta_G^{-1/2}(h)\sigma(h)f(g); \tag{9}$$

the representation is by right translations:

$$(\rho(g_0)f)(g) = f(gg_0). \tag{10}$$

We also use $\rho(g)$, $g \in G_r$, to denote an operator by right translation in other contexts. In particular, if $G = G_r(F)$ and $H = Q(F)$ where Q is the subgroup (1), an m-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of smooth representations of $G_{r_i}(F)$, $1 \leq i \leq m$, defines a representation σ of $Q(F)$, namely,

$$g \mapsto \sigma_1(g_1) \otimes \sigma_2(g_2) \otimes \cdots \otimes \sigma_m(g_m). \tag{11}$$

Then we write

$$\xi = \text{Ind}(G, Q, \sigma_1, \sigma_2, \ldots, \sigma_m).$$

Here the space of ξ consists of all right-smooth functions of satisfying (9). We use a similar notation for the group $\bar{Q}$.

If π is a representation of G_r, then $\tilde{\pi}$ will denote the representation contragredient to π. We also denote by π^ι, the representation, acting on the same space as π, and defined by $\pi^\iota(g) = \pi(g^\iota)$ for $g \in G_r$. If π is admissible, we will denote by $\mathfrak{A}(\pi)$ the space of bi-K-finite coefficients of π.

We denote by ψ a nontrivial additive character of F. Then we define a character $\theta_{\psi,r}$ (or simply θ_r or even θ) of N_r by

$$\theta(n) = \prod_{i=1}^{r-1} \psi(n_{i,i+1}).$$

The representation of P_r that θ induces is denoted by τ_r; here the space of τ_r consists of all smooth functions on P_r, transforming on the left under N_r according to θ and with compact support mod N_r.

We denote by $\mathcal{S}(F^r)$ the space of Schwartz-Bruhat functions on F^r and similarly by $\mathcal{S}(a \times b, F)$ the space of Schwartz-Bruhat functions on the space $M(a \times b, F)$ of matrices with a rows and b columns. We also denote by $\mathcal{S}(F^\times)$ the space of smooth functions of compact support on $F^\times$. The Fourier-transform of an element Φ of that space is defined, depending on the context, by $\Phi(x) = \int \Phi(y)\psi(\mathrm{tr}(yx))dy$ or $\int \Phi(y)\psi(\mathrm{tr}({}^tyx))dy$, the Haar-measure dy being self-invariant.

If π is an (admissible) representation of $G_r(F)$ and χ a character of $F^\times$, we denote by $\pi \otimes \chi$ the representation of $G_r(F)$ on the same space defined by:

$$\pi \otimes \chi(g) = \pi(g)\chi(\deg g).$$

Finally, if π is irreducible, the operators $\pi(a)$ for $a \in Z_r$ are scalar. We denote by ω_π the central character of π, that is, the morphism $\omega_\pi : F^\times \to \mathbf{C}$ such that $\pi(a \cdot 1_r) = \omega_\pi(a)1$.

If G is a group and π_1, π_2 are representations of G on V_1 and V_2 respectively, we denote by $\mathrm{Bil}_G(\pi_1, \pi_2)$ the space of bilinear forms β on $V_1 \times V_2$ such that

$$\beta(\pi_1(g)v_1, \pi_2(g)v_2) = \beta(v_1, v_2), \qquad \text{for all} \quad g \in G_F.$$

1. Representations of P_r.

(1.1). We recall facts we need concerning the representation theory of P_r.

The representation which will be of most interest to us is the smooth representation τ_r of P_r induced by θ_r:

$$\tau_r = \mathrm{Ind}(P_r, N_r, \theta_r). \tag{1}$$

The following simple remark is also basic. Since U_r is normal in P_r, P_r acts on U_r by conjugation and hence by duality on the dual group of U_r. There are two orbits in the dual group, one represented by the trivial character and the other, say, by $\chi_r = \theta_r | U_r$. In the latter case the isotropy group of χ_r is the subgroup $P_{r-1}U_r$, the group P_{r-1} being identified with a subgroup of P_r via the map

$$p \mapsto \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

We begin with the following Proposition.

PROPOSITION. *Let ξ be a smooth, irreducible representation of P_r. Then either ξ has a trivial restriction to U_r or it has the form*

(2) $$\xi = \mathrm{Ind}(P_r, P_{r-1}U_r, \sigma \otimes \chi_r)$$

where σ is a smooth, irreducible representation of P_{r-1} and $\chi_r = \theta_r \mid U_r$. Moreover, $\xi \simeq \tau_r$ if and only if $\sigma \simeq \tau_{r-1}$.

For a proof see [B-Z] Section 5.12. We remark that if $\xi \mid U_r$ is trivial, we may and will regard ξ as a representation of G_{r-1}. As such it is smooth and irreducible. Thus it is also admissible ([H.J. II], [B-Z]).

(1.2). We may use the Proposition to introduce an invariant, the *index*, which will be a non-negative integer $\leq r - 1$, associated to an irreducible representation ξ of P_r. If ξ has a U_r-fixed vector we write index$(\xi) = 0$. Otherwise we represent ξ in the form (1.1.1). Then assuming the index of σ has already been defined and satisfies $0 \leq$ index$(\sigma) \leq r - 2$, we set index$(\xi) =$ index$(\sigma) + 1$.

Next let us denote by $H_r^{(j)} (0 \leq j \leq r - 1)$ the subgroup of G_r consisting of all matrices of the form

(1) $$h = \begin{pmatrix} g & x \\ 0 & n \end{pmatrix},$$

where $g \in G_{r-1-j}$, $x \in M(r - 1 - j \times j + 1)$ and $n \in N_{j+1}$. Given an irreducible admissible representation ν of G_{r-1-j} we define a representation $\sigma = [\nu, r]$ of $H_r^{(j)}$ by setting with h as in (1):

(2) $$\sigma(h) = \nu(g)\theta_{j+1}(n).$$

Note that $H_r^{(j)} = P_{r-j}N_r$. In particular $H_r^{(r-1)} = N_r$. Then we have the following Theorem.

THEOREM. *Every smooth, irreducible representation ξ of P_r is of the form*

(3) $$\xi = \mathrm{Ind}(P_r, H_r^{(j)}; \sigma),$$

where $\sigma = [\nu, r]$ *and* ν *is irreducible and admissible. Moreover* j *and* σ *are uniquely determined by* ξ.

Except for notation this is the corollary to Proposition (5.12) in [B-Z]. Note then that $\mathrm{index}(\xi) = j$ is well-defined.

(1.3). We state here two versions of Frobenius' reciprocity which we shall need later.

Let then G be a p-adic group and H a closed subgroup. Let ξ and ξ' be smooth representations of G and suppose that ξ' has the form

(1) $$\xi' = \mathrm{Ind}(G, H; \sigma').$$

The first version (of Frobenius' reciprocity) asserts that

(2) $$\mathrm{Bil}_G(\xi, \xi') \simeq \mathrm{Bil}_H(\xi \,|\, H, \sigma' \otimes \delta_G^{1/2} \delta_H^{-1/2})$$

(isomorphism of complex vector-spaces) (c.f. for example [B-Z] pg. 23 ff.).

Next suppose that ξ is also an induced representation:

(3) $$\xi = \mathrm{Ind}(G, H; \sigma).$$

Let σ (resp. σ') operate on the complex vector-space $\mathcal{V}$ (resp. $\mathcal{V}'$) and set $\mathcal{U} = \mathcal{V} \otimes \mathcal{V}'$. The another version of Frobenius' reciprocity asserts that $\mathrm{Bil}_G(\xi, \xi')$ *is isomorphic, as a complex vector space, to the space of all* $\mathcal{U}$-*valued distributions* T *on* G *satisfying*

(1) $$dT(xh) = \delta_G^{1/2}(h)\delta_H^{-1/2}(h)\{1 \otimes {}^t(\sigma')(h^{-1})\}dT(x)$$

(2) $$dT(h^{-1}x) = \delta_G^{-1/2}(h)\delta_H^{-1/2}(h)\{{}^t\sigma(h^{-1}) \otimes 1\}dT(x)$$

for all $x \in G$, $h \in H$.
(c.f. for example [G.W.] Section 5.3).

(1.4). We will need the following analogue of Theorem (1.2) for which unfortunately no reference could be found. We use the same notation as in (1.2).

PROPOSITION. *Suppose* ξ *and* ξ' *as in* (1.2.3). *Then* $\mathrm{Bil}_p(\xi, \xi') = 0$ *unless* $j = j'$ *and* $\nu \simeq \tilde{\nu}'$, *in which case* $Bil_p(\xi, \xi')$ *is one dimensional.*

Proof. Suppose first of all that ξ has a U_r-fixed vector (or equivalently that ξ has a trivial restriction to U_r). Then if $\mathrm{Bil}_G(\xi, \xi')$ is non-zero, ξ' also has a U_r-fixed vector. Otherwise by Proposition (1.1), ξ' has the form

$$\xi' = \mathrm{Ind}(P_r, P_{r-1}U_r, \sigma' \otimes \chi_r), \tag{3}$$

and applying Frobenius' reciprocity we find that, with $H = P_{r-1}U_r$ and $\mu = \sigma' \otimes \chi_r$, the space $\mathrm{Bil}_H(\xi \,|\, H, \mu \otimes \delta_{P_r}^{1/2}\delta_H^{-1/2})$ is non-zero. If then $B \neq 0$ belongs to this space we have in particular, for $u \in U_r$,

$$B(\xi(u)v, \chi_r(u)w) = B(v, w), \tag{4}$$

if v (resp. w) belongs to the space of the representation ξ (resp. σ'). But by hypothesis $\xi(u) = 1$ for $u \in U_r$; since χ_r is non-trivial, we find from (4) that $B = 0$. This contradiction proves our assertion. Thus, in the case at hand, we have $\mathrm{index}(\xi) = \mathrm{index}(\xi') = 0$ and regarding ξ and ξ' as *admissible* representations of G_{r-1} we have

$$\mathrm{Bil}_P(\xi, \xi') = \mathrm{Bil}_{G_{r-1}}(\xi, \xi') = \mathrm{Hom}_{G_{r-1}}(\xi, \tilde{\xi}')$$

from which our assertion follows.

Thus we may assume that ξ and ξ' have positive index and that ξ (resp. ξ') has the form (1.1.1) (resp. (3)). Let $\mathcal{V}$ (resp. $\mathcal{V}'$) denote the space of σ (resp. σ') and set $\mathcal{U} = \mathcal{V} \otimes \mathcal{V}'$. Then applying the second version of Frobenius' reciprocity with $G = P_r$, $H = P_{r-1}U_r$, we find a $\mathcal{U}$-valued distribution $T \neq 0$ on P_r satisfying

$$dT(xg) = \delta_{P_r}^{1/2}(g)\delta_H^{-1/2}(g)\{1 \otimes ({}^t\sigma')(g^{-1})\}dT(x) \tag{5}$$

$$dT(g^{-1}x) = \delta_{P_r}^{-1/2}(g)\delta_H^{-1/2}(g)\{{}^t\sigma(g^{-1}) \otimes 1\}dT(x) \tag{6}$$

for $x \in P_r$, $g \in P_{r-1}$, and

$$dT(xu) = dT(ux) = \chi_r(u)dT(x) \tag{7}$$

for $u \in U_r$, $x \in P_r$.

We prove first that dT has support in $H = P_{r-1}U_r$. In fact, given $F \in C_c^\infty(P_r) \otimes \mathcal{U}$, let us set

(8) $$f_F(p) = \int_{U_r} F(pu)\bar{\chi}_r(u)du.$$

Since dT satisfies $dT(xu) = \chi_r(u)dT(x)$, there is a distribution $d\lambda$ on P_r so that

(9) $$\int F(p)dT(p) = \int f_F(p)d\lambda(p).$$

Next if we set, for $u \in U_r$, $p \in P_r$,

(10) $$h_u(p) = \chi_r(p^{-1}up),$$

we find directly from (7) that if we replace F by $L_{u^{-1}}F$ then f_F is replaced by $h_u f_F$. Note also that $F \to f_F$ commutes with multiplication by h_u. If then we replace F by $L_{u^{-1}}F$ in (8) and use the fact that $dT(ux) = \chi_r(u)dT(x)$, we find that for all $u \in U_r$,

$$\chi_r(u)dT(p) = h_u(p)dT(p).$$

Since the set of common zeroes of the functions

$$p \mapsto h_u(p) - \chi_r(u) \ (u \in U_r)$$

is exactly $P_{r-1}U_r$, we see indeed that dT has support in this subgroup. Thus we may and will regard dT as a $\mathfrak{U}$-valued distribution on $H = P_{r-1}U_r$.

Next given $f \in C_c^\infty(H) \otimes \mathfrak{U}$, set

(11) $$\psi_f = \int_H \delta_{P_r}^{1/2}(h)\delta_H^{-1/2}(h)1 \otimes \mu(h)f(h)d_r(h),$$

where as before $\mu = \sigma' \otimes \chi_r$. It follows from (5) and (7) (c.f. [G.W.] Section 5.2) that there is a linear form λ on $\mathfrak{U}$ so that

(12) $$\int_H f(h)dT(h) = (\lambda, \psi_f).$$

If now we combine (6) and (12), we find after a straight-forward computation that λ necessarily satisfies

$$\lambda(\sigma(g) \otimes \sigma'(g)w) = \lambda(w),$$

for all $g \in P_{r-1}$ and $w \in \mathcal{U}$, or if we view λ as a bilinear form on $\mathcal{V} \times \mathcal{V}'$ that $\lambda \in \mathrm{Bil}_{P_{r-1}}(\sigma, \sigma')$.

The proof of the Proposition now follows by induction on r. In fact, if $r = 1$ there is nothing to prove ($P_1 = \{e\}$). We have already considered the case when $\mathrm{index}(\xi) = 0$. If $\mathrm{index}(\xi) > 0$, we have with our previous notation $\mathrm{Bil}_{P_r}(\xi, \xi') = \mathrm{Bil}_{P_{r-1}}(\sigma, \sigma')$. If then this space is non-trivial, we have by induction, index σ = index $\sigma' = k$ say, and then j = index $\xi = k + 1$ = index ξ'. Thus by Theorem (1.2)

$$\sigma = \mathrm{Ind}(P_{r-1}, H_{r-1}^{(k)}; \mu), \qquad \sigma' = \mathrm{Ind}(P_{r-1}, H_{r-1}^{(k)}; \mu'),$$

where μ is an irreducible (smooth) representation of H of the form $\mu = [\nu, r - 1]$, ν an irreducible admissible representation of $GL(r - 2 - k) = GL(r - 1 - j)$. Similarly $\mu' = [\nu', r - 1]$. Again by induction we must have $\nu = \tilde{\nu}'$. On the other hand if we set $\rho = [\nu, r]$ and $\rho' = [\nu', r]$ then by transitivity of induction we have

$$\xi = \mathrm{Ind}(P_r, H_r^{(j)}; \rho), \qquad \xi' = \mathrm{Ind}(P_r, H_r^{(j)}; \rho'),$$

and, by the uniqueness assertion of Theorem (1.2), the first part of our Proposition is now completely proved. The second part now also follows by induction on r. □

(1.5). We derive two corollaries of Proposition (1.4).

By Theorem (1.2) there is, up to equivalence, exactly one representation of P_r of index $r - 1$, namely the representation

$$\tau_r = \mathrm{Ind}(P_r, N_r, \theta_r).$$

(The fact that τ_r is irreducible is an immediate consequence of the lemma of Gelfand-Kajdan (c.f. Section 2.2).)

Next given a smooth representation $(\xi, \mathcal{V})$ of P_r we denote, by $\mathcal{V}_\theta^*$ or ξ_θ^* the space of linear forms λ on $\mathcal{V}$ satisfying

$$\lambda(\sigma(n)v) = \theta(n)\lambda(v),$$

for $n \in N_r$, $v \in \mathcal{V}$. As is well-known the functor $\mathcal{V} \to \mathcal{V}_\theta^*$ is exact (c.f. for example [B-Z] pg. 25). The first corollary is then as follows:

COROLLARY 1. *Suppose ξ is an irreducible representation of P_r. Then* $\dim \xi_\theta^* \leq 1$; *moreover* $\dim \xi_\theta^* = 1$ *if and only if* $\xi \simeq \tau_r$.

Proof. In fact

$$\xi_{\theta_r}^* = \mathrm{Bil}_{N_r}(\xi \,|\, N_r, \bar{\theta}_r),$$

and by Frobenius' reciprocity the latter space is isomorphic, as a vector space, to

$$\mathrm{Bil}_{P_r}(\xi, \mathrm{Ind}(P_r, N_r; \bar{\theta}_r)) = \mathrm{Bil}(\xi, \tau_r).$$

By Proposition (1.4) either the latter space is trivial or $\mathrm{index}(\xi) = \mathrm{index}(\tau_r) = r - 1$ in which case $\xi \simeq \tau_r$. □

The second corollary is as follows:

COROLLARY 2. *Suppose ξ and ξ' are two irreducible representations of P_r. Then*

$$\mathrm{Bil}_{P_r}(\xi \otimes \alpha^s, \xi') = 0 \tag{1}$$

for almost all q^{-s}, unless $\xi \simeq \xi' \simeq \tau_r$.

Proof. Write

$$\xi = \mathrm{Ind}(P_r, H_r^{(j)}; \rho), \qquad \xi' = \mathrm{Ind}(P_r, H_r^{(j')}; \rho'),$$

where say $\rho = [\nu, r]$ and $\rho' = [\nu', r]$. Then:

$$\xi \otimes \alpha^s = \mathrm{Ind}(P_r, H_r^{(j)}; \rho \otimes \alpha^s),$$

$$\rho \otimes \alpha^s = [\nu \otimes \alpha^s, r].$$

Thus, by Proposition (1.4), if $\mathrm{Bil}_{P_r}(\xi \otimes \alpha^s, \xi')$ is non-zero, we must have $j = j'$ and then $\nu \otimes \alpha^s \simeq \nu'$. If then $j < r - 1$, ν and ν' being representations of $GL(r - 1 - j, F)$ this can only happen for finitely many q^{-s}.

On the other hand if $j = j' = r - 1$, we must have $\xi \simeq \xi' \simeq \tau_r$ as claimed. □

2. The Functional Equations.

(2.1). Let F be a non-archimedean local field and π an *admissible representation of finite type* of the group $G_r(F)$ on a vector space $\mathcal{V} \neq \{0\}$. As in the case of Section 1, we denote by $\mathcal{V}_\theta^*$ or π_θ^* the space of all linear forms λ on $\mathcal{V}$ satisfying

$$\lambda[\pi(n)v] = \theta(n)\lambda(v), \tag{1}$$

for all $v \in \mathcal{V}$, $n \in N_r$. We note again that $\mathcal{V} \to \mathcal{V}_\theta^*$ is an exact functor. We will also assume that π satisfies the following condition:

$$\dim \mathcal{V}_\theta^* = 1. \tag{2}$$

We shall refer in what follows to such representations, that is, admissible of finite type and satisfying (2), as representations of *Whittaker type*. For example, if $r = 1$, then N_r is trivial and all linear forms λ satisfy (2). Thus the condition (2) is equivalent to $\dim \mathcal{V} = 1$, that is to say π is a (quasi-) character of $G_1(F) = F^\times$. If $r \geq 1$ and π is irreducible, then $\dim \mathcal{V}_\theta^* = 0$ or 1 [G-K]; in the latter case we say that π is generic. The irreducible generic representations are the principal examples that we will consider.

We remark that if λ is a linear form satisfying (1), then the linear form μ defined by $\mu(v) = \lambda(\pi(a)v)$ with $a \in A_r$ satisfies:

$$\mu[\pi(n)v] = \theta'(n)\mu(v),$$

where θ' is the character of N_r defined by $\theta'(n) = \theta(ana^{-1})$. Thus we can replace θ by θ' in condition (2). In particular we may replace θ by $\bar{\theta}$, or, what amounts to the same, ψ by $\bar{\psi}$.

If π is an admissible representation on a space $\mathcal{V}$, we denote by π^ι the representation on the same space defined by $\pi^\iota(g) = \pi(g^\iota)$. Evidently if λ satisfies (1), then

$$\lambda[\pi^\iota(w_r n w_r^{-1})v] = \theta(w_r n^\iota w_r)\lambda[v].$$

As $\theta(w_r n^\iota w_r^{-1}) = \bar{\theta}(n)$ we see from our preceding remarks that if π satisfies (2), then so does π^ι.

If π satisfies (2), we denote by $\mathcal{W}(\pi; \psi)$ the space of functions W on G of the form

$$W(g) = \lambda[\pi(g)v] \tag{3}$$

where λ is in $\mathcal{V}_\theta^*$ and $v \in \mathcal{V}$. Each W satisfies $W(ng) = \theta(n)W(g)$ for $n \in N_r$, $g \in G_r$. The space $\mathcal{W}(\pi; \psi)$ is invariant by right translations, and the corresponding representation of G_r is equivalent to a quotient of π. If $W \in \mathcal{W}(\pi; \psi)$, then the function $\tilde{W}$ defined by

$$\tilde{W}(g) = W(w_r g^\iota) \tag{4}$$

is in $\mathcal{W}(\pi^\iota; \bar{\psi})$. If $r = 1$, one has

$$\mathcal{W}(\pi; \psi) = \mathbf{C}\pi \quad \text{and} \quad \mathcal{W}(\pi^\iota; \psi) = \mathbf{C}\pi^\iota = \mathbf{C}\pi^{-1}.$$

If π is irreducible, then the representation of G_r on $\mathcal{W}(\pi; \psi)$ is equivalent to π and π^ι is equivalent to the representation $\tilde{\pi}$ contragredient to π([G-K]).

(2.2). Let us denote by $\mathcal{K}(\pi; \psi)$ the space of functions $W|P_r$ where W varies $\mathcal{W}(\pi; \psi)$. Then, of course, $\mathcal{K}(\pi; \psi)$ is invariant by right translations by elements of P_r. Let $\mathcal{K}_0(\psi) = \mathcal{K}_0(\psi; r)$ be the space of all smooth functions ϕ on P_r satisfying

$$\phi(np) = \theta(n)\phi(p), \qquad n \in N_r, \qquad p \in P_r,$$

and compactly supported mod N_r. Then, according to [G-K], we have $\mathcal{K}_0(\psi) \subseteq \mathcal{K}(\pi; \psi)$ (of course $\mathcal{K}_0(\psi)$ is the space of the representation τ_r of P_r (c.f. (1.5).). Moreover, if π is supercuspidal, then we actually have $\mathcal{K}_0(\psi) = \mathcal{K}(\pi; \psi)$.

(2.3). Let Q be a lower standard parabolic subgroup of type $(r_1, r_2, \ldots, r_j)$. For each i, $1 \leq i \leq j$, let π_i be an admissible representation of finite type of G_{r_i}. Let π be the corresponding representation of G_r:

$$\pi = \mathrm{Ind}(G, Q; \pi_1, \pi_2, \ldots, \pi_j). \tag{1}$$

π is again admissible of finite type.

PROPOSITION. (Rodier). *With the preceding notations, π satisfies condition* (2) *if and only if each π_i does also.*

We refer to [F.R.] for the proof, remarking that in view of the exactness of the functor $\pi \to \pi_\theta^*$ it suffices to treat the case when each π_i is irreducible.

(2.4). We now introduce the integrals which are our primary object of study.

Let $r \geq 1$ and $t \geq 1$ be two integers. Let π (resp. σ) be a representation of G_r (resp. G_t) which is of Whittaker type. Let $W \in \mathcal{W}(\pi; \psi)$ and $W' \in \mathcal{W}(\sigma; \bar{\psi})$.

If $r = t$, we set for $\Phi \in \mathcal{S}(F^r)$,

$$\Psi(s, W, W'; \Phi) = \int W(g)W'(g)\Phi(\eta_r g)|\det g|^s dg, \tag{1}$$

where η_r is the row-vector of length r

$$\eta_r = (0, 0, \ldots, 1),$$

and the integration is over $N_r\backslash G_r$.

If $r = t - 1$, we set

$$\Psi(s, W, W') = \int_{N_t\backslash G_t} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g)|\det g|^{s-1/2} dg. \tag{2}$$

If $1 \leq t \leq r - 1$, and j is an integer for which $0 \leq j \leq r - t - 1$, we set, with $k = r - t - 1 - j$,

(3)

$$\Psi(s, W, W'; j) = \iint W\left[\begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{k+1} \end{pmatrix}\right] W'(g)|\det g|^{s-(r-t)/2} dx dg,$$

the integration being over g in $N_t\backslash G_t$ and x in $M(j \times t, F)$. We have, of course, $\Psi(s, W, W'; 0) = \Psi(s, W, W')$ if $r = t - 1$.

We shall also have to consider certain auxiliary integrals similar to those in (3). Accordingly, for $\Phi \in \mathcal{S}(F^t)$, we set

$$\Psi(s, W, W'; j, \Phi) = \iiint W\left[\begin{pmatrix} g & 0 & 0 & 0 \\ x & 1_j & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_k \end{pmatrix}\right] \times W'(g)\,|\det g|^{s-(r-t)/2} dg dx \Phi(y) dy, \tag{4}$$

the integration being as before over g in $N_t \backslash G_t$, x in $M(j \times t, F)$ and ${}^t y$ in F^t.

(2.5). To consider questions of convergence of the above integrals, we recall certain simple results of [J-P-S]. A gauge ξ on G_r is a positive function, invariant on the left under N and on the right under K, and given on A by:

$$\xi(a) = \phi(\alpha_1(a), \ldots, \alpha_{r-1}(a))\,|\alpha_1(a) \cdots \alpha_{r-1}(a)|^{-t}, \tag{1}$$

where t is ≥ 0 and ϕ is a ≥ 0 Schwartz-Bruhat function on F^{r-1}. If π satisfies the conditions of (2.1) and if, moreover, the central character of π is unitary, then each $W \in \mathcal{W}(\pi; \psi)$ is majorized by a gauge. In fact, the following more precise statement is true. Recall that a finite function on a locally compact group is a continuous function whose translates span a finite dimensional vector space. Then, if π is generic, there exists a finite set X of finite functions on A_r (depending only on π) such that for all $W \in \mathcal{W}(\pi; \psi)$ one has

$$W(a) = \sum_{\chi \in X} \phi_\chi(\alpha_1(a), \ldots, \alpha_{r-1}(a))\chi(a) \tag{2}$$

where $a \in A_r$ and $\phi_\chi \in \mathcal{S}(F^{r-1})$ (loc. cit. Proposition (2.2)). Note then that for any $W \in \mathcal{W}(\pi; \psi)$ we have a majorization of the form

$$|W(a)| \leq \sum_{\eta \in Y} \phi_\eta(\alpha_1(a), \ldots, \alpha_{r-1}(a))\eta(a), \tag{3}$$

where now $\phi_\eta \in \mathcal{S}(F^{r-1})$ is ≥ 0 and η varies is another finite set Y of finite functions on A_r. We also note, for future reference, that any finite function on A_r is dominated by a finite sum of quasi-characters.

(2.6). Our convergence questions will be answered in what follows by the following lemma.

LEMMA. *Let ξ be a gauge on G_r and $j_1 > 0, j_2 \geq 0, k \geq 0$ be integers such that $j_1 + j_2 + k = r - 1$. Then the relation*

$$\xi\begin{pmatrix} g_1 & 0 & 0 & 0 \\ x & 1_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_2 \end{pmatrix} \neq 0,\ g_1 \in G_{j_1},\ g_2 \in G_{j_2},\ x \in M(j_1 \times k, F),$$

implies that x belongs to a fixed compact set independent of g_1 and g_2. Moreover, there is a gauge ξ' on G_r such that

$$\xi\begin{pmatrix} g_1 & 0 & 0 & 0 \\ x & 1_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_2 \end{pmatrix} \leq \xi'\begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1_{k+1} & 0 \\ 0 & 0 & g_2 \end{pmatrix}.$$

Proof. The second assertion follows from the first assertion and Lemma (2.3.5) of [J-P-S]. We prove the first assertion. Let

$$g = \begin{pmatrix} g_1 & 0 & 0 \\ x & 1_k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write $g = n_1 a_1 k_1 (n_1 \in N_t, a_1 \in A_t, k_1 \in K_t)$ with $t = j_1 + k + 1$. Similarly $g_2 = n_2 a_2 k_2$. If we let

$$a_1 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_t),$$

then the relation

$$\xi\begin{pmatrix} g & 0 \\ 0 & g_2 \end{pmatrix} = \xi\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \neq 0$$

implies that

$$(1) \qquad |\lambda_i/\lambda_{i+1}| \leq c, \qquad 1 \leq i \leq t-1,$$

c being a (positive) constant depending only on the gauge ξ. We will deduce from this that x lies in a fixed compact set. Let $e_1, e_2, \ldots, e_t$ be the canonical basis of the space $\mathcal{V}$ of row vectors of length t. Let $\Lambda^j\mathcal{V}$ be the j^{th} exterior power of $\mathcal{V}$ and $\| \ \|$ the sup-norm with respect to the canonical basis of $\Lambda^j\mathcal{V}$. Put, for $1 \leq j \leq t$,

$$(2) \qquad \epsilon_j = e_{t-j+1} \wedge e_{t-j+2} \wedge \cdots \wedge e_t$$

and let G_t operate on $\mathcal{V}$ (and hence, $\Lambda^j\mathcal{V}$) on the right. Then

$$(3) \qquad \|\epsilon_j g\| = |\lambda_{t-j+1} \cdot \lambda_{t-j+2} \cdots \lambda_t|.$$

Next, we have evidently $e_t g = e_t$ and, therefore, from (3), $|\lambda_t| \leq 1$. Whence, by (1), $|\lambda_i| \lesssim 1$, for $1 \leq i \leq t$, and hence by (3)

$$\|\epsilon_j g\| \lesssim 1.$$

Each coefficient of the matrix x is a coefficient of one of the row vectors $\epsilon_j g$. Thus, the coefficients of x are bounded in absolute value as claimed. □

(2.7). Our first main result is the following Theorem.

THEOREM. *Let π (resp. π') be a representation of G_r (resp. G_t) of Whittaker type. Let $W \in \mathcal{W}(\pi; \psi)$ and $W' \in \mathcal{W}(\pi'; \bar{\psi})$. Then*:

(i) *Each of the integrals $\Psi(s, W, W'; \Phi)$ $(r = t)$ and the integrals $\Psi(s, W, W'; j)$ $(1 \leq t \leq r-1, 0 \leq j \leq r-t-1)$ is absolutely convergent for $Re(s)$ large.*

(ii) *They are rational functions of q^{-s}. More precisely, if $r = t$, for $W \in \mathcal{W}(\pi; \psi)$, $W' \in \mathcal{W}(\pi'; \bar{\psi})$, and for $\Phi \in \mathcal{S}(F^r)$, the integrals*

$\Psi(s, W, W', \Phi)$ span a fractional ideal $\mathbf{C}[q^{-s}, q^s]L(s, \pi \times \pi')$ of the ring $\mathbf{C}[q^s, q^{-s}]$; the factor $L(s, \pi \times \pi')$ has the form $P(q^{-s})^{-1}$ where $P \in \mathbf{C}[X]$ and $P(0) = 1$. If $t \leq r - 1$, there is a similar factor, $L(s, \pi \times \pi')$, independent of j, generating the ideal spanned by the integrals $\Psi(s, W, W'; j)$. The same results are true for the pair of representations $(\pi^\iota, (\pi')^\iota)$.

(iii) *Suppose $r = t$. Then there is a factor $\epsilon(s, \pi \times \pi', \psi)$ of the form cq^{-ns} such that*

$$\begin{aligned} &\Psi(1 - s, \tilde{W}, \tilde{W}', \hat{\Phi})/L(1 - s, \pi^\iota \times (\pi')^\iota) \\ &\qquad = \omega_{\pi'}(-1)^{r-1}\epsilon(s, \pi \times \pi', \psi)\Psi(s, W, W', \Phi)/L(s, \pi \times \pi'). \end{aligned} \tag{1}$$

Similarly, if $t \leq r - 1$, set $k = r - t - 1 - j$. Then

$$\begin{aligned} &\Psi(1 - s, \rho(w_{r,t})\tilde{W}, \tilde{W}'; k)/L(1 - s, \pi^\iota \times (\pi')^\iota) \\ &\qquad = \omega_{\pi'}(-1)^{r-1}\epsilon(s, \pi \times \pi', \psi)\Psi(s, W, W'; j)/L(s, \pi \times \pi'), \end{aligned} \tag{2}$$

the factor $\epsilon(s, \pi \times \pi', \psi)$ being independent of j.

Recall that in general $\rho(g)$, $g \in G_r$, denotes the operator of right translation (c.f. (0.3)).

Proof. We prove (i) and (ii) simultaneously. Let us consider for example the integral $\Psi(s, W, W'; \Phi)$ given by (2.4.1). We proceed formally at first. Using the Iwasawa decomposition and the fact that W, W' and Φ are K-finite, we get at once

$$\Psi(s, W, W'; \Phi) = \sum_i \int_{A_r} W_i(a)W_i'(a)\Phi_i(a_r)\delta_r^{-1}(a)\,|\det a|^s d^\times a, \tag{3}$$

where again $W_i \in \mathcal{W}(\pi; \psi)$, $W_i' \in \mathcal{W}(\pi'; \bar{\psi})$, $\Phi_i \in \mathcal{S}(F)$, and we have written

$$a = \operatorname{diag}(a_1a_2 \ldots a_r, a_2 \ldots a_r, \ldots, a_r). \tag{4}$$

Next, we write each W_i and W_i' in the form (2.5.2). Substituting in the above expression, we get for our integral a finite sum of the form

$$\sum_j \int_{A_r} \phi_j\phi_j'(\alpha_1(a), \alpha_2(a), \ldots, \alpha_{r-1}(a))\Phi_j(a_r) \tag{5}$$
$$\chi_j(a)\chi_j'(a)\delta_r^{-1}(a)\,|\det a|^s d^\times a,$$

where $\phi_j, \phi_j' \in \mathcal{S}(F^{r-1})$, $\Phi_j \in \mathcal{S}(F)$, and χ_j resp. χ_j') belongs to a finite set $X(\pi)$ (resp. $X(\pi')$) of finite functions on A_r depending only on π (resp. π'). With a given by (2), we may write each factor $\chi_j\delta_r^{-1/2}$ as a finite sum of products $\eta_1(a_1)\eta_2(a_2)\cdots\eta_r(a_r)$ where each η is a finite function on $F^\times$ and the η's vary in a finite set, again depending only on π. Similarly for χ_j'. Then each integral appearing in (3) is a finite sum of integrals of the form

$$\int \phi\phi'(a_1, a_2, \ldots, a_{r-1})\Phi(a_r)\eta_1(a_1)\eta_2(a_2)\cdots\eta_r(a_r)$$
$$|a_1a_2^2\cdots a_r^r|^s \otimes d^\times a_i,$$

ϕ, ϕ' and Φ being Schwartz functions. Finally, the latter integral is a sum of products of the form

$$\prod_i \int \Phi_i(a_i)\eta_i(a_i)|a_i^i|^s d^\times a_i, \tag{6}$$

with $\Phi_i \in \mathcal{S}(F)$. Note that this converges for $Re(s)$ large.

Now replacing W, W', and Φ by their absolute values in (2.4.1) and replacing the equality (2.5.2) by (2.5.3), the same argument shows first that the integrals $\Psi(s, W, W'; \Phi)$ are indeed convergent for $Re(s)$ large. This justifies our formal calculations.

Next, if we write $X = q^{-s}$, each integral in (6) has the form $Q_i(X)\cdot(1 - \alpha_iX)^{-k_i}$, where Q_i is a polynomial in X, and α_i depends only on η_i. Thus, the integrals $\Psi(s, W, W'; \Phi)$ represent rational functions of X having a common denominator independent of W, W' and Φ.

Moreover, the subspace of $\mathbf{C}(X)$ generated by these fractions is an ideal for the ring $\mathbf{C}[X, X^{-1}]$. To see this it suffices to remark that if

$$W_1(g) = W(gh), \qquad W_1'(g) = W'(gh),$$
$$\Phi_1(x) = \Phi(xh),$$

then

$$\Psi(s, W_1, W_1', \Phi_1) = |\det h|^{-s}\Psi(s, W, W'; \Phi).$$

Finally, one can choose W, W', Φ such that $\Psi(s, W, W'; \Phi) = 1$. In fact, we have in general

$$\Psi(s, W, W'; \Phi) = \int_K dk \int_{N_r\backslash P_r} d_r p\,|\det p|^{s-1} W'(pk)W(pk) \times \int_{F^\times} \Phi(\eta ak)\,|a|^{rs}\,\omega'\omega(a)d^\times a,$$

ω (resp. ω') being the central character of π (resp. π'). Given $\phi \in \mathcal{K}_0(\psi)$ and $\phi' \in \mathcal{K}_0(\bar{\psi})$ choose $W \in \mathcal{W}(\pi; \psi)$ and $W' \in \mathcal{W}(\pi'; \bar{\psi})$ such that $W|P_r = \phi$ and $W'|P_r = \phi'$. Then choose m so that both W and W' are invariant on the right under K_m. If we let Φ be the characteristic function of ηK_m, and if $\Phi(\eta ak) \neq 0$, we find first of all that $a \in \underline{R}^\times$ and then that $ak \in (P \cap K)K_m$. Then, with these choices, our integral reduces to

$$c \cdot \int_{N_r\backslash P_r} d_r(p)\,|\det p|^{s-1}\phi(p)\phi'(p)$$

with $c > 0$, and we may choose ϕ and ϕ' so that this integral is one. Thus, the ideal contains 1 and admits a unique generator of the form $P(X)^{-1}$ with $P \in \mathbf{C}[X]$ and $P(0) = 1$. The proof of (2.7)(i) and (2.7)(ii) is now complete in the case $r = t$.

If $r > t$, we use the fact (c.f. (2.6)) that

$$W\left[\begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \neq 0, \qquad g \in G_t,$$

implies that x belongs to a compact set independent of $g \in G_t$. Thus, it is clear that the integral (2.4.3) for $\Psi(s, W, W'; j)$ is a finite sum of integrals of the form

(7) $$\int_{N_t\backslash G_t} W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-t}\end{pmatrix}\right] W'(g)\,|\det g|^{s-(r-t)/2}dg,$$

with $W \in \mathcal{W}(\pi;\psi)$, $W' \in \mathcal{W}(\pi';\bar{\psi})$. We may proceed exactly as before, using (2.5.2), to see that the integrals (7) converge for $Re(s)$ large and represent rational functions—with a fixed denominator. Thus, the same is true for the integrals $\Psi(s, W, W';j)$. As before, they span a fractional ideal I_j of the ring $\mathbf{C}[X, X^{-1}]$. Again using the fact that $\mathcal{K}_0(\psi) \subset \mathcal{W}(\pi;\psi)\,|\,P$, we see that $I_j \ni 1$. In fact, I_j contains all of the functions of s (or q^{-s}) represented by the integrals

$$\int dx \int_{G_t} \phi\left[\begin{pmatrix} g & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1\end{pmatrix}\right] W'(g)\,|\det g|^{s-(r-t)/2}dg,$$

with ϕ arbitrary in $C_c^\infty(G)$. Thus $I_j = (P_j(x)^{-1})$ with a unique $P_j \in \mathbf{C}[X]$ satisfying $P_j(0) = 1$. We prove now that P_j is independent of j $(0 \le j \le r-t-1)$. For that we proceed as in [J-P-S] Prop. (4.1.4), using Lemma (2.6) and the following Lemma:

LEMMA. *Let H be a function on G, right K-finite and satisfying*

$$H(ng) = \theta(n)H(g); \qquad n \in N, \qquad g \in G.$$

Set for $1 \le j \le r-t-1$, $\Phi \in \mathcal{S}(F^t)$, $g \in G_r$,

$$H^1(g) = \int_{F^t} H\left[g\begin{pmatrix} 1_t & 0 & z & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-t-j-1}\end{pmatrix}\right]\hat{\Phi}(-{}^tz)dz.$$

Then

$$\int_{F^t} H^1\left[\begin{pmatrix} g & 0 & 0 & 0 \\ b & 1_{j-1} & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-t-j-1}\end{pmatrix}\right] dx$$

$$= \int_{F^t} H\left[\begin{pmatrix} g & 0 & 0 & 0 \\ b & 1_{j-1} & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-t-j-1} \end{pmatrix}\right] \Phi(x)dx.$$

Proof. The proof which we omit is entirely similar to that of Lemma (4.1.5) of [J-P-S]. □

We return to the proof of Theorem (2.7). As above the integrals

$$\Psi(s, W, W'; j, \Phi) \quad \text{(c.f. (1.4))}$$

span a fractional ideal, say I_j^*, of $\mathbf{C}[X, X^{-1}]$. Then exactly as in [J-P-S], only now using (2.5.2) and Lemma (2.6) of the present paper, one proves that, for $1 \leq j \leq r - t - 1$,

$$I_j \subseteq I_{j-1}^* \subseteq I_{j-1} \subseteq I_{j-1}^* \subseteq I_j.$$

In particular $I_j = I_{j-1}$ and, therefore, $P_j = P_{j-1}$ $(1 \leq j \leq r - t - 1)$ as required. The proof of (2.7)(i) and (ii) is now complete in all cases.

(2.8). *Proof of Theorem* (2.7)(iii). *First step.* We remark first that if $t \leq r - 1$ then it is enough to treat the single case $j = 0$. In fact the validity of any one of the $r - t - 1$ functional equations (2.7.2) implies the validity of all of the others—with the same constant of proportionality. For $t = 1$ this is the exact content of Theorem (4.5) in [J-P-S]. The proof for general t is similar, one need only replace them (4.1.5) (loc. cit.) by lemma (2.7) above.

(2.9). *We prove that if $\epsilon(s, \pi \times \pi', \psi)$ exists then it is necessarily a monomial.* Suppose, for example $r = t$. In (2.7.1) we substitute π^ι for π, $(\pi')^\iota$ for π'. Then $\tilde{W} \in \mathcal{W}(\pi^\iota; \bar{\psi})$ and $(W')^\sim \in \mathcal{W}((\pi')^\iota; \psi)$. Thus, substituting $\bar{\psi}$ for ψ and s for $1 - s$ in (2.7.1), we get, noting that $\omega_\pi \cdot \pi_{\pi^\iota} = 1$,

(1) $\Psi(s, W, W', \Phi)/L(s, \pi \times \pi')$

$$= \omega_{\pi'}(-1)^{r-1}\epsilon(1 - s, \pi^\iota \times (\pi')^\iota, \bar{\psi})$$

$$\times \Psi(1 - s, \tilde{W}, \tilde{W}', \Phi)/L(1 - s, \pi \times \pi').$$

Using (2.7.1) again we find immediately

(2) $$\epsilon(1 - s, \pi^{\iota} \times (\pi')^{\iota}, \bar{\psi})\epsilon(s, \pi \times \pi', \psi) = 1.$$

Since both factors in (2) belong to $\mathbf{C}[X, X^{-1}]$, they must both be monomials. The proof for $r > t$ is similar.

(2.10). *Proof of Theorem* (2.7)(iii). *Second step.*

We consider first the case $r = t$, again making free use of standard facts from "distribution theory." Recall that, for Re(s) large,

(1) $$\Psi(s, W, W', \Phi) = \int_{N_r \backslash G_r} W(g)W'(g)\Phi(\eta_r g)\,|\det g|^s\, dg.$$

We may assume without loss of generality that π is the representation given by right translations acting on $\mathcal{W}(\pi; \psi)$. Then π^{ι} acts similarly on $\mathcal{W}(\pi^{\iota}; \bar{\psi})$. Similarly for π'. Now it is clear from (1) that if s is not a pole of $\Psi(s, W, W', \Phi)$ then

(2) $$\Psi(s, \pi(g)W, \pi'(g)W', \rho(g)\Phi) = |\det g|^{-s}\Psi(s, W, W', \Phi).$$

Similarly the identity

$$(\rho(g)\Phi)^{\wedge} = |\det g|^{-1}\rho(g^{\iota})\hat{\Phi},$$

leads at once to

(3) $$\Psi(1 - s, (\pi(g)W)^{\sim}, (\pi'(g)W')^{\sim}, (\rho(g)\Phi)^{\wedge}) = |\det g|^{-s}\Psi(s, \tilde{W}, \tilde{W}', \hat{\Phi}).$$

Thus, our assertion will be a consequence of the following Proposition, which holds for all pairs (π, π') of representations of Whittaker type.

PROPOSITION. *With the exception of finitely many values q^{-s}, the space of trilinear forms B on $\mathcal{W}(\pi; \psi) \times \mathcal{W}(\pi', \bar{\psi}) \times \mathcal{S}(F^r)$ satisfying*

(4) $$B(\pi(g)W, \pi'(g)W', \rho(g)\Phi) = |\det g|^{-s}B(W, W', \Phi)$$

has at most dimension one.

We abbreviate

$$\mathcal{W} = \mathcal{W}(\pi; \psi), \qquad \mathcal{W}' = \mathcal{W}(\pi'; \bar{\psi}), \qquad \mathcal{V} = \mathcal{W} \otimes \mathcal{W}'.$$

Proof. Let $\mathcal{S}^+$ denote the space of $\Phi \in \mathcal{S}(F^r)$ satisfying $\Phi(0) = 0$. Let B^+ denote the restriction of B to $\mathcal{W} \times \mathcal{W}' \times \mathcal{S}^+$. The map $g \mapsto \eta_r g$ induces an isomorphism $P\backslash G \rightrightarrows F^r - \{0\}$. Thus, we may identify $\mathcal{S}^+$ with $C_c^\infty(P\backslash G)$. Thus we may regard B^+ as an element of the space

$$\text{(5)} \qquad \mathrm{Bil}_{G_r}(\pi \otimes \pi' \otimes \alpha^s, \mathrm{Ind}(G_r, P_r, \delta_P^{-1/2})).$$

By Frobenius' reciprocity, this space is isomorphic as a complex vector space to

$$\mathrm{Bil}_{P_r}(\pi \otimes \pi' \otimes \alpha^s | P_r, \delta_P^{-1})$$

or, what amounts to the same, to

$$\text{(6)} \qquad \mathrm{Bil}_{P_r}(\pi \otimes \alpha^{s-1} | P_r, \pi' | P_r).$$

We shall prove now that, except for a finite set of q^{-s}, this space has dimension ≤ 1.

By Corollary 5.22 of [B-Z] both representations $\pi | P_r$ and $\pi' | P_r$ have composition series of finite length. Thus,

$$\mathcal{W} = \bigcup_{0 \leq \alpha \leq l} \mathcal{W}_\alpha, \qquad \mathcal{W}' = \bigcup_{0 \leq \alpha \leq m} \mathcal{W}'_\alpha,$$

where $\mathcal{W}_{\alpha+1} \supseteq \mathcal{W}_\alpha$, $\mathcal{W}'_{\alpha+1} \supseteq \mathcal{W}'_\alpha$, $\mathcal{W}_0 = \mathcal{W}'_0 = \{0\}$, $\mathcal{W}_\alpha$ and $\mathcal{W}'_\alpha$ are stable by P_r, and each successive quotient $\mathcal{W}_{\alpha+1}/\mathcal{W}_\alpha$ (resp. $\mathcal{W}'_{\alpha+1}/\mathcal{W}'_\alpha$) affords an irreducible representation say ξ_α (resp. ξ'_α) of P_r. Since $\dim \mathcal{W}^*_\theta = 1$, there is a unique index $j \leq l$ such that $\xi_j \simeq \tau_r$. Similarly, $\xi'_k \simeq \tau_r$ for a unique $k \leq m$.

Suppose $B \in \mathrm{Bil}_{P_r}(\pi \otimes \alpha^s | P_r, \pi' | P_r)$ is non-zero. Among all pairs (μ, ν) choose one so that $B | \mathcal{W}_\mu \times \mathcal{W}'_\nu \neq 0$ and such that $\mu + \nu$ is minimal. Then $\mu > 0$, $\nu > 0$ and B is trivial on both spaces $\mathcal{W}_{\mu-1} \times \mathcal{W}'_\nu$ and $\mathcal{W}_\mu \times \mathcal{W}'_{\nu-1}$. Thus B defines a (non-zero) element of the space $\mathrm{Bil}_{P_r}(\xi_\mu \otimes \alpha^s, \xi'_\nu)$ and, by Proposition (1.4), we must have $\mathrm{index}(\xi_\mu) = \mathrm{index}(\xi'_\nu)$. Moreover, by (1.5) Corollary 2, if we exclude a finite set of q^{-s}, we see that this index must be $r - 1$. Thus, $\mu = j$, $\nu = k$, and any $B \in \mathrm{Bil}_{P_r}(\pi \otimes \alpha^s | P_r, \pi' | P_r)$ vanishes on $\mathcal{W}_j \times \mathcal{W}'_{k-1} + \mathcal{W}_{j-1} \times \mathcal{W}'_k$. Thus, dim $\mathrm{Bil}_{P_r}(\tau_r \otimes \alpha^s, \tau_r)$ being one, given B and $B' \in \mathrm{Bil}_{P_r}(\pi \otimes \alpha^s | P_r, \pi' | P_r)$, there is a constant c so that $B = cB'$ on $\mathcal{W}_j \times \mathcal{W}'_k$. Set $B'' = B - cB$. Then since B'' vanishes on $\mathcal{W}_j \times \mathcal{W}'_k$, after excluding a finite set of q^{-s}, it must be identically zero.

Now suppose we have two trilinear forms B and B' as in the proposition. Then by what we have proved there is a constant c such that $B_0 = B - cB'$ vanishes on $\mathcal{W} \times \mathcal{W}' \times \mathcal{S}^+$. Thus B_0 is necessarily of the form

$$B_0(W, W', \Phi) = \gamma(W, W')\Phi(0),$$

and we clearly have

$$\gamma(\pi(g)W, \pi'(g)W') = |\det g|^{-s}\gamma(W, W')$$

for all $g \in G_r$. Now γ must be zero, unless $\pi \otimes \alpha^s \simeq \tilde{\pi}'$, and this can happen at most for a finite set of q^{-s}. Thus except for these values of s, $B_0 = 0$, $B = cB'$, and the proposition is proved. □

(2.11). We continue with the proof of (2.7). We may assume $t \leq r - 1$. Then by our remark in (2.8) we may assume $j = 0$. We have then

$$(1) \qquad \Psi(s, W, W'; 0) = \int_{N_t \backslash G_t} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W'(g) |\det g|^{s-(r-t)/2} dg.$$

We may assume as before that π acts by right translations on $\mathcal{W}(\pi; \psi)$. Similarly for, π^ι, π' and $(\pi')^\iota$. Then except for finitely many q^{-s}, we have

$$(2) \qquad \Psi(s, \pi(g)W, \pi'(g)W'; 0) = |\det g|^{(r-t)/2-s}\Psi(s, W, W'; 0),$$

and similarly for $\Psi(1 - s, \rho(w_{r,t})\tilde{W}, \tilde{W}'; r - t - 1)$. Next let $U_{t,r}$ denote the subgroup of G_r consisting of all matrices of the form

$$(3) \qquad u = \begin{pmatrix} 1_t & u_2 \\ 0 & u_1 \end{pmatrix},$$

where $u_2 \in M(t \times r - t)$ and $u_1 \in N_{r-t}$. Then we also have an identity

$$(4) \qquad \Psi(s, \pi(u)W, W'; 0) = \theta_r(u)\Psi(s, W, W'; 0),$$

for all $u \in U_{t+1,r}$. Similarly for $\Psi(1 - s, \rho(w_{r,t})\tilde{W}, \tilde{W}'; r - t - 1)$. Thus exactly as in (2.10) the existence of the functional equation (2.7.2) (for $j = 0$, $k = r - t - 1$) will be a trivial consequence of the following Proposition.

PROPOSITION. *Let π (resp. π') be an irreducible, generic representation of G_r (resp. G_t). Then except for finitely many q^{-s}, the space of bilinear forms B belonging to* $\mathrm{Bil}_{G_t}(\pi \otimes \alpha^s, \pi')$ *and satisfying the additional condition*

$$(5) \qquad B(\pi(u)W, W') = \theta_r(u)B(W, W'), \qquad \text{for} \quad u \in U_{t+1,r},$$

has dimension ≤ 1.

Proceeding as in the proof of Proposition (2.10), using again the fact that $\pi | P_r$ has a (finite) composite series, we are reduced to proving the following lemma.

LEMMA. *Suppose ξ is a smooth, irreducible representation of P_r. Let π' be an irreducible, admissible representation of G_t. Then, if ξ is equivalent to τ_r and if π' is generic, the space of $B \in$* $\mathrm{Bil}_{G_t}(\xi \otimes \alpha^s, \pi')$ *and satisfying* (5) *has dimension one* (for all s). *Otherwise, there is a finite subset S of* $\mathbf{C}$, *such that for all $q^{-s} \notin S$, the space of such bilinear forms reduces to zero.*

Proof. The proof is similar to the proof of Proposition (1.4). We content ourselves therefore with a sketch.

We will prove the lemma by induction on r starting with $r = 2$. Thus we assume at first $r = 2$, $t = 1$. In this case condition (5) is vacuous. Moreover π', being a representation of $G_1 \simeq F^\times$, is generic by definition. Next it follows, say from Proposition (1.1), that either $\xi \simeq \tau_2$ or ξ is actually a representation of $P_2/U_2 \simeq G_1$. In the first case the restriction of ξ to G_1 is equivalent to the action of G_1 on $\mathcal{S}(F^\times)$ by translations. Our assertion is now clear in the first case and equally clear in the second.

Thus we may suppose $r > 2$ and the lemma true for $p < r$. Let then $\mathcal{W}$ be the space on which ξ acts and set $\mathcal{W}' = \mathcal{W}(\pi'; \bar{\psi})$. We may regard any bilinear form B under question as a linear form λ on $\mathcal{W} \otimes \mathcal{W}'$. Next, by Theorem (1.2), ξ is an induced representation

$$(6) \qquad \xi = \mathrm{Ind}(P_r, H_r^{(j)}; \sigma),$$

where $\sigma = [\nu, r]$, ν is an irreducible admissible representation of G_{r-1-j}, and $j = \text{index}(\xi)$. Let $\mathcal{V}$ be the space on which ν operates. Then $\mathcal{W} \otimes \mathcal{W}'$ is a quotient of $C_c^\infty(P_r) \otimes \mathcal{V} \otimes \mathcal{W}'$ and we may lift λ to a $\mathcal{V} \otimes \mathcal{W}'$-valued distribution T on P_r. It is easy to see that T satisfies the following three conditions:

(7) $$dT(hp) = \delta_{P_r}^{1/2}(h)\delta_H^{1/2}(h)\{{}^t\sigma(h) \otimes 1\}dT(p),$$

for all $p \in P_r$, $h \in H_r^{(j)}$;

(8) $$dT(pu) = \bar{\theta}_r(u)dT(p),$$

for all $p \in P_r$, $u \in U_{t+1,r}$;

(9) $$\{1 \otimes {}^t\pi'(g)\}dT(pg) = |\det g|^s dT(p),$$

for all $p \in P_r$, $g \in G_t$.

There are several cases to be considered. First suppose $r - j \geq t + 1$. Then $U_{r-j,r} \subset U_{t+1,r}$; and, since $U_{r-j,r}$ is also contained in $H_r^{(j)}$, we deduce from (7) and (8) the condition

(10) $$dT(up) = dT(pu) = \bar{\theta}_r(u)dT(p),$$

for $p \in P_r$, $u \in U_{r-j,r}$. As in (1.4) this implies that T has support in the subgroup $P_{r-j}N_r$. In fact this is proved there for $j = 1$. The general case follows by an easy induction argument. That being the case we may therefore view T as a distribution on $P_{r-j}N_r$. Note that this group is the same as the group we have denoted by $H_r^{(j)}$.

Let us set then, for $f \in C_c^\infty(H_r^{(j)}) \otimes \mathcal{V} \otimes \mathcal{W}'$,

(11) $$F_f(h) = \int_{H_r^{(j)}} \delta_{P_r}^{1/2}(h)\delta_H^{1/2}(h)\{\sigma^{-1}(h) \otimes 1\}f(h)d_l(h).$$

Then from (7) we deduce immediately that

(12) $$\int_{H_r^{(j)}} f(h)dT(h) = (\mu, F_f),$$

where μ is an appropriate linear form on $\mathcal{V} \otimes \mathcal{W}'$.

Next since $U_{t+1,r}$ is contained in $H_r^{(j)}$, we easily deduce from (8), (11) and (12) that μ satisfies the condition

$$\{{}^t\sigma(u)^{-1} \otimes 1\}\mu = \theta_r(u)\mu, \tag{13}$$

for all $u \in U_{t+1,r}$.

Suppose now that $r - j > t + 1$. Let $U_r^{(j)}$ denote the group of $r \times r$ matrices of the form

$$u = \begin{pmatrix} 1_{r-1-j} & v \\ 0 & 1_{j+1} \end{pmatrix},$$

where v is an $(r - 1 - j) \times (j + 1)$ matrix with an arbitrary first column and whose other columns are zero. With our hypothesis we have $U_r^{(j)} \subset U_{t+1,r}$. On the other hand since σ has a trivial restriction to $U_r^{(j)}$ and θ_r does not, we deduce from (13) that $\mu = 0$. Thus T and hence λ also vanish.

Next suppose $r - j = t + 1$. In particular G_t is contained in $H_r^{(j)}$. We find then, from (9), (11) and (12), that if $\mu \neq 0$, then

$$\operatorname{Hom}_{G_t}(\nu \otimes (\alpha^s \chi), (\pi')^{\sim}) \neq 0. \tag{14}$$

Here χ is a fixed quasi-character of G_t (depending only on r and t). Thus, as in (2.10), if λ or equivalently μ is non-zero, we deduce from (14) that q^{-s} is restricted to a finite set.

Finally suppose that $r - j < t + 1$. Then $\sigma(u) = \theta_r(u)$ for all $u \in U_{t,r}$, and from (7) we get

$$dT(up) = \bar{\theta}_r(u)dT(p), \tag{15}$$

again for all $u \in U_{t,r}$. This together with (8) implies that T has support in $P_{t+1}N_r$. Thus we may view T as a distribution on the group $P_{t+1}N_r$. Next observe that $P_{t+1}N_r = G_t U_{t,r}$. If we combine this with (9) and (15) we find the following expression for T:

$$\int f(p)dT(p) = (\mu, \Phi_f), \tag{16}$$

where μ is a linear form on $\mathcal{V} \otimes \mathcal{W}'$ and where Φ_f is an element of the same space, defined by the expression

$$\Phi_f = \int_{G_t \times U_{t,r}} \theta_r(u) |\det g|^s \{1 \otimes \pi'(g)\} f(ug) du dg. \tag{17}$$

We now view μ as a bilinear form B' on the space $\mathcal{V} \times \mathcal{W}'$. If we combine (16) and (7) we arrive at the following conditions on B':

$$B'(\nu(g)v, \pi'(g)w) = |\det g|^{-(s+r-t)} B'(v, w), \tag{18}$$

for $g \in G_{r-1-j}$;

$$B'(v, \pi'(u)w) = B'(v, w), \tag{19}$$

for $u \in U_r^{(j)}$;

$$B'(v, \pi'(u)w) = \bar{\theta}_r(u) B'(v, w), \tag{20}$$

for $u \in U_{r-j,t}$, $v \in \mathcal{V}$, $w \in \mathcal{W}'$.

Suppose first that $j < r - 1$. We are going to prove then *that any bilinear form B' on $\mathcal{V} \times \mathcal{W}'$ satisfying these three conditions is identically zero* (except for a finite set of q^{-s}). We start with the fact that the restriction of π' to P_t has a finite composition series. It is then clearly enough to prove the corresponding assertion for a bilinear form B'' satisfying (18), (19) and (20) when π' is replaced by a smooth irreducible representation, say ξ', of P_t. But then conditions (18) and (20) are the precise conditions of the Lemma for the pair $(t, r - 1 - j)$, with ξ' replacing ξ, and ν replacing π'. Thus, since $t < r$, our induction hypothesis implies that (except for a finite set of q^{-s}) the bilinear form $B'' = 0$ unless ν is generic and ξ' is isomorphic to τ_t. In the latter case we may view B'' as bilinear form on the space $\mathcal{W}(\nu; \bar{\psi}) \times \mathcal{K}_0(\psi; t)$. (c.f. (2.2)). It follows from the uniqueness assertion of the lemma (for the pair $(t, r - 1 - j)$) that B'', if it is not zero, has the form

$$B'(W, \phi) = c \int_{N_{r-1-j} \backslash G_{r-1-j}} \phi\left[\begin{pmatrix} g & 0 \\ 0 & 1_p \end{pmatrix}\right] W(g) |\det g|^{s+r-t} dg, \tag{21}$$

where $p = t - r + j + 1$, $\phi \in \mathcal{K}_0(\psi; t)$, $W \in \mathcal{W}(\nu; \psi)$ and c is a nonzero constant. But then condition (19), or rather its analogue for ξ', implies that the integral in (21) does not change if we replace $W(g)$ by

$\psi[\eta_{r-1-j}gx]W(g)$, for any column vector x of height $r-1-j$. It follows at once that $W=0$ for all $W \in \mathcal{W}(\nu; \psi)$. This contradiction implies that $B''=0$. Since this is now true for *each* composition factor ξ' of π', we see that B' also vanishes as claimed. In turn dT and B also vanish in this case.

Finally suppose that $j = \text{index}(\xi) = r-1$. Then $\dim \mathcal{V} = 1$, condition (20) implies that π' is generic and the uniqueness of B' (or B) is then equivalent to the uniqueness of the Whittaker model of π'. The proof of the lemma and the Proposition is now complete.

We have already seen that the existence of any of the functional equations (2.7.2) is independent of j. Thus we have completed the proof of its existence in all cases.

The proof of Theorem (2.7) is now complete.

(2.12). We note that the L-factors $L(s, \pi \times \sigma)$ do not depend on the additive character ψ. In fact replacing ψ by another non-trivial character entails replacing each W in $\mathcal{W}(\pi; \psi)$ by its left translate $\lambda(a)W$ by a fixed element $a \in A_r$. Then for example the integral in (2.4.1) is replaced by the integral

$$(1) \qquad \int_{N_r/G_r} W(ag)W'(ag)\Phi(\eta_r g)\,|\det g|^s\,dg,$$

which after a change of variables becomes

$$|\det a|^{-s}\delta_r(a)\int_{N_r/G_r} W(g)W'(g)\Phi(a_r\eta_r g)\,|\det g|^s\,dg,$$

if $a = \text{diag}(a_1, a_2, \ldots, a_r)$. It is clear that the ideal spanned by the integrals (1) is the same as that spanned by the $\Psi(s, W, W', \Phi)$. This is equivalent to our assertion for $r=t$; the other cases are similar. In particular we may interchange the roles played by ψ and $\bar{\psi}$ to conclude that, for $r=t$, $L(s, \pi \times \sigma) = L(s, \sigma \times \pi)$. If $r > t$, we write by convention

$$L(s, \sigma \times \pi) = L(s, \pi \times \sigma).$$

It is easy to see directly that if $r=t$, then $\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \sigma \times \pi, \psi)$. Thus, if $r > t$, we may also define

$$\epsilon(s, \sigma \times \pi, \psi) = \epsilon(s, \pi \times \sigma, \psi).$$

(2.13). The following Proposition will be used in section 7.

PROPOSITION. *If π and σ are given, then $L(s, (\pi \otimes \mu) \times \sigma) = 1$, provided the conductor of μ is sufficiently large.*

Proof. Suppose for example that $r > t$. Then if $W \in \mathcal{W}(\pi; \psi)$, the function $W \otimes \mu$ (defined by $W \otimes \mu(g) = W(g)\mu(\det g)$) is in $\mathcal{W}(\pi \otimes \mu; \psi)$. Proceeding exactly as in (2.7), we may express the integral $\Psi(s, W \otimes \mu, W')$ as a finite sum of integrals of the form

$$\int \phi(a_1, a_2, \ldots, a_t)|a_1|^s|a_2|^{2s} \cdots |a_t|^{ts}$$

$$\Pi\, \eta_i(a_i)\mu(a_i) \otimes d^\times a_i,$$

where $\phi \in \mathcal{S}(F^t)$, η_i is a finite-function of $F^\times$ belonging to a *fixed finite* set. We may choose μ so that each $\mu\eta_i$ is ramified. Then each of the above integrals is a polynomial in q^s, q^{-s}. In particular $L(s, (\pi \otimes \mu) \times \sigma)$ is itself such a polynomial and therefore must be 1. □

3. Statement of Theorem (3.1).

(3.1). As in section 2 our representations π will be assumed to be of Whittaker type.

Let again r and t be integers ≥ 0, $(t_1, t_2, \ldots, t_p)$ a partition of t, π (resp. σ_i) a representation of Whittaker type of G_r (resp. G_{t_i}). Let Q be the *lower* standard parabolic subgroup of G_t of type $(t_1, t_2, \ldots, t_p)$. Let

$$\sigma = \mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2, \ldots, \sigma_p). \tag{1}$$

Then by Proposition (2.3) σ is of Whittaker type if and only if each σ_i is.

In general if π and σ are of Whittaker type, the factors $L(s, \pi \times \sigma)$ and $\epsilon(s, \pi \times \sigma, \psi)$ have been defined (Theorem 2.7)).

We also set

$$\gamma(s, \pi \times \sigma, \psi) = \epsilon(s, \pi \times \sigma, \psi)L(1 - s, \pi^\iota \times \sigma^\iota)/L(s, \pi \times \sigma). \tag{2}$$

With these definitions we then have the following Theorem.

THEOREM. *Let π and σ respectively be representations of Whittaker type of G_r and G_t. Suppose σ has the form* (1). *Then*:

(i) $$\gamma(s, \pi \times \sigma, \psi) = \prod_{1 \leq i \leq p} \gamma(s, \pi \times \sigma_i, \psi),$$

(ii) $$L(s, \pi \times \sigma) \in \prod_{1 \leq i \leq p} L(s, \pi \times \sigma_i)\mathbf{C}[q^{-s}, q^s].$$

(3.2). The proof of Theorem (3.1) will occupy section 3 to section 7 of the paper. A more definitive result concerning $L(s, \pi \times \sigma)$, when π and σ are irreducible, will appear in sections 8 and 9. We show now that *the truth of Theorem* (3.1) *for* $p = 2$ *implies its truth in general.* We proceed by induction on p. Suppose that the Theorem is true for any pair (r, t) and any partition of t into l integers, $2 \leq l < p$. Then π, $\sigma_i (1 \leq i \leq p)$, and σ being given, let

$$\sigma_1' = \operatorname{Ind}(G_{t-t_p}, Q'; \sigma_1, \sigma_2, \ldots, \sigma_{p-1}),$$

where Q' is of type $(t_1, t_2, \ldots, t_{p-1})$. Then

$$\sigma = \operatorname{Ind}(G_t, R; \sigma_1', \sigma_p),$$

where R' is of type $(t - t_p, t_p)$. Then by the induction hypothesis

$$\gamma(s, \pi \times \sigma_1', \psi) = \prod_{1 \leq j \leq p-1} \gamma(s, \pi \times \sigma_j, \psi),$$

and by the case $p = 2$

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi \times \sigma_1', \psi)\gamma(s, \pi \times \sigma_p, \psi).$$

The relation (3.1)(i) follows immediately. One proves the second relation in the same way.

From now on we may suppose $p = 2$. We will first prove Theorem (3.1) in three particular cases. The general case will be treated in section 7.

(3.3) *Remark.* In general suppose π is a representation of G_r of Whittaker type. Let π^0 be the representation of G_r by right translations on the space $\mathcal{W}(\pi; \psi)$. The representation π^0, being a quotient of π for which $\dim(\pi^0)_\theta^* = 1$, is again of Whittaker type. It is also obvious that if σ is a representation of, say, G_t of Whittaker type then:

$$L(s, \pi \times \sigma) = L(s, \pi^0 \times \sigma).$$

Moreover, we also have (c.f. (2.1)) $(\pi^0)^\iota = (\pi^\iota)^0$ and it follows that

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi^0 \times \sigma, \psi).$$

Thus, when convenient we may suppose for any representation ξ under consideration that $\xi = \xi^0$.

4. The case $t \leq r$.

(4.1). By a formal Laurent series (with complex coefficients) we shall mean a formal sum

$$s(X) = \Sigma\, a_m X^m \tag{1}$$

where $a_m \in \mathbf{C}$ and the sum is extended over all $m \in \mathbf{Z}$. In an obvious way we may regard the set of such series as a module (with torsion) over $\mathbf{C}[X, X^{-1}]$.

Suppose we are given a measure space $(\Lambda, d\lambda)$ and a formal Laurent series

$$s(\lambda, X) = \Sigma\, a_m(\lambda) X^m$$

whose coefficients are measurable functions on Λ. We shall say that $s(\lambda, X)$ is integrable if and only if each coefficient $a_m(\lambda)$ is integrable (with respect to $d\lambda$) and shall then write

$$\oint s(\lambda, X) d\lambda = \Sigma\, X^m \int a_m(\lambda) d\lambda.$$

Note that if $s(\lambda, X)$ is integrable we may interchange integration and multiplication by a polynomial in X.

(4.2). We may associate to each of the integrals in (2.4) a formal Laurent series as follows. Consider for example the integral (2.4.1). For $m \in \mathbf{Z}$, set $G_r^m = \{g \mid |\det g| = q^{-m}\}$. Set also

$$\Psi^m(W, W'; \Phi) = \int_{N_r \backslash G_r^m} W(g) W'(g) \Phi(\eta_r g) dg. \tag{1}$$

Since the integral (2.4.1) converges for Re(s) large, by Fubini, each of the integrals in (1) is convergent for all s and, moreover,

(2) $$\Psi(s, W, W'; \Phi) = \sum_{m \in \mathbf{Z}} q^{-ms} \Psi^m(W, W'; \Phi).$$

The series we have in mind is obtained by formally substituting X for q^{-s} in (2). By abuse of language we will also denote it by $\Psi(s, W, W'; \Phi)$. Note that in any case *there are only finitely many* (non-vanishing) *negative terms* ($m < 0$) *in* (2). In fact, this follows readily from (2.7.3). In more detail, as in (2.7),

(3) $$\Psi^m(W, W'; \Phi) = \sum_i \int_{A_r^m} W_i(a) W_i'(a) \Phi_i(a_r) \delta_r^{-1}(a) d^\times a,$$

with $A_r^m = A_r \cap G_r^m$. Write $a = \operatorname{diag}(a_1 a_2 \cdots a_r, a_2 \cdots a_r, \ldots, a_r)$. Then, since W_i, say, is dominated by a gauge, its support is contained in the set of a for which $|a_i| \leq 1$ for $1 \leq i \leq r - 1$. In addition, if $\Phi_i(a_r) \neq 0$, then $|a_r| \lesssim 1$. In particular each integral in (3) vanishes unless $|\det a| \lesssim 1$ and our assertion follows. Similarly, we may associate to each of the integrals (2.4.2), (2.4.3), and (2.4.4) a formal Laurent series with finitely many negative terms.

(4.3). We will make frequent use of the following remark. Consider for example the functional equation for $t = r - 1$:

(1) $$L(1 - s, \pi^\iota \times \sigma^\iota)^{-1} \Psi(1 - s, \tilde{W}, \tilde{W}') = \omega_\sigma(-1)^{r-1} \epsilon(s, \pi \times \sigma, \psi) L(s, \pi \times \sigma)^{-1} \Psi(s, W, W'),$$

where now $\Psi(1 - s, W, W')$ and $\Psi(s, W, W')$ are regarded as analytic functions. Then the same identity is true in the sense of formal power series. In fact, by the very definition of the L-factor,

(2) $$L(s, \pi \times \sigma)^{-1} \Psi(s, W, W') = Q(X, X^{-1})$$

is a polynomial in X, X^{-1}. Similarly, the left side of (4) is a polynomial $Q^1(X, X^{-1})$, and (4) is equivalent to

(3) $$Q(X, X^{-1}) = Q^1(X, X^{-1})$$

in either sense. Thus either interpretation of (4)—as an identity of analytic functions or an identity of power series—is entirely equivalent to the other.

(4.4). We now turn to the proof of Theorem 3.1 in the case $t \leq r$. We are also assuming $p \leq 2$. The principle of the proof, in the case $r = t$, with which we begin, is to establish the following identity:

$$(1)\quad L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1} L(1-s, \pi^\iota \times \sigma_2^\iota)^{-1} \Psi(1-s, \tilde{W}, \tilde{W}'; \hat{\Phi})$$

$$= \omega_\sigma(-1)^{r-1} \epsilon(s, \pi \times \sigma_1, \psi) \epsilon(s, \pi \times \sigma_2, \psi) L(s, \pi \times \sigma_1)^{-1}$$

$$\cdot L(s, \pi \times \sigma_2)^{-1} \Psi(s, W, W'; \Phi),$$

the identity being interpretated as *an equality of formal Laurent series* (in $X = q^{-s}$). Note that as opposed to the situation in (4.3) we do not know a priori that the left side of (1) is a polynomial (in X, X^{-1}). However, once (1) is established this readily follows. In fact, we have seen that $\Psi(s, W, W'; \Phi)$ has the form $\Sigma\, a_n X^n$ with $a_n = 0$ if $n \ll 0$. As the other factors on the right are polynomials in X, X^{-1}, the right side itself is such a series. Similarly, the left side is a series $\Sigma\, a_n X^n$ with $a_n = 0$ for $n \gg 0$. In other words, once (1) is established both sides are a posteriori polynomials in (X, X^{-1}). Then, as in (4.3), (1) may be regarded as an identity of analytic functions. Thus (3.1)(i) will follow immediately and taking $\Psi(s, W, W'; \Phi) = L(s, \pi \times \sigma)$ we will also get (3.1)(ii).

(4.5). We will prove now that (4.4.1) follows from the functional equations for the pairs (π, σ_1) and (π, σ_2). In the process we will replace σ_1 by the representation $\sigma_1 \otimes \alpha^{-u}$ with $u \gg 0$, making an "auxiliary" analytic continuation in u. For that we will need the following Lemma, essentially due to W. Casselman [C-S, section 2].

LEMMA. Set

$$\sigma_u = \operatorname{Ind}(G_t, Q; \sigma_1 \otimes \alpha^u, \sigma_2).$$

Then given $W \in \mathcal{W}(\sigma; \psi)$ *there exists for each* $u \in \mathbf{C}$ *a function* W_u *in* $\mathcal{W}(\sigma_u; \psi)$ *such that*:

(1) $W_0 = W$.

(2) *There is a fixed open compact subgroup* K_0 *such that*

$$W_u(gk) = W_u(g)$$

for all $g \in G_r$, $k \in K_0$, $u \in \mathbf{C}$.

(3) *For each* g, $u \mapsto W_u(g)$ *is analytic in* u.

The lemma is a trivial consequence of Proposition (3.2) of [F.S. I] and its corollaries, especially Corollary (3.4). We would however like to make the following explication. Let, for $\nu \in \mathbf{Z}$, N^ν be the subgroup of N consisting of those matrices $n = (n_{ij})$ for which $n_{ij} \in \underline{P}^{-(j-i)\nu}$. We have $N^m \subset N^\nu$ if $m \leq \nu$ and $N = \bigcup_\nu N^\nu$. Let $\bar{U}_Q$ denote the unipotent radical of $\bar{Q}$ and set $V^\nu = N^\nu \cap \bar{U}_Q$. Note that, since N normalizes $\bar{U}_Q$, N^m normalizes V^ν if $\nu \geq m$. In our notation, the groups V^ν play the role of the $\{N_i\}$ in (loc. cit.).

We apply the lemma to $W' \in \mathcal{W}(\sigma; \bar{\psi})$. Then the integral $\Psi^m(W, W'_u; \Phi)$ is analytic in u. In fact, using (4.5.2), we may write it as a finite sum of integrals of the form

$$\int_{A_r^m} W(ag)W'_u(ag)\Phi'(a_r)\delta_r^{-1}(a)d^\times a.$$

As in (4.2), for a in the support of the integral, we have

$$|a_1| \leq |a_2| \leq \cdots \leq |a_r|;$$

but since $|a_1 a_2^2 \cdots a_r^r| \asymp 1$ for $a \in A_r^m$, we also have

$$|a_1| \asymp |a_2| \asymp \cdots \asymp |a_r| \asymp 1,$$

in other words, the integrand has compact support independent of u. Thus $\{\sigma_1$ being replaced by $\sigma_1 \otimes \alpha^u$ and W' by $W'_u\}$ in (4.4.1), the resulting Laurent series $\Psi(s, W, W'; \Phi)$ and $\Psi(1 - s, \tilde{W}, (\tilde{W}'_u)^\sim; \hat{\Phi})$ have coefficients which are analytic in u. In addition $L(s, \pi \times \sigma_1 \otimes \alpha^u)^{-1} = L(s + u, \pi \times \sigma_1)^{-1}$ is a polynomial in X with coefficients analytic in u and

$$L(1 - s, \pi^\iota \times (\sigma_1 \otimes \alpha^u)^\iota)^{-1} = L(1 - s - u, \pi^\iota \times \sigma_1^\iota)^{-1}$$

is a polynomial in X^{-1} with analytic coefficients. Similarly,

$$\epsilon(s, \pi \times (\sigma_1 \otimes \alpha^u), \psi) = \epsilon(s + u, \pi \times \sigma_1, \psi).$$

Thus, the resulting identity is equivalent to an (infinite) number of identities each of which has the form

$$\sum_{|j|\le m} a_j(u)c_j = 0,$$

where m is independent of u and where again the a_j are analytic in u. We see then that in proving (4.4.1) we may assume, say, that u is real, negative and large in absolute value.

(4.6). To simplify notation we will write $\sigma_1 = \tau_1 \otimes \alpha^{-u}$, $\sigma_2 = \tau_2 \otimes \alpha^u$ where τ_1 and τ_2 are fixed and u is a positive real number sufficiently large. We also set

$$\sigma_u = \operatorname{Ind}(G_t, Q; \sigma_1 \times \sigma_2).$$

Recall that Q is a lower standard parabolic subgroup of G_t of type (t_1, t_2). By (3.3) we may suppose that σ_i is the representation of G_{t_i} acting by right translations on $\mathcal{W}(\sigma_i; \bar{\psi})$. Then $\sigma_1 \times \sigma_2$ operates on $\mathcal{W}(\sigma_1; \bar{\psi}) \otimes \mathcal{W}(\sigma_2; \bar{\psi})$.

Let f be an element of the space of σ_u. Then f takes values in $\mathcal{W}(\sigma_1; \bar{\psi}) \otimes \mathcal{W}(\sigma_2; \bar{\psi})$, a space of functions on $G_{t_1} \times G_{t_2}$. We denote by $f(g; h_1, h_2)$ the value of that function at a point g of G_t. Then one has, for $a_1, h_1 \in G_{t_1}$, $a_2, h_2 \in G_{t_2}$,

(1)

$$f\left[\begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} g; h_1, h_2\right] = f[g; h_1 a_1, h_2 a_2] \, |\det a_1|^{-t_2/2} |\det a_2|^{t_1/2}.$$

We will show now that if $u \gg 0$, the integral

$$\int f\left[\begin{pmatrix} 1_{t_1} & x \\ 0 & 1_{t_2} \end{pmatrix} g; e, e\right] \theta\left[\begin{pmatrix} 1_{t_1} & x \\ 0 & 1_{t_2} \end{pmatrix}\right] dx \tag{2}$$

converges. It suffices to prove this for $g = e$.

With the notation as above let us denote by ϕ_u the positive function on $G_t \times G_{t_1} \times G_{t_2}$ defined by

$$\phi_u\left[\begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} k; h_1, h_2\right] = |\det a_2 a_1^{-1}|^u |\det h_2 h_1^{-1}|^u, \tag{3}$$

for $k \in K_r$. It is clear that the map $f_0 \mapsto f_0\phi_u$ defines a bijection between $\mathrm{Ind}(G_t, Q; \tau_1, \tau_2)$ and $\mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2)$.

Next consider the function $h(g) = f_0(g; e, e)$. Suppose g of the form $g = \bar{n}a$, $n \in \bar{N}$, $a \in A$, and further write

$$\bar{n} = \begin{pmatrix} u_1 & 0 \\ * & u_2 \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

with $u_1 \in N_{t_1}$, $u_2 \in N_{t_2}$, $a_1 \in A_{t_1}$, $a_2 \in A_{t_2}$. Then since $\bar{N}A$ is contained in Q one has

$$h(\bar{n}a) = \delta_Q^{1/2}(a) f_0(e; u_1a_1, u_2a_2).$$

Since f_0 takes values in $\mathcal{W}(\tau_1; \bar{\psi}) \otimes \mathcal{W}(\tau_2; \bar{\psi})$, using (2.5) we deduce a majorization

$$|h(\bar{n}a)| \preceq \xi(a),$$

where ξ is a finite sum of positive (quasi-) characters of A_t. We are thereby reduced to proving the following assertion. Let η be a positive quasi-character on A_t. With the notation as before, let $\tilde{\eta}$ be the function on G_t defined by

$$\tilde{\eta}(\bar{n}ak) = \det|a_2a_1^{-1}|^u \eta(a), \qquad k \in K_t;$$

then the integral

$$\int \tilde{\eta}\left[\begin{pmatrix} 1_{t_1} & x \\ 0 & 1_{t_2} \end{pmatrix}\right] dx$$

converges for $u \gg 0$. In fact, this is a special case of a well-known result.

That being the case, the function W' of g defined by (2) evidently belongs to $\mathcal{W}(\sigma_u; \bar{\psi})$. Moreover, all elements of $\mathcal{W}(\sigma_u; \bar{\psi})$ are of this form. Substituting then in (4.2.1) we get

(4)

$$\Psi^m(W, W'; \Phi) = \int_{N\backslash G^m} \Phi(\eta g) W(g) \int_{\bar{U}_Q} f\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g; e, e\right] \theta\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} dx.$$

Using the fact that $\Phi(\eta_r vg) = \Phi(g)$ for $v \in \bar{U}_Q$, we get formally for this integral

$$\int_{N_{t_1,t_2}\backslash G_r^m} W(g) f[g; e, e]\Phi(\eta g)dg,$$

where N_{t_1,t_2} is the group of matrices of the form

$$(5) \qquad \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}, \qquad n_i \in N_{t_i}.$$

It is easy to see (c.f. (4.2)) that the function $W(g)\Phi(\eta_r g)$ on G_r^m is compactly supported modulo N_r. Thus the integral in (4) is absolutely convergent, and our step is justified. To proceed, we make use of the following general formula

$$(6) \qquad \int_{N_{t_1,t_2}\backslash G_r^m} f(g)dg = \int_{G_{t_1}^0 N_{t_1,t_2}\backslash G_r^0} dh \int_{N_{t_1}\backslash G_{t_1}^m} f\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right] d^\times a$$

(Here we have $r = t = t_1 + t_2$). If we combine this with (1), we get

$$(7) \quad \Psi^m(W, W'; \Phi)$$

$$= \int_{G_{t_1}^0 N_{t_2,t_2}\backslash G_r^0} \Phi(\eta h)dh \int_{N_{t_1}\backslash G_{t_1}^m} W\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} h\right] f[h; a, e] |\det a|^{-t_2/2} d^\times a.$$

(4.7). Expliting the functional equation for the pair (π, σ_1), we will deduce from (4.6.7) the following identity between formal Laurent series:

$$(1) \quad L(s, \pi \times \sigma_1)^{-1}\epsilon(s, \pi \times \sigma_1, \psi)\Psi(s, W, W'; \Phi)$$

$$= L(1 - s, \pi^\iota \times \sigma_1^\iota)^{-1}\omega_{\sigma_1}(-1)^{r-1}$$

$$\int_{H_{t_1,t_2}\backslash G_r} f[g; e, e] W\left[w_r \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix} g\right] |\det g|^s \cdot (g\Phi)^\sim [\zeta_{t_1}, \eta_{t_2}]dg,$$

where H_{t_1,t_2} is the unipotent subgroup of G_r consisting of all matrices of the form

$$\begin{pmatrix} n_1 & 0 \\ x & n_2 \end{pmatrix}, \qquad n_i \in N_{t_i};$$

and we have set $\zeta_{t_1} = (1, 0, \ldots, 0) \in F^{t_1}$, $\eta_{t_2} = (0, 0, \ldots, 1) \in F^{t_2}$, and

$$\Phi^{\sim}(\xi, v) = \int_{F^{t_1}} \Phi[(w, v)]\psi(\xi^t w)dw,$$

for $\xi, w \in F^{t_1}$, $v \in F^{t_2}$. We prove first that the expression on the right in (1) has a sense, that is to say, that for $u \gg 0$ the integral we obtain by replacing the integral in (1) by the corresponding integral over $H_{t_1,t_2}\backslash G_r^m$ is convergent. Again, we make use of a general integration formula, namely:

$$\int_{H_{t_1,t_2}\backslash G_r^m} f(g)dg = \int_K dk \int_{A_r^m} f(ak)\mu(a)d^\times a,$$

where μ is a (fixed) positive quasi-character. If then we write

$$w' = w_r \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix}, \qquad a = \operatorname{diag}(a_1, a_2, \ldots, a_r),$$

we see that our integral is a finite sum of *iterated* integrals of the form

$$(2) \quad \int_{A_r^m} f(a; e, e)\mu(a)W(w'a(w')^{-1})$$

$$\int_{F^{t_1}} \Phi[u_1a_1, \ldots, u_{t_1}a_{t_1}, 0, 0, \ldots, a_r]\psi[u_1]du_1du_2 \cdots du_{t_1}.$$

Here we may assume that Φ is of the form

$$\Phi = \Phi_1 \otimes \Phi_2 \otimes' \cdots \Phi_r.$$

Then evaluating the inner integral we obtain for (2) an integral of the form

(3) $$\int_{A_r^m} f(a: e, e) W(w'a(w')^{-1}) \Phi(a_1^{-1}, a_r) \rho(a) d^\times a,$$

with $\Phi \in \mathcal{S}(F^2)$, ρ a positive character. We need only prove that the integral (3) is absolutely convergent. As we have seen in (4.6), we have a majorization, for $a \in A_r$,

(4) $$|f(a; e, e)| \leq |a_{t_1+1} \cdots a_r a_1^{-1} \cdots a_{t_1}^{-1}|^u \eta(a),$$

where as before η is a finite sum of positive quasi-characters. Using (2.5) to estimate W, we find that the integral in question is dominated by an integral of the form

(5) $$\int \nu(a) |a_1 \cdots a_{t_1}|^{-u} |a_{t_1+1} \cdots a_r|^u \bigotimes_{i=1}^{r} d^\times a_i,$$

where the integration is extended over the domain defined by

$$1 \leq |a_1| \leq |a_2| \leq \cdots \leq |a_{t_1}|, |a_{t_1+1}| \leq |a_{t_1+2}| \leq \cdots \leq |a_r| \leq 1,$$

and where again ν is a positive quasi-character. It is clear finally that (5) is convergent for $u \gg 0$.

(4.8). *We show now that the identity* (4.7.1) *implies the relation* (4.4.1). As before, let f belong to the space of the representation $\mathrm{Ind}(G_r, Q; \sigma_1, \sigma_2)$ and set, for $h_i \in G_{t_i}$ $(i = 1, 2)$,

$$\tilde{f}[g; h_2, h_1] = f[w_r^t g^{-1}; h_1^\iota, h_2^\iota].$$

Since f takes values in $\mathcal{W}(\sigma_1; \bar{\psi}) \otimes \mathcal{W}(\sigma_2; \bar{\psi})$, $\tilde{f}$ takes values in $\mathcal{W}(\sigma_2^\iota; \psi) \otimes \mathcal{W}(\sigma_1^\iota; \psi)$. Moreover, if we set $Q^\iota = w^t Q w^{-1}$, then Q^ι is again a lower parabolic subgroup (of type (t_2, t_1)), and it is clear that $\tilde{f}$ belongs to the space of the representation $\mathrm{Ind}(G_r, Q^\iota; \sigma_2^\iota, \sigma_1^\iota)$. Note that $\sigma_2^\iota = \tau_2^\iota \otimes \alpha^{-u}$, $\sigma_1^\iota = \tau_1^\iota \otimes \alpha^u$. Finally we have

(1) $$\tilde{W}'(g) = W'(w_r^t g^{-1})$$

$$= \int f\left[\begin{pmatrix} 1_{t_1} & x \\ 0 & 1_{t_2} \end{pmatrix} w_r^t g^{-1}; e, e\right] \theta\left[\begin{pmatrix} 1_{t_1} & x \\ 0 & 1_{t_2} \end{pmatrix}\right] dx$$

$$= \int \tilde{f}\left[\begin{pmatrix} 1_{t_2} & x \\ 0 & 1_{t_1} \end{pmatrix} g; e, e\right] \bar{\theta}\left[\begin{pmatrix} 1_{t_2} & x \\ 0 & 1_{t_1} \end{pmatrix}\right] dx.$$

Thus, replacing (t_1, t_2) by (t_2, t_1), (σ_1, σ_2) by $(\sigma_2^\iota, \sigma_1^\iota)$, W by $\tilde{W}$, W' by $\tilde{W}'$, Φ by $\hat{\Phi}$, and ψ by $\bar{\psi}$ in (1), we obtain

(2) $L(1 - s, \pi^\iota \times \sigma_2^\iota)\epsilon(1 - s, \pi^\iota \times \sigma_1^\iota, \bar{\psi})\Psi(1 - s, \tilde{W}, \tilde{W}'; \hat{\Phi})$

$$= L(s, \pi \times \sigma_2)^{-1}\omega_{\sigma_2}(-1)^{r-1}$$

$$\int_{H_{t_2,t_1}\backslash G_r} \tilde{f}[g; e, e]\tilde{W}\left[w_r \begin{pmatrix} w_{t_2} & 0 \\ 0 & w_{t_1} \end{pmatrix} g\right]$$

$$\cdot |\det g|^{1-s}(g\hat{\Phi})^*[\zeta_{t_2}, \eta_{t_1}]dg,$$

where

$$\Phi^*(\eta, w) = \int_{F^{t_2}} \Phi[v, w]\bar{\psi}(\eta\, {}^t v)dv.$$

Next, as a simple consequence of Fourier inversion, we obtain the identity

(3) $$(w_r^t g^{-1}\hat{\Phi})^*[\zeta_{t_2}, \eta_{t_1}]\,|\det g|^{-1} = (g\Phi)^{\sim}[\zeta_{t_1}, \eta_{t_2}].$$

Thus, replacing g by $w_r^t g^{-1}$ in the integral in the right side of (2), we obtain:

(4)

$$\int_{H_{t_1,t_2}\backslash G} f[g; e, e]W\left[w_r \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix} g\right] |\det g|^s (g\Phi)^{\sim}[\zeta_{t_1}, \eta_{t_2}]dg,$$

an expression identical with "the integral" on the right side of (4.7.1). Finally, from (2.9.2) we have

(5) $$\epsilon(1 - s, \pi^\iota \times \sigma_2^\iota, \bar{\psi})\epsilon(s, \pi \times \sigma_2, \psi) = 1.$$

Thus, combining (4.7.1) and (2) noting that $\omega_\tau = \omega_{\tau_1}\cdot\omega_{\sigma_2}$, we indeed obtain the relation (4.4.1).

(4.9). It remains to prove the relation (4.7.1). We start with the identity (4.6.7). Following our conventions (c.f. (4.1)), we deduce the following identity:

$$(1)\qquad L(s, \pi \times \sigma_1)^{-1}\epsilon(s, \pi \times \sigma_1, \psi)\Psi(s, W, W'; \Phi)$$

$$= \oint_{G^0_{t_1}N_{t_1,t_2}\backslash G^0_r} \Phi(\eta h)dhL(s, \pi \times \sigma_1)^{-1}\epsilon(s, \pi \times \sigma_1, \psi)$$

$$\int_{N_{t_1}\backslash G_{t_1}} W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right] f[h; a, e]\,|\det a|^{s-t_2/2} d^\times a.$$

To proceed further, we use the following integration formula:

$$(2)\qquad \int_{G^0_{t_1}N_{t_1,t_2}\backslash G^0_r} F(h)dh = \int_{L\backslash G^0_r} dh \int F\left[\begin{pmatrix} 1_{t_1} & 0 \\ z & 1_{t_2} \end{pmatrix} h\right] dz,$$

where L is the (unimodular) group of matrices of the form

$$\begin{pmatrix} g_1 & 0 \\ z & n_2 \end{pmatrix}; \qquad g_1 \in G^0_{t_1}, \qquad n_2 \in N_{t_2}.$$

Then, writing

$$z = \begin{bmatrix} y \\ x \end{bmatrix}, \qquad y \in M(t_2 - 1 \times t_1), \qquad x \in F^{t_1},$$

we get for (1):

(3)

$$\oint_{L\backslash G^0_r} dh \oint dx\, \Phi[(x, \eta_{t_2})h] \oint dy\cdot L(s, \pi \times \sigma_1)^{-1}\epsilon(s, \pi \times \sigma_1, \psi)$$

$$\int_{N_{t_1}\backslash G_{t_1}} W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{t_1} & 0 & 0 \\ y & 1_{t_2-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{t_1} & 0 & 0 \\ 0 & 1_{t_2-1} & 0 \\ x & 0 & 1 \end{pmatrix} h\right]$$

$$\cdot f[h; a, e] \,|\det a|^{s-t_2/2} d^\times a.$$

Now we apply the functional equation (2.7.2) for the pair (π, σ_1) with $j = t_2 - 1$. Then $k = r - t_1 - t_2 = 0$, and we get after changing a to $w_{t_1}{}^t a^{-1}$,

$$\text{(4)} \qquad \oint_{L\backslash G_r^0} dh \oint dx \Phi[(x, \eta_{t_2})h]\omega_{\sigma_1}(-1)^{r-1} L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1}$$

$$\int_{N_{t_1}\backslash G_{t_1}} W\left[w_r\begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1_{t_2} \end{pmatrix}\begin{pmatrix} 1_{t_1} & 0 & 0 \\ 0 & 1_{t_2-1} & 0 \\ x & 0 & 1 \end{pmatrix} h\right]$$

$$\cdot f[h; a, e] \,|\det a|^{s-1+t_2/2} d^\times a.$$

Next, we note that, for h fixed, the function

$$\Phi[(x, \eta_{t_2})h] W\left[w_r\begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1_{t_1} & 0 & 0 \\ 0 & 1_{t_2-1} & 0 \\ x & 0 & 1 \end{pmatrix} h\right]$$

of (x, a) has compact support—for $x \in F^{t_1}$ and $a \in G_{t_1}^m$ (taken modulo N_{t_1}). Thus, we may interchange multiplication by the polynomial $L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1}$ and integration with respect to dx. Then, if we replace x by xa, we obtain

(5)

$$\omega_{\sigma_1}(-1)^{r-1} \oint_{L\backslash G_r^0} dh L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1} \int_{N_{t_1}\backslash G_{t_1}} f[h; a, e] \,|\det a|^{s+t_2/2}$$

$$W\left[w_r\begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix}\begin{pmatrix} 1_{r-1} & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right] d^\times a \int \Phi[(xa, \eta_{t_2})h] dx.$$

Next with

$$w' = w_r \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix},$$

we have

(6) $$\theta\left[w'\begin{pmatrix} 1_{r-1} & 0 \\ x & 1 \end{pmatrix}(w')^{-1}\right] = \psi(\zeta_{t_1}{}^t x).$$

Thus, our integral is

(7) $$\omega_{\sigma_1}(-1)^{r-1} \oint_{L\backslash G_r^0} dh L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1}$$

$$\int_{N_{t_1}\backslash G_{t_1}} f\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h;\, e,\, e\right] |\det a|^{s+t_2}$$

$$\cdot W\left[w_r \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{t_2} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right]\left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\Phi\right\}^{\sim} [\zeta_{t_1}, \eta_{t_2}] d^\times a.$$

Finally, we make use of the formula

(8)

$$\int_{H_{t_1,t_2}\backslash G_r} F(g)dg = \int_{L\backslash G_r^0} dh \int_{N_{t_1}\backslash G_{t_1}} F\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h\right] |\det a|^{t_2} d^\times a.$$

Since we have already proved the convergence of the integral in (4.7.1), we may then interchange multiplication by $L(1-s, \pi^\iota \times \sigma^\iota)^{-1}$ and integration over $L\backslash G_r^0$ in integral (7). Then using formula (8), we arrive directly at (4.7.1). The proof of (4.4.1) is now complete.

(4.10). We now consider *the proof of Theorem* (3.1) *in the case* $r > t$. It suffices then to deduce the *formal* identity

(1) $L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1} L(1-s, \pi^\iota \times \sigma_2^\iota)^{-1} \Psi(1-s, \tilde{W}, \tilde{W}'; k)$

$$= \omega_\sigma(-1)^{r-1}\epsilon(s, \pi \times \sigma_1, \psi)\epsilon(s, \pi \times \sigma_2, \psi)L(s, \pi \times \sigma_1)^{-1}$$

$$\cdot L(s, \pi \times \sigma_2)^{-1}\Psi(s, W, W'; j),$$

where as before σ is a representation of G_t and π a representation of G_r. The proof of this identity is entirely analogous to the case $r = t$. Let us consider for example the case $k = 0$. Then exploiting the functional equation for the pair (π, σ_1) one first deduces the following identity between formal Laurent series:

(2)

$$L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1}\omega_{\sigma_1}(-1)^{r-1}\int_{H_{t_1,t_2}\backslash G_t} dg \int dydxf(g;e,e)\,|\det g|^{s-(r-t)/2+j}$$

$$W\left[w_r \begin{pmatrix} 1_{t_1} & x & 0 & 0 & 0 \\ 0 & 1_k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1_j & y \\ 0 & 0 & 0 & 0 & 1_{t_2} \end{pmatrix} \begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{r-t_1} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right].$$

Proceeding as in (4.8), replacing the pair (σ_1, σ_2) by the pair $(\sigma_2^\iota, \sigma_1^\iota)$, we get immediately from (2) the identity

(3) $$L(1-s, \pi^\iota \times \sigma_2^\iota)^{-1}\epsilon(1-s, \pi^\iota \times \sigma_2^\iota, \bar{\psi})\Psi(1-s, \rho(w_{r,t})\tilde{W}, \tilde{W}'; k)$$

$$= L(s, \pi \times \sigma_2)^{-1}\cdot\omega_{\sigma_2}(-1)^{r-1}\int_{H_{t_2,t_1}\backslash G_t} dg$$

$$\cdot \int dydx\tilde{f}(g; e, e)\,|\det g|^{(1-s)-(r-t)/2+k}$$

$$\tilde{W}\left[w_r \begin{pmatrix} 1_{t_2} & x & 0 & 0 & 0 \\ 0 & 1_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1_k & y \\ 0 & 0 & 0 & 0 & 1_{t_1} \end{pmatrix} \begin{pmatrix} w_{t_2} & 0 \\ 0 & w_{r-t_2} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w_{r-t} \end{pmatrix}\right].$$

To conclude the proof of Theorem (3.1) in this case, we need only verify the equality of the integrals in (2) and (3). This follows directly from the relation $\tilde{f}(g; e, e) = f(w_t^t g^{-1}; e, e)$, the identity (2.9.2) and the identity

$$\begin{pmatrix} w_{t_2} & 0 \\ 0 & w_{r-t_2} \end{pmatrix}\begin{pmatrix} w_t & 0 \\ 0 & w_{r-t} \end{pmatrix} = w_r\begin{pmatrix} w_{t_1} & 0 \\ 0 & w_{r-t_1} \end{pmatrix}.$$

5. The case $r = 1$.

(5.1). We complete *the proof of Theorem* (3.1) *when* $r = 1$. Then π is just a quasi-character of $GL_1 \simeq F^\times$, and we may derive without difficulty from the definitions that $L(s, \pi \times \sigma) = L(s, (\sigma \otimes \pi) \times 1)$ and

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, (\sigma \otimes \pi) \times 1, \psi),$$

σ as before being a representation of G_t. We shall write simply $L(s, \sigma)$ and $\epsilon(s, \sigma, \psi)$ for $L(s, \sigma \times 1)$ and $\epsilon(s, \sigma \times 1, \psi)$. In (5.2) we complete certain results in [J-P-S I]. The proof will then be reduced to the analogous known assertions for the Zeta-integrals introduced in [R.G.-J] (c.f. (5.3), (5.4) below).

(5.2). As before σ is a representation of G_t of Whittaker type. We also assume that σ acts on $\mathcal{W}(\sigma; \psi)$. In addition to the integrals

(1) $$\Psi(s, W) = \int_{F^\times} W\left[\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right] |a|^{s-(t-1)/2} d^\times a$$

we consider as well the integrals

(2) $$Z(\Phi, s, f) = \int_{G_t} \Phi(g) f(g) |\det g|^s dg,$$

where Φ is a Schwartz-Bruhat function on $M(t \times t, F)$, $f \in \mathfrak{A}(\sigma)$, the space of bi-$K$-finite coefficients of the representation σ, and the integrals

(3) $$Z(\Phi, s, W) = \int_{G_t} \Phi(g) W(g) |\det g|^s dg,$$

with Φ as above and $W \in \mathcal{W}(\sigma; \psi)$. These integrals converge for Re(s) large, define rational functions of $X = q^{-s}$ and admit a "g.c.d.". Let $I_1 = I_1(\sigma)$ be the (fractional) ideal of $\mathbf{C}[X, X^{-1}]$ generated by the integrals (1), $I_2 = I_2(\sigma)$ the ideal generated by the integrals $Z(\Phi, s + (t - 1)/2, f)$ and $I_3 = I_3(\sigma)$ that generated by the integrals $Z(\Phi, s + (t - 1)/2, W)$. *We prove now that* $I_1 = I_2$.

We have seen in [J-P-S I] that $I_1 = I_3$ (Theorem (4.3) and Remark (4.2)) and also that $I_3 \subset I_2$ (loc. cit. pg. 186). We prove here that $I_2 \subset I_3$. If Ω is an open compact subgroup of G_t and $h \in G_t$ then the linear form μ on $\mathcal{W}(\sigma; \psi)$ defined by

$$\mu(W) = \int_\Omega W(h\omega)d\omega$$

is evidently Ω-invariant and therefore K-finite. If W is annihilated by all these forms then evidently $W = 0$; equivalently they span the vector space of K-finite linear forms on $\mathcal{W}(\sigma; \psi)$. Thus any element of $\mathfrak{A}(\sigma)$ is a finite sum of functions of the form $f(g) = \mu(\sigma(g)W) = \int W(h\omega g)d\omega$. In that case we have the identity

$$Z(\Phi, s, f) = \int_{G_t} \Phi(g)|\det g|^s dg \int_\Omega W(h\omega g)d\omega,$$

and, if Re(s) is large, we may change variables to obtain

$$Z(\Phi, s, f) = \int W_1(g)\Phi_1(g)|\det g|^s dg,$$

where $W_1(g) = W(gh)$, and

$$\Phi_1(x) = \int_\Omega \Phi(\omega h^{-1}xh)d\omega.$$

As W_1 is in $\mathcal{W}(\sigma; \psi)$ and Φ_1 in $\mathcal{S}(r \times r, F)$ we see that $Z(\Phi, s + (t - 1)/2, f)$ defines an element of I_3. Thus $I_2 \subset I_3$ and $I_1 = I_2$ as claimed.

(5.3). We now turn to *the proof of assertion* (ii) *of Theorem* (3.1) *in the case* $r = 1$. We may assume that $p = 2$. (c.f. Section 3). Thus *let* σ_i *be a representation of* G_{t_i} *of Whittaker type* and, with $t = t_1 + t_2$, set

$$\sigma = \mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2).$$

In addition let σ_2' the representation of G_{t_i} (by right translations) on $\mathcal{W}(\sigma_i; \psi)$ and set

$$\sigma' = \mathrm{Ind}(G_t, Q; \sigma_1', \sigma_2').$$

Since σ_i' is a quotient of σ_i, σ' is a quotient of σ. Moreover, by Rodier's theorem (Section (2.3)), σ' is of Whittaker type and hence $\mathcal{W}(\sigma; \psi) = \mathcal{W}(\sigma'; \psi)$. Thus we may assume $\sigma_1 = \sigma_1'$, $\sigma_2 = \sigma_2'$. Next we apply (5.2) successively with t_1 and then t_2 equal to zero. We see then that $L(s, \sigma_i)$ generates the ideal $I_2(\sigma_i)$. We may now apply Theorem (3.4) of [R.G.-J] to deduce that $L(s, \sigma_1)L(s, \sigma_2)$ generates $I_2(\sigma)$. Finally let σ_0 be the representation of G_t on $\mathcal{W}(\sigma; \psi)$. Since σ_0 is a quotient of σ, we have $\mathcal{A}(\sigma_0) \subseteq \mathcal{A}(\sigma)$ and therefore $I_2(\sigma) \supseteq I_2(\sigma_0)$. But by (5.2) applied to σ_0, $I_2(\sigma_0)$ contains $L(s, \sigma_0) = L(s, \sigma)$. Thus $L(s, \sigma)$ belongs to $I_2(\sigma)$ and is divisible by $L(s, \sigma_1)L(s, \sigma_2)$ as claimed.

(5.4). We turn to the proof of the first assertion of Theorem (3.1) ($r = 1$). We let

$$\sigma = \mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2).$$

Then we have an identity of the form

$$\Psi(1 - s, \rho(w_{t,1})\tilde{W}; k) = \gamma(s, \sigma, \psi)\Psi(s, W; j), \tag{1}$$

$j + k = t - 2$ and $W \in \mathcal{W}(\sigma; \psi)$. We also have a similar statement for the representations σ_1 and σ_2. In case σ is irreducible not only is there a relation of this form but there is also a related functional equation:

$$Z(\hat{\Phi}, 1 - s + (t - 1)/2, f^{\iota}) = \gamma(s, \sigma, \psi)Z(\Phi, s + (t - 1)/2, f), \tag{2}$$

here $f \in \mathcal{A}(\sigma)$, $f^{\iota} \in \mathcal{A}(\sigma^{\iota})$ is the function defined by $f^{\iota}(g) = f({}^t g^{-1})$ and $\hat{\Phi}$ is the Fourier-transform of Φ:

$$\hat{\Phi}(x) = \int \Phi(y)\psi(\mathrm{tr}(y^t x))dy.$$

The equality for irreducible representations of the γ-factors in (1) and (2), is, except for notation, the statement of Theorem (4.5) of [J-P-S].

Let σ' be the representation of G_t on $\mathcal{W}(\sigma; \psi)$. Clearly $\gamma(s, \sigma, \psi) = \gamma(s, \sigma', \psi)$. In general the action of G_t on $\mathcal{W}(\sigma; \psi)$ is not irreducible. However, the exactness of the functor $\pi \to \pi_\theta^*$ implies at once the existence of a unique minimal invariant subspace. Let σ_0 be the corresponding representation. Since σ_0 is realized on a subspace of $\mathcal{W}(\sigma; \psi)$, σ_0 is generic and we have at once

$$\gamma(s, \sigma_0, \psi) = \gamma(s, \sigma, \psi). \tag{3}$$

Similarly $(\sigma_i)_0$ is a component of σ_i and, since σ_0 is the unique generic constituent of σ, σ_0 must be a constituent of the representation

$$\mathrm{Ind}(G_t, Q; (\sigma_1)_0, (\sigma_2)_0). \tag{4}$$

Since we may now apply Theorem (3.4) of [R.G.-J], we have

$$\begin{aligned}(5)\quad & Z(\hat{\Phi}, 1 - s + (t - 1)/2, f^\iota) \\ &= \gamma(s, (\sigma_1)_0, \psi)\gamma(s, (\sigma_2)_0, \psi)Z(\Phi, s + (t - 1)/2, f)\end{aligned}$$

for all coefficients f of the representation (4), in particular for $f \in \mathcal{A}(\sigma_0)$. Thus by (3) (applied twice)

$$\gamma(s, \sigma, \psi) = \gamma(s, (\sigma_1)_0, \psi)\gamma(s, (\sigma_2)_0, \psi) = \gamma(s, \sigma_1, \psi)\gamma(s, \sigma_2, \psi)$$

as claimed. The proof of Theorem (3.1) in the case $r = 1$ is now complete.

(5.5). The following lemma will be used in Section 7 as a step in the proof of Theorem (3.1)(ii).

LEMMA. *Let the notation be as in* (3.1). *Suppose in addition that* $t_2 = 1$ *and that* $r \geq t + 1$ $(t = t_1 + t_2)$. *Then with* σ *as the induced representation*

$$\sigma = \mathrm{Ind}(G_t, Q; \sigma_1, \sigma_2),$$

we have

$$L(s, \pi \times \sigma_1) = L(s, \pi \times \sigma)P(q^{-s})$$

where P *is a polynomial.*

Proof. Let λ be the (essentially) unique element of $(\sigma_\theta)^*$. As in Proposition (3.2) of [F.S. I], we have an explicit formula for λ as follows. Let $\mathcal{U}$ denote the space of the induced representation σ and $\mathcal{V}$ the space of $\sigma_1 \otimes \sigma_2$. Then if μ is a non-zero Whittaker functional for $\sigma_1 \otimes \sigma_2$, we may take for λ the limit

$$(1) \qquad \lambda(f) = \lim_{\nu \to \infty} \int_{V^\nu} \mu(f(v))\bar{\theta}(v)dv,$$

a limit which exists for all $f \in \mathcal{U}$ (c.f. (4.5)). Then if we set $W_f(g) = \lambda(\sigma(g)f)$, the map $f \to W_f$ is a surjection from $\mathcal{U}$ to $\mathcal{W}(\sigma; \psi)$. Note that since $t_2 = 1$, we may regard f as taking values in $\mathcal{W}(\sigma_1; \psi)$; we have then $\mu(f(v)) = f(v; e)$. Recall that $\bigcup_\nu V^\nu = \bar{U}_Q$.

This being so we have for Re(s) large the identity

$$(2) \qquad \Psi(s, W, W_f; 0) = \int_{N_t \backslash G_t} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W_f(g) |\det g|^{s-(r-t)/2} dg;$$

and, if we formally substitute

$$(3) \qquad W_f(g) = \int_{\bar{U}_Q} f(vg; e)\bar{\theta}(v)dv,$$

noting that $\bar{U}_Q \simeq N_{t_1,t_2} \backslash N_t$, we find the relation

$$(4) \qquad \Psi(s, W, W_f; 0) = \int_{N_{t_1,t_2} \backslash G_t} W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] f(g; e) |\det g|^{s-(r-t)/2} dg.$$

In what follows we shall choose $W \in \mathcal{W}(\pi; \psi)$ and f in $\mathcal{U}$ so that the integral in (4) is absolutely convergent (for Re(s) large). We may then apply Fubini's Theorem to deduce that the integral in (3) is convergent, at first for almost all g in the support of W but then, since f is locally constant, for all such g. The identity (3) and hence (4) will then follow from Lebesque's Theorem.

We prove first that given $W_1 \in \mathcal{W}(\sigma_1; \psi)$ there exists a function $\phi \in C_c^\infty(\bar{U}_Q)$ and an element f in $\mathcal{U}$ so that

(5) $$\int_{\bar{U}_Q} \phi(v)dv = 1$$

and

(6) $$f(u; h) = W_1(h)\phi(u)$$

for all $h \in G_{t_1}$, $u \in \bar{U}_Q$. In fact if we write an element of Q in the form

$$q = \begin{pmatrix} 1_{t_1} & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & a \end{pmatrix}; \qquad m \in G_{t_1}, \qquad a \in F^\times,$$

then for an arbitrary $F \in C_c^\infty(G_t) \otimes \mathcal{W}(\sigma_1, \psi)$ the function f on $G_t \times G_{t_1}$ defined by

$$f(g; h) = \int_Q \sigma_2^{-1}(a)\delta_Q^{-1/2}(q)F(qg; hm^{-1})d_r(q)$$

is in $\mathcal{U}$. We choose F supported in $Q\bar{U}_Q$ of the form

$$F = \beta \otimes \phi \otimes W_1$$

where $\beta \in C_c^\infty(Q)$, $\phi \in C_c^\infty(\bar{U}_Q)$ satisfies (5) and W_1 is as given. Then we see that for $u \in \bar{U}_Q$,

$$f(u; h) = \phi(u) \int \sigma_2^{-1}(a)|a|^{-1/2}|\det m|^{-1/2}\beta(q)W_1(hm^{-1})dyd^\times adm;$$

and, since W_1 is right K-finite, this reduces to (6) for an appropriate choice of β.

Next, since we are still free to choose the support of ϕ, given $W \in \mathcal{W}(\pi; \psi)$ we may assume that

(7) $$W\left[g\begin{pmatrix} u & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] = W(g),$$

for all u in the support of ϕ.

If then we write

$$g = \begin{pmatrix} m & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{t_1} & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1_{t_1} & y \\ 0 & 1 \end{pmatrix},$$

where m varies in $N_{t-1}\backslash G_{t-1}$, we obtain for the integral in (4)

(8)

$$\int W\left[\begin{pmatrix} m & 0 & 0 \\ x & a & 0 \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right] W_1(m)\,|\det m|^{s-(r-t_1)/2}|a|^{s-(r-1)/2}\sigma_2(a)dmd^{\times}adx.$$

We shall prove momentarily that given $\epsilon > 0$ and $W_2 \in \mathcal{W}(\pi;\,\psi)$ there exists $W \in \mathcal{W}(\pi;\,\psi)$ so that

$$(9) \qquad W\left[\begin{pmatrix} m & 0 & 0 \\ x & a & 0 \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right] = W_2\left[\begin{pmatrix} m & 0 & 0 \\ x & a & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\sigma_2^{-1}(a)$$

if $|a - 1| < \epsilon$ and is zero otherwise. For such W the integral (8) converges. This implies that the integral in (4) also converges and that, moreover,

$$\Psi(s,\, W,\, W_f;\, 0) = \Psi(s,\, W_2,\, W_1;\, 1).$$

Since $W_1 \in \mathcal{W}(\sigma_1;\,\psi)$ and $W_2 \in \mathcal{W}(\pi;\,\psi)$ are arbitrary, we conclude that $L(s,\,\pi \times \sigma_1)$ belongs to the ideal generated by $L(s,\,\pi \times \sigma)$.

Returning to (9), let ϕ_1 be the element of $\mathcal{S}(F)$ defined by $\hat{\phi}_1(a) = \sigma_2^{-1}(a)$ if $|a - 1| < \epsilon$ and equal to zero otherwise. Let

$$W(g) = \int \phi_1(u)W_2\left[g\begin{pmatrix} 1_{t-1} & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right]du.$$

Then

$$W\left[\begin{pmatrix} m & 0 & 0 \\ x & a & 0 \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right] = W_2\left[\begin{pmatrix} m & 0 & 0 \\ x & a & 0 \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right]\hat{\phi}_1(a),$$

whence the relation (9) and the lemma.

6. Induced representations. 2nd step.

(6.1). We continue now with the proof of Theorem (3.1). As before, we may assume $p = 2$. We treat here the case $t = r + 1$, $t_1 = r$, $t_2 = 1$. As in Section 4 the idea of the proof is to establish the formula

$$\text{(1)}\quad L(1 - s, \pi^\iota \times \sigma_1^\iota)^{-1} L(1 - s, \pi^\iota \times \sigma_2^\iota)^{-1}\Psi(1 - s, \tilde{W}, \tilde{W}')$$
$$= \omega_\pi(-1)^r \epsilon(s, \pi \times \sigma_1, \psi)\epsilon(s, \pi \times \sigma_2, \psi)L(s, \pi \times \sigma_1)^{-1}$$
$$\cdot L(s, \pi \times \sigma_2)^{-1}\Psi(s, W, W'),$$

where σ is a representation of G_t, π a representation of G_r, $W \in \mathcal{W}(\sigma; \psi)$, $W' \in \mathcal{W}(\pi; \bar{\psi})$, and where $\sigma = \operatorname{Ind}(G_t, Q; \sigma_1, \sigma_2)$. As always we may suppose that σ (resp. σ_i, $i = 1, 2$) operates by right translations on $\mathcal{W}(\sigma; \psi)$ (resp. $\mathcal{W}(\sigma_i; \psi)$). Again (1) is to be regarded as an equality of formal Laurent series and, as in (4.5), we may write $\sigma_2 = \sigma_0 \otimes \alpha^u$ where σ_0 is fixed (and say unitary) and restrict ourselves to u large and positive.

The proof depends on an integral representation for the elements of $\mathcal{W}(\sigma; \psi)$ which we now describe.

We have a natural action of G_t on $M(t - 1 \times t)$ given by $m \to mg^{-1}(m \in M(t - 1 \times t), g \in G_t)$ and the dual action on $\mathcal{S}(t - 1 \times t, F)$. Thus $(g\Phi)(m) = \Phi(mg)$ for $\Phi \in \mathcal{S}(t - 1 \times t, F)$.

We will often write $m = [x, y]$ where x is the $(t - 1) \times (t - 1)$ matrix whose columns are the first $t - 1$ columns of x and y is the last column of m. For Φ in $\mathcal{S}(t - 1 \times t, F)$ we set

$$\hat{\Phi}[x, y] = \int_{F^{t-1}} \Phi[x, u]\bar{\psi}[{}^t yu]du.$$

We may twist the natural action of G_t on $\mathcal{S}(t - 1 \times t, F)$ by this partial Fourier transform to obtain a new action which we denote by ρ_0. Thus:

$$\rho_0(g)(\Phi^{\sim}) = (g\Phi)^{\sim}. \tag{2}$$

We will use the same notation for vector-valued functions. One easily deduces then the following formulas:

$$\text{if} \quad g = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \qquad h \in G_{t-1}, \qquad \text{then} \quad \rho_0(g)\Phi = g\Phi; \tag{3}$$

$$\text{if} \quad u = \begin{pmatrix} 1_{t-1} & v \\ 0 & 1 \end{pmatrix} \tag{4}$$

and $m = [x, y]$ as above, then

$$\rho_0(u)\Phi(m) = \Phi(m)\psi({}^t yxv);$$

$$\text{if } g = \operatorname{diag}(1, 1, \ldots, a), \text{ then} \tag{5}$$

$$\rho_0(g)\Phi = |a|^{-(t-1)} g^{-1}\Phi.$$

Finally, let $\Phi^\wedge$ denote the usual Fourier transform of Φ:

$$\hat{\Phi}(x) = \int \Phi(y)\psi(\operatorname{tr}(y^t x))dy. \tag{6}$$

Then

$$(g\Phi)^\wedge = |\det g|^{-(t-1)} g^\iota(\Phi^\wedge).$$

Using the fact that $(\Phi^{\sim})^\wedge = (\Phi^\wedge)^{\sim}$, it easily follows that

$$(\rho_0(g)\Phi)^\wedge = |\det g|^{-(t-1)} \rho_0(g^\iota)(\Phi^\wedge). \tag{7}$$

We also let ϵ be the transpose of the row vector $\eta = (0, 0, \ldots, 0, 1)$ of length $t - 1$. With this notation we have:

PROPOSITION. *Let Ψ belong to $\mathcal{S}(t - 1 \times t; F) \otimes \mathcal{W}(\sigma_1; \psi)$, the space of Schwartz-functions with values in $\mathcal{W}(\sigma_1; \psi)$. Then the integral*

(8) $W_{\Psi,\sigma_2}(g)$

$$= \sigma_2\alpha^{(t-1)/2}(\det g)\int_{G_{t-1}}(\rho_0(g)\Psi)[h,\, h^t\epsilon;\, h^{-1}]\sigma_2\alpha^{t/2-1}(\det h)dh$$

converges. If $u \gg 0$, it represents an arbitrary element of $\mathcal{W}(\sigma;\, \psi)$.

Proof. It will be sufficient to consider special Ψ of the form $\Psi = \Psi_1 \otimes \Psi_2 \otimes W$, where $\Psi_1 \in \mathcal{S}(t-1 \times t-1, F)$, $\Psi_2 \in \mathcal{S}(F^{t-1})$ and $W \in \mathcal{W}(\sigma_1;\, \Psi)$. For the proof of convergence we may also assume $g = e$. Then since Ψ_1, Ψ_2, and W are right K-finite, we may use the Iwasawa decomposition to write the integral (8) as a finite sum of integrals of the form

$$\int_{A_{t-1}}\int_{N_{t-1}}\Psi_1(an)\Psi_2(a^t n^t \epsilon)\sigma_2\alpha^{t/2-1}\det(a)\delta_r(a)W(a^{-1})\bar{\theta}(n)dnd^\times a. \tag{9}$$

Write $a = \operatorname{diag}(a_1, a_2, \ldots, a_{t-1})$. Then if a belongs to the support of the integrand in (9), W being dominated by a gauge, we have on the one hand

$$|a_1|^{-1} \lesssim |a_2|^{-1} \lesssim \cdots \lesssim |a_{t-1}|^{-1} \lesssim 1,$$

and on the other hand, since an belongs to a compact subset of $M(t-1 \times t-1, F)$, we have

$$|a_1| \lesssim 1, \qquad |a_2| \lesssim 1, \qquad |a_{t-1}| \lesssim 1.$$

Thus $|a_i| \asymp 1$ and n also varies in a compact set. The convergence of (9) and therefore (8) is now clear.

For the second part let μ_0 denote the element of $M(t-1 \times t,\, F)$ defined by

$$\mu_0 = \begin{pmatrix} 1 & 0 & 0 \\ \cdot & & 0 \\ & \cdot & \cdot \\ & \cdot & \cdot \\ 0 & 1 & 0 \end{pmatrix}.$$

The G_t-orbit $\mu_0 G_t$ through μ_0 is open. Thus, if we denote by V the isotropy group at μ_0, this orbit is equivalent to $V\backslash G_t$ as a G_t-space. Moreover, if $\pi : G_t \to V\backslash G_t$ is the canonical projection then $\pi | G_{t-1}$ is a homeomorphism.

Next let Φ by any element of $\mathcal{S}(t-1 \times t, F)$ with support in $\mu_0 G_t$. Let also W_1 be an element of $\mathcal{W}(\sigma_1; \psi)$. Then with $g \in G_t$, $m \in G_{t-1}$, the integral

$$(10) \qquad f(g; m) = \alpha^{(t-1)/2}(\det g)\sigma_2(\det g) \cdot \int_{G_{t-1}} \Phi(h\mu_0 g)W_1(mh^{-1})\sigma_2\alpha^{t/2}(\det h)dh$$

is convergent. In fact, replacing Φ by a right translate we may assume $g = e$. Then in the integral $h\mu_0$ may be restricted to a compact set in $\mu_0 G_t \simeq V\backslash G_t$ and from our earlier remarks it follows that h is restricted to a compact in G_t. A simple calculation shows that

$$(11) \qquad f\left[\begin{pmatrix} a & 0 \\ u & b \end{pmatrix} g; m\right] = \sigma_2\alpha^{(t-1)/2}(b)\alpha^{-1/2}(\det a)f[g; ma],$$

$a \in G_{t-1}$, $b \in F^\times$, or in other words f defines an element of the induced representation $\sigma = \mathrm{Ind}(G, Q; \sigma_1, \sigma_2)$.

Next set

$$(12) \qquad W_f(g) = \int_{F^{t-1}} f\left[\begin{pmatrix} 1_{t-1} & x \\ 0 & 1 \end{pmatrix} g; e\right] \bar{\theta}\left[\begin{pmatrix} 1_{t-1} & x \\ 0 & 1 \end{pmatrix}\right] dx$$

and formally substitute for f its expression (10) to get

$$(13) \quad W_f(g) = \alpha^{(t-1)/2}(\det g)\sigma_2(\det g) \int_{F^{t-1}} \bar{\psi}(\eta x) \int_{G_{t-1}} (g\Phi)[h, hx]W_1(h^{-1})\sigma_2\alpha^{t/2}(\det h)dhdx.$$

Now if u is sufficiently large the double integral in (13) is absolutely convergent. In fact, if W_1 is majorized by a gauge ξ, then after taking $g = e$ and changing x to $h^{-1}x$, we are reduced to proving the convergence of the integral

$$\int_{G_{t-1}} \Phi(h)\xi(h^{-1})\,|\det h|^{t/2-1+u}dh$$

for $u \gg 0$, and Φ in $\mathcal{S}(t-1 \times t-1, F)$, an easy exercise left to the reader. Thus the integral in (13) is convergent, and it is clear from (12) that the function $W_f(g)$ it defines belongs to $\mathcal{W}(\sigma; \psi)$. Moreover, if we change $x \to h^{-1}x$ in the integral in (13) and integrate first over F^{t-1}, we obtain the integral in (8) with $\Psi = \Phi^{\sim} \otimes W_1$. Since on the one hand an arbitrary Ψ is a finite sum of such pure tensors, and on the other, the functions f given by (10) span the space of σ, our conclusion follows.

(6.2). The functional equation (6.1.1) will in part be deduced from the following formula, stated formally below as a Proposition.

We define a map $\Phi \to \Phi^*$ of $\mathcal{S}(t-1 \times t, F) \otimes \mathcal{W}(\sigma_1, \bar{\psi}) \to \mathcal{S}(t-1 \times t, F) \otimes \mathcal{W}(\tilde{\sigma}_1, \psi)$ so that on pure tensors we have $(\phi \otimes W_1)^* = \hat{\phi} \otimes \tilde{W}_1$; $\phi \in \mathcal{S}(t-1 \times t, F)$, $W_1 \in \mathcal{W}(\sigma_1, \bar{\psi})$. We also let $\zeta \in G_t$ be the diagonal matrix

$$\zeta = \mathrm{diag}(-1, -1, \ldots, -1, 1).$$

With these notations:

PROPOSITION. *For all $\Phi \in \mathcal{S}(t-1 \times t, F) \otimes \mathcal{W}(\sigma_1, \psi)$, $g \in G_t$, we have*

$$(W_{\Phi,\sigma_2})^{\sim}(g) = W_{\Phi^*,\sigma_2^{-1}}(\zeta g). \tag{1}$$

This identity has been established for G_3 in [J-S I]. In fact except for a change of notation it is exactly the same as the identity (4.5) in (loc. cit.). The proof in general is completely analogous. One need only substitute for Lemma (3.2) in loc. cit. the following generalization.

LEMMA. *Suppose Φ is in $\mathcal{S}(t-1 \times t, F)$. Then*

$$\int (w_t \zeta \phi)^{\sim} [n, \epsilon]\bar{\theta}(n)dn = \int (\tilde{\phi})^{\wedge}[w_{t-1}n, w_{t-1}\epsilon]\theta(n)dn. \tag{2}$$

Proof. We sketch a proof for the reader's convenience. If we take ϕ of the form $\phi[x, y] = \phi_1(x)\phi_2(y)$, then after a simple calculation we see that (2) reduces to the identity

$$\int \phi_1[x\zeta]\psi(\mathrm{tr}(w_{t-1}x))d\mu(x) = \int_{N_{t-1}} \hat{\phi}_1[w_{t-1}n]\theta(n)dn, \tag{3}$$

the first integral taken over the subspace of $M(t-1 \times t-1, F)$ consisting of all matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \cdots & x_{1t-1} \\ x_{21} & x_{22} & x_{23} \cdots & 1 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_{t-1,1} & 1 & 0 \cdots & 0 \end{pmatrix}$$

with respect to the canonical Lebesque measure $d\mu(x) = \otimes\, dx_{ij}$ on this subspace. The identity (3) follows directly from the Fourier inversion formula. □

(6.3). We turn now to the proof of (6.1.1). We take W in the space $\mathcal{W}(\sigma; \psi)$, where

$$\sigma = \operatorname{Ind}(G, Q; \sigma_1, \sigma_2),$$

and $W' \in \mathcal{W}(\pi; \bar{\psi})$. Without loss of generality we may assume $u \gg 0$ (c.f. (6.1)). By Proposition (6.1) we may suppose W is of the form W_{Ψ,σ_2} with $\Psi \in \mathcal{S}(t-1 \times t, F) \otimes \mathcal{W}(\sigma_1; \psi)$. Without loss of generality we may also assume that $\Psi = \phi \otimes W_1$ with $W_1 \in \mathcal{W}(\sigma_1; \psi)$ where $\phi \in \mathcal{S}(t-1 \times t; F)$ has the form

$$\phi[x, y] = \phi_1(x)\phi_2(y),$$

$\phi_1 \in \mathcal{S}(t-1 \times t-1, F)$, $\phi_2 \in \mathcal{S}(F^{t-1})$. We are also assuming $\sigma_2 = \sigma_0 \otimes \alpha^u$ with u large.

Substituting directly in (2.4.2) and using (6.1.3), we find then that

$$(1) \quad \Psi(s, W, W') = \int_{N_{t-1}\backslash G_{t-1}} W'(g)\sigma_2\alpha^{s+t/2-1}(\det g)dg$$

$$\int_{G_{t-1}} \phi_1(hg)\phi_2(h^{\iota}\epsilon)W_1(h^{-1})\sigma_2\alpha^{t/2-1}(\det h)dh.$$

As always we interpret (1) as an identity of Laurent series. Thus if we write $X = q^{-s}$ and accordingly

(2) $$\Psi(s, W, W') = \Sigma X^m \Psi_m(W, W'),$$

then we have the identity

(3) $$\Psi_m(W, W') = \int_{N_{t-1}\backslash G_{t-1}^m} W'(g)\sigma_2\alpha^{t/2-1}(\det g)dg \int_{G_{t-1}} \phi_1(hg)\phi_2(h^\iota \epsilon)W_1(h^{-1})\sigma_2\alpha^{t/2-1}(\det h)dh.$$

To proceed, in the inner integral change h to hg^{-1}. We get then the integral

(4)
$$\int_{N_{t-1}\backslash G_{t-1}^m} W'(g)dg \int_{G_{t-1}} \phi_1(h)\phi_2(h^\iota({}^tg)\epsilon)W_1(gh^{-1})\sigma_2\alpha^{t/2-1}(\det h)dh.$$

We shall prove later that the double integral in (4) is absolutely convergent. Taking that for granted, after changing g to gh in the double integral, we obtain, with χ_m the characteristic function of G_{t-1}^m the integral

(5)
$$\int_{N_{t-1}\backslash G_{t-1}} W_1(g)\phi_2({}^tg\epsilon)dg \int_{G_{t-1}} \chi_m(gh)W'(gh)\phi_1(h)\sigma_2\alpha^{t/2-1}(\det h)dh.$$

We now consider the inner integral in this expression:

(6) $$\int_{G_{t-1}} \chi_m(gh)W'(gh)\phi_1(h)\sigma_2\alpha^{t/2-1}(\det h)dh.$$

There is an open compact subgroup Ω of G_{t-1} such that ${}^t\Omega = \Omega$, $\phi_1(\omega h) = \phi_1(h)$, and $\sigma_2(\det \omega) = 1$ for $\omega \in \Omega$. Let $d\omega$ be the Haar measure on Ω normalized so that $\mathrm{vol}(\Omega) = 1$. Then replacing h by ωh in (5) and integrating over all of Ω, we find the equivalent expression

$$\int_{G_{t-1}} \chi_m(gh)\phi_1(h)\sigma_2\alpha^{t/2-1}(\det h)dh \int_\Omega W'(g\omega h)d\omega.$$

Next we consider the integral over Ω. Let λ be the linear functional on $\mathcal{W}(\pi; \bar{\psi})$ defined by setting

$$\lambda(W') = W'(e), \qquad W' \in \mathcal{W}(\pi; \bar{\psi}).$$

Let $\mathfrak{A}$ be the subspace of $\mathcal{W}(\bar{\pi}; \bar{\psi})$ consisting of all elements fixed by Ω. The operator $P = \int_\Omega \pi(\omega)d\omega$ is a projection onto this subspace:

$$P: \mathcal{W}(\pi; \bar{\psi}) \to \mathfrak{A}.$$

Since π is admissible $\mathfrak{A}$ is finite dimensional. Thus if $\{W_1', W_2', \ldots, W_p'\}$ is a basis of $\mathfrak{A}$, we have

$$(7) \quad \int_\Omega W'(g\omega h)d\omega = \int_\Omega \lambda(\pi(g)\pi(\omega)\pi(h)W')d\omega$$
$$= \lambda(\pi(g)P\pi(h)W') = \lambda(\pi(g) \Sigma f_j(h)W_j')$$
$$= \Sigma\, \lambda(\pi(g)W_j')f_j(h) = \sum_{1\le j\le p} W_j'(g)f_j(h),$$

where we have written

$$P\pi(h)W' = \Sigma f_j(h)W_j'.$$

The f_j are of course in $\mathfrak{A}(\pi)$. Thus (6) and hence (5) may be written as

$$\sum_j W_j'(g) \int_{G_{t-1}} \chi_m(gh)f_j(h)\phi_1(h)\sigma_2\alpha^{t/2-1}(\det h)dh.$$

Substituting this for the inner integral in (5), we arrive at the identity (of Laurent series)

$$(8) \quad \Psi(s, W, W') = \sum_j \sum_m X^m \int_{N_{t-1}\backslash G_{t-1}} \chi_m(gh)W_1(g)W_j'(g)\phi_2({}^tg\epsilon)dg$$
$$\int_{G_{t-1}} \phi_1(h)f_j(h)\sigma_2\alpha^{t/2-1}(\det h)dh.$$

Next write

(9)
$$\chi_m(gh) = \sum_{\mu+\nu=m} \chi_\mu(g)\chi_\nu(h).$$

Since ϕ_1 is compactly supported on $M(t-1 \times t-1, F)$, in the integrand in (8) we may suppose $|\det h| \leq 1$. Similarly since W_1, say, is dominated by a gauge and ϕ_2 has compact support, we may also assume $|\det g| \leq 1$. Thus when we substitute the expression (9) for χ_m in (8), we may effectively suppose that μ and ν are bounded from below. Referring then to (2.4.1) and (5.2.2), we find immediately the identity.

(10)
$$\Psi(s, W, W') = \sum_{1\leq j\leq p} \Psi(s, W_1, W_j'; {}^t\phi_2)Z(\phi_1, s+t/2-1, f_j \otimes \sigma_2).$$

where ${}^t\phi_2(x) = \phi_2({}^tx)(x \in F^{t-1})$, where $f_j \otimes \sigma_2(g) = f_j(g)\sigma_2(\det g)$, and again both sides are regarded as Laurent series (with finitely many non-zero negative coefficients).

To justify the steps we have taken thus far, we need only prove the absolute convergence of the integral (4) or equivalently of the integral

(11)
$$\int_{G_{t-1}} \phi_1(h)\sigma_2\alpha^{t/2-1}(\det h)dh \int_{N_{t-1}\backslash G_{t-1}} W'(gh)\phi_2({}^tg\epsilon)W_1(g)X_m(gh)dg.$$

For this we may replace W' and W_1 respectively by the gauges ξ' and ξ_1, assume that ϕ_1 and ϕ_2 are ≥ 0, and replace σ_2 by a positive quasi-character. If we use the Iwasawa decomposition for integrals in (11), we are then reduced to considering the integral

$$\int_{N_{r-1}} \int_{A_{r-1}} \int_{A_{r-1}} \chi_m(aa')\xi'(aa')\phi_1(na')\phi_2(a\epsilon)\xi_1(a)\mu'(a)\nu'(a')d^\times a d^\times a' dn,$$

where μ' and ν' are positive quasi-characters. Next we majorize ξ' by a sum of positive quasi-characters. We easily obtain then a majorization of the preceding integral by

$$I_m = \int_{A_{r-1}} \int_{A_{r-1}} \chi_m(aa')\mu'(a)\nu'(a')\Phi_1[a'_1, \ldots, a'_{t-1}]$$

$$\cdot \Phi_2(a_{t-1})\xi_1(a)d^\times a d^\times a',$$

where we have written

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_{t-1}), \qquad a' = \mathrm{diag}(a'_1, a'_2, \ldots, a'_{t-1}),$$

and where μ' and ν' are again (possibly different) quasi-characters. We shall prove more than we need, namely that the series $\Sigma I_m q^{-ms}$ is convergent for $s \gg 0$; in fact this series is the product of two integrals

$$\int |\det a|^s \mu(a)\Phi_2(a_{t-1})\xi_1(a)d^\times a \cdot \int |\det a'|^s \nu(a')\Phi_1[a'_1, \ldots, a'_{t-1}]d^\times a',$$

and both of these integrals are clearly convergent for s large.

We return to the proof of (6.1.1). By Theorem (2.7) each *integral* $\Psi(s, W_1, W'_j; {}^t\phi_2)$ is of the form $L(s, \sigma_1 \times \pi)P(q^{-s})$, P a polynomial; and, as we have seen in Section 5, each integral $Z(\phi_1, s + t/2 - 1, f_j)$ has the form $L(s, \pi \otimes \sigma_2)Q(q^{-s})$, Q a polynomial. It is a simple consequence whose derivation we leave to the reader that, in the sense of formal power series, both $L(s, \sigma_1 \times \pi)^{-1}\Psi(s, W_1, W'_j, {}^t\phi_2)$ and $L(s, \pi \otimes \sigma_2)^{-1} \cdot Z(\phi_1, s + t/2, f_j)$ are polynomials (in q^{-s}). In the same way (c.f. (2.7.1) and (5.4.2)), we have the two functional equations

$$(12) \quad L(1 - s, \sigma_1^\iota \times \pi^\iota)^{-1}\Psi(1 - s, \tilde{W}_1, (W'_j)^\sim; ({}^t\phi_2)^\wedge)$$

$$= \omega_\pi(-1)^{r-1}\epsilon(s, \sigma_1 \times \pi, \psi)L(s, \sigma_1 \times \pi)^{-1}\Psi(s, W_1, W'_j; {}^t\phi_2),$$

and

$$(13) \quad L(1 - s, \pi^\iota \otimes \sigma_2^{-1})^{-1}Z(\hat{\phi}_1, 1 - s + t/2 - 1, f_j^\iota)$$

$$= \epsilon(s, \pi \otimes \sigma_2, \psi)L(s, \pi \otimes \sigma_2)^{-1}Z(\phi_1, s + t/2 - 1, f_j).$$

The proof of the functional equation (6.1.1) is now an easy consequence of (10), (12), (13) and Proposition (6.2). To begin with recall that W is of the form W_{Ψ,σ_2} where $\Psi[x, y; m] = \phi_1(x)\phi_2(y)W_1(m)$. Then

$$\Psi^*[x, y; m] = \hat{\phi}_1(x)\hat{\phi}_2(y)\tilde{W}_1(m)$$

and by Proposition (6.2) we have, for $g \in G_{t-1}$,

$$\tilde{W}(g) = W_{\Psi^*,\sigma_2^{-1}}(\zeta g) = \sigma_2(\det \zeta)\sigma_2^{-1}\alpha^{(t-1)/2}(\det g)$$

$$\int_{G_{t-1}} \hat{\phi}_1[(-1_{t-1})hg]\hat{\phi}_2(h^\iota \epsilon)\tilde{W}_1(h^{-1})\sigma_2^{-1}\alpha^{t/2-1}(\det h)dh,$$

the last identity stemming from (6.1.3) and (6.1.8). If, for any $\phi \in \mathcal{S}(t-1 \times t-1, F)$, we set $\rho(-1)\,\phi(x) = \phi(-x)$, we see then that replacing W by $\tilde{W}$ amounts to replacing ϕ_1 by $\rho(-1)\hat{\phi}_1$, ϕ_2 by $\hat{\phi}_2$, W_1 by $\tilde{W}_1$, σ_2 by σ_2^{-1} and multiplying by the constant $\sigma_2(\det \zeta)$. Moreover, noting in (7) that $\Omega = {}^t\Omega$, we get at once

$$\int_\Omega (W')^\sim(g\omega h)d\omega = \int_\Omega W'(w_{t-1}{}^t g^{-1}\omega^t h^{-1})d\omega = \sum_{1\le j\le p} \tilde{W}_j'(g)f_j^\iota(h).$$

Thus replacing W by $\tilde{W}$, W' by $(W')^\sim$, and s by $1-s$ in (10), we find immediately the identity

$$(14)\quad \Psi(1-s, \tilde{W}, (W')^\sim)$$
$$= \sigma_2(\det \zeta) \sum_{1\le j\le p} \Psi(1-s, (W_1)^\sim, (W_j')^\sim; {}^t(\hat{\phi}_2))$$
$$\cdot Z(\rho(-1)\hat{\phi}_1, 1-s+t/2-1, f_j^\iota \otimes \sigma_2^{-1}).$$

We have trivially ${}^t(\hat{\phi}_2) = ({}^t\phi_2)^\wedge$ and $(f_j \otimes \sigma_2)^\iota = f_j^\iota \otimes \sigma_2^{-1}$. It is also easy to see that in general given π and $f \in \mathcal{A}(\pi)$ then

$$Z(\rho(-1)\Phi, s, f) = \omega_\pi(-1)Z(\Phi, s, f).$$

(c.f. (5.2)). If in this identity we replace π by $\tilde{\pi} \otimes \sigma_2^{-1}$, $\omega_\pi(-1)$ is replaced by $\omega_\pi(-1)\sigma_2(\det \zeta)$. We may now apply the identities (12) and (13). Thus multiplying both sides of (14) by $L(1-s, \sigma_1^\iota \times \pi^\iota)^{-1}L(1-s, \pi^\iota \otimes \sigma_2^{-1})^{-1}$ we get the identity

$$L(1-s, \pi^\iota \times \sigma_1^\iota)^{-1}L(1-s, \pi^\iota \times \sigma_2^\iota)^{-1}\Psi(1-s, \tilde{W}, \tilde{W}')$$

$$= \sum_{1 \leq j \leq p} L(1-s, \pi^{\iota} \times \sigma_1^{\iota})^{-1} \Psi(1-s, \tilde{W}_1, (W_j')^{\sim}; ({}^t\phi_2)^{\wedge})$$

$$\cdot \omega_{\pi}(-1)^r L(1-s, \pi^{\iota} \otimes \sigma_2^{-1})^{-1} Z(\hat{\phi}_1, 1-s+t/2-1, (f_j \otimes \sigma_2)^{\iota})$$

$$= \omega_{\pi}(-1)^r \epsilon(s, \pi \times \sigma_1, \psi) \epsilon(s, \pi \otimes \sigma_2, \psi)$$

$$\sum_{1<j<p} L(s, \pi \times \sigma_1)^{-1} \Psi(s, W_1, W_j'; {}^t\phi_2)$$

$$\cdot L(s, \pi \times \sigma_2)^{-1} Z(\phi_1, s+t/2-1, f_j \otimes \sigma_2),$$

and applying (10) once more, we get finally

$$L(1-s, \pi^{\iota} \times \sigma_1^{\iota})^{-1} L(1-s, \pi^{\iota} \times \sigma_2^{\iota})^{-1} \Psi(1-s, \tilde{W}, \tilde{W}')$$

$$= \omega_{\pi}(-1)^r \epsilon(s, \sigma_1 \times \pi, \psi) \epsilon(s, \pi \otimes \sigma_2, \psi) L(s, \pi \times \sigma_1)^{-1}$$

$$\cdot L(s, \pi \times \sigma_2)^{-1} \Psi(s, W, W').$$

Thus indeed we have established (6.1.1) in the case at hand. □

7. Induced representations: The general case.

(7.1). In this section we complete the proof of Theorem (3.1). As we have seen, we may assume $p = 2$. Thus we are given a representation π of G_r of Whittaker type, two representations σ_1 and σ_2 respectively of G_{t_1} and G_{t_2}, again of Whittaker type. With

$$\sigma = \operatorname{Ind}(G_t, Q; \sigma_1, \sigma_2),$$

$t = t_1 + t_2$, we have to verify the identities (3.1)(i) and (3.1)(ii). Recall that $\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \sigma \times \pi, \psi)$ (c.f. (2.11).

The assertions of the Theorem have been established in the following cases:

(1) $$r \geq t$$

in Section 4,

(2) $$r = t-1, \qquad t_1 = t-1, \qquad t_2 = 1,$$

in Section 6,

(3) $$r = 1$$

in Section 5.

We begin by establishing the Theorem in the case $t_2 = 1$ and r arbitrary. If $r \geq t$ or $r = t - 1$, the Theorem is true (cases (1) and (2)); we fix t then and proceed by descending induction on r. We assume then that $r \leq t - 2$ and that the Theorem is true for $r + 1$ (and $t_2 = 1$). Let then μ be a quasi-character of $F^\times$ and consider the representation

$$\pi' = \operatorname{Ind}(G_{r+1}, R; \pi, \mu),$$

where R is a lower parabolic of type $(r, 1)$. By the induction hypothesis we have

(4) $$\gamma(s, \pi' \times \sigma, \psi) = \gamma(s, \pi' \times \sigma_1, \psi)\gamma(s, \pi' \times \sigma_2, \psi).$$

Next, since by assumption $r + 1 \leq t - 1$, we have by case (1):

(5) $$\gamma(s, \pi' \times \sigma_1, \psi) = \gamma(s, \pi \times \sigma_1, \psi)\gamma(s, \sigma_1 \otimes \mu, \psi).$$

Since $r + 1 \leq t$ as well, we can apply case (1) again to obtain

(6) $$\gamma(s, \pi' \times \sigma, \psi) = \gamma(s, \pi \times \sigma, \psi)\gamma(s, \sigma \otimes \mu, \psi).$$

Since $t_2 = 1$, by case (3) we have

(7) $$\gamma(s, \pi' \times \sigma_2, \psi) = \gamma(s, \pi \times \sigma_2, \psi)\gamma(s, \mu\sigma_2, \psi),$$

and again by case (3)

(8) $$\gamma(s, \sigma \otimes \mu, \psi) = \gamma(s, \sigma_1 \otimes \mu, \psi)\gamma(s, \mu\sigma_2, \psi).$$

Finally comparing (4) and (6) we get

(9) $$\gamma(s, \pi \times \sigma, \psi)\gamma(s, \sigma \otimes \mu, \psi) = \gamma(s, \pi' \times \sigma_1, \psi)\gamma(s, \pi' \times \sigma_2, \psi).$$

By (5), (7) and (8) this is equivalent to

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi \times \sigma_1, \psi)\gamma(s, \pi \times \sigma_2, \psi) \tag{10}$$

which is the second assertion of the theorem.

We pass now to the assertion relative to the L-factor. We proceed as above. Note that since the quasi-character μ plays only an auxiliary role, we may choose its conductor to be as large as necessary.

By the induction hypothesis we have

$$L(s, \pi' \times \sigma) = L(s, \pi' \times \sigma_1)L(s, \pi' \times \sigma_2)P_1(q^{-s}), \tag{11}$$

where P_1 is a polynomial. Moreover, as $r + 1 \leq t - 1$ by case (1) of the theorem:

$$L(s, \pi' \times \sigma_1) = L(s, \pi \times \sigma_1)L(s, \mu \otimes \sigma_1)P_2(q^{-s}) \tag{12}$$

where P_2 is a polynomial. Taking the conductor of μ sufficiently large, we may suppose $L(s, \sigma_1 \otimes \mu) = 1$ (Proposition (2.13)). Thus

$$L(s, \pi' \times \sigma_1) = L(s, \pi \times \sigma_1)P_2(q^{-s}). \tag{13}$$

As $r + 1 < t$ again by case (1)

$$L(s, \pi' \times \sigma) = L(s, \pi \times \sigma)L(s, \mu \otimes \sigma)P_3(q^{-s}),$$

and again if the conductor of μ is large enough, we have a relation

$$L(s, \pi' \times \sigma) = L(s, \pi \times \sigma)P_3(q^{-s}). \tag{14}$$

Next by case (3) we obtain

$$L(s, \pi' \times \sigma_2) = L(s, \pi \times \sigma_2)L(s, \sigma_2\mu)P_4(q^{-s}), \tag{15}$$

and we can suppose $L(s, \sigma_2\mu) = 1$. Whence

$$L(s, \pi' \times \sigma_2) = L(s, \pi \times \sigma_2)P_4(q^{-s}). \tag{16}$$

Comparing (11) and (14) we arrive at

(17)

$$L(s, \pi \times \sigma)P_3(q^{-s}) = L(s, \pi \times \sigma_1)L(s, \pi \otimes \sigma_2)P_1(q^{-s})P_2(q^{-s})P_4(q^{-s}).$$

On the other hand, we have from Lemma (5.5) a relation

(18) $$L(s, \sigma \times \pi) = L(s, \sigma \times \pi')R(q^{-s})$$

where R is a polynomial. Comparing with (14) we see at once that P_3 is equal to 1. Thus $L(s, \pi \times \sigma)$ belongs to

$$L(s, \pi \times \sigma_1)L(s, \pi \otimes \sigma_2)\mathbf{C}[q^{-s}, q^s],$$

and we have finished the proof of the Theorem in the case $t_2 = 1$.

(7.2). We prove the Theorem in general. It has now been established in the following cases:

(1) $r \geq t$,
(2) r arbitrary and $t_2 = 1$,
(3) $r = 1$ and t arbitrary.

Since the theorem is true for $r \geq t$, we may proceed as above, reasoning by descending induction on r with t fixed. We assume then that $r < t$ and that the Theorem is true for any pair $(r + 1, t)$. We introduce again the auxiliary representation

$$\pi' = \mathrm{Ind}(G_{r+1}, R; \pi, \mu),$$

where again R is a lower parabolic of type $(r, 1)$.

As before, we consider first the γ-factor. By the induction hypothesis we have

(4) $$\gamma(s, \pi' \times \sigma, \psi) = \gamma(s, \pi' \times \sigma_1, \psi)\gamma(s, \pi' \times \sigma_2, \psi).$$

By case (2) we also have:

(5)

$$\gamma(s, \pi' \times \sigma_i, \psi) = \gamma(s, \sigma_i \times \pi', \psi) = \gamma(s, \pi \times \sigma_i, \psi)\gamma(s, \sigma_i \otimes \mu, \psi)$$

for $i = 1, 2$. In exactly the same way we obtain

$$\gamma(s, \pi' \times \sigma, \psi) = \gamma(s, \pi \times \sigma, \psi)\gamma(s, \sigma \otimes \mu, \psi), \tag{6}$$

and by case (3) we have

$$\gamma(s, \sigma \otimes \mu, \psi) = \gamma(s, \sigma_1 \otimes \mu, \psi)\gamma(s, \sigma_2 \otimes \mu, \psi). \tag{7}$$

Finally comparing (4) and (6) we get

$$\gamma(s, \pi \times \sigma_1, \psi)\gamma(s, \sigma_1 \otimes \mu, \psi)\gamma(s, \pi \times \sigma_2, \psi)\gamma(s, \sigma_2 \otimes \mu, \psi)$$
$$= \gamma(s, \pi \times \sigma, \psi)\gamma(s, \sigma_1 \otimes \mu, \psi)\gamma(s, \sigma_2 \otimes \mu, \psi),$$

or

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, \pi \times \sigma_1, \psi)\gamma(s, \pi \times \sigma_2, \psi). \tag{8}$$

This is the first assertion of Theorem (3.1).

Finally we prove the second assertion. By the induction hypothesis we have

$$L(s, \pi' \times \sigma) = L(s, \pi' \times \sigma_1)L(s, \pi' \times \sigma_2)P(q^{-s}) \tag{9}$$

with $P \in \mathbf{C}[X]$, and by case (2) we also have

$$L(s, \pi' \times \sigma_i) = L(s, \pi \times \sigma_i)L(s, \sigma_i \otimes \mu)P_i(q^{-s}), \tag{10}$$

for $i = 1, 2$. Choosing the conductor of μ sufficiently large, this may be written as

$$L(s, \pi' \times \sigma_i) = L(s, \pi \times \sigma_i)P_i(q^{-s}). \tag{11}$$

In exactly the same way we obtain

$$L(s, \pi' \times \sigma) = L(s, \pi \times \sigma)P_3(q^{-s}) \tag{12}$$

with $P_3 \in \mathbf{C}[X]$. As in (7.1) we may appeal to Lemma (5.5) to obtain

(13) $$L(s, \pi \times \sigma) = L(s, \pi' \times \sigma)R(q^{-s}).$$

Hence $P_3 = 1$, $L(s, \pi \times \sigma) = L(s, \pi' \times \sigma)$ and from (9) and (11) we get

(14) $$L(s, \pi \times \sigma) = L(s, \pi \times \sigma_1)L(s, \pi \times \sigma_2)P_1(q^{-s})P_2(q^{-s}).$$

This completes the proof of Theorem (3.1). □

8. Computation of the L-factor: tempered representations.

(8.1). In this section, we compute the L-factor for tempered representations. We assume first that $r \geq t$ and that π is irreducible supercuspidal. Then π is generic and the restrictions $W|P_r$, for $W \in \mathcal{W}(\pi; \psi)$, then have compact support mod N_r (c.f. (2.2)); in other words, the functions $W \in \mathcal{W}(\pi; \psi)$ have compact support modulo N_rZ_r.

PROPOSITION. *Suppose π is an irreducible, supercuspidal representation of G_r and σ a representation of Whittaker type of G_r.*

(i) *If $r > t$, then $L(s, \pi \times \sigma) = 1$.*
(ii) *If $r = t$ and σ is irreducible, then $L(s, \pi \times \sigma) = \Pi(1 - aq^{-s})^{-1}$,*

where the product is taken over all complex numbers a of the form $a = q^{-u}$ such that $\pi \otimes \alpha^{-u} \simeq \sigma$.

Proof. If $r > t$, the integrand in $\Psi(s, W, W'; 0)$ has compact support (modulo N_t). Hence $\Psi(s, W, W'; 0)$ is a polynomial in q^{-s}, q^s for all W and W'. This is the same as saying that $L(s, \pi \times \sigma) = 1$.

For $r = 2$, except for trivial modifications, the second part of the Proposition is identical with Proposition (1.2) of [S.G.-J]. The proof in the general case is entirely analogous and is consequently omitted.

(8.2). We now compute the L-factor of pairs for square-integrable representations. Recall that an *irreducible* admissible representation is *square-integrable* if its central character ω_π is unitary and

$$\int_{Z_r \backslash G_r} |\langle \pi(g)v, \tilde{v}\rangle|^2 dg < \infty,$$

for any pairs of vectors v, $\tilde{v}$, v in the space of π, $\tilde{v}$ in the space of $\tilde{\pi}$. A representation of the form $\pi \otimes \alpha^u$, where π is square-integrable, is said to be *quasi-square-integrable*. An irreducible cuspidal representation is always square-integrable.

Bernstein and Zelevinsky have completely determined the (quasi-) square-integrable representation of GL_r ([B-Z], [Z]). Their result is as follows. Let a be a divisor of r (possibly 1 or r), π_0 an irreducible supercuspidal representation of G_a, m the quotient r/a, Q the (upper) standard parabolic subgroup of G_r of type $(a, a, \ldots, a)$; set $\pi_i = \pi_0 \otimes \alpha^{(m+1)/2-i}$, $1 \leq i \leq m$. Then the induced representation

$$\xi = \mathrm{Ind}(G, Q; \pi_1, \pi_2, \ldots, \pi_m) \tag{1}$$

has a unique component which is quasi-square-integrable. We denote this component by

$$\pi = \sigma(\pi_1, \pi_2, \ldots, \pi_m). \tag{2}$$

(We have also denoted this representation by $\sigma_m(\pi_0)$ in the introduction.) The representation π is unitary (or equivalently square-integrable) if and only if π_0 is. All irreducible quasi-square-integrable representations have the form (2); the integer m and the class of the representation π_0 being uniquely determined by π.

Recall that any (quasi-) square-integrable representation π is automatically generic [H.J. II]. Moreover any element W of $\mathcal{W}(\pi; \psi)$ satisfies

$$\int_{Z_r N_r \backslash G_r} |W(g)|^2 dg < \infty. \tag{3}$$

Thus π may also be described as the unique generic component of ξ (c.f. (2.3)). We remark in passing that π is also the unique minimal component of ξ (see also [Z], (9.1)).

Suppose now that π is of the form (1). Assume similarly $t = bn$, σ_0 is an irreducible cuspidal representation of G_b and

$$\sigma = \sigma(\sigma_1, \sigma_2, \ldots, \sigma_n), \tag{4}$$

where $\sigma_i = \sigma_0 \otimes \alpha^{(n+1)/2-i}$ $(1 \leq i \leq n)$. Finally suppose $t \leq r$.

THEOREM. *With the above assumptions,*

$$L(s, \pi \times \sigma) = \prod_{i=1}^{n} L(s, \pi_1 \times \sigma_i). \tag{5}$$

Proof. The critical point is to prove the following statement:

(6) *If π and σ are square-integrable, then $L(s, \pi \times \sigma)$ has no poles in the half-plane* $\mathrm{Re}(s) > 0$.

Indeed, let us assume this for now and derive the Theorem. Consider first the case $n = 1$. Then $\sigma = \sigma_1$ is supercuspidal and what we have to prove is the identity

$$L(s, \pi \times \sigma) = L(s, \pi_1 \times \sigma). \tag{7}$$

If $m = 1$ there is nothing to prove. So we may assume $m > 1$ proceed by induction on m. Thus, if we set

$$\pi' = \sigma(\pi_1, \pi_2, \ldots \pi_{m-1}), \tag{8}$$

then π' is a quasi-square integrable representation of G_{r-a}, and we may assume that

$$L(s, \pi' \times \sigma) = L(s, \pi_1 \times \sigma). \tag{9}$$

Now π is a component, in fact the unique generic component, of the induced representation

$$\mathrm{Ind}(G_r, Q; \pi', \pi_m), \tag{10}$$

where Q has type $(r - a, a)$. Hence by Theorem (3.1), we have

$$L(s, \pi \times \sigma) = P(q^{-s}, q^s)L(s, \pi' \times \sigma)L(s, \pi_m \times \sigma), \tag{11}$$

where $P \in \mathbf{C}[q^{-s}, q^s]$. At this point there is no harm in assuming that π_0 and σ have unitary central characters. Then, by Proposition (8.1), the poles of $L(s, \pi_m \times \sigma)$, which are simple, are on the line $\mathrm{Re}(s) = (m - 1)/2$. On the other hand, by (6), $L(s, \pi \times \sigma)$ has no poles in $\mathrm{Re}(s) > 0$. Thus in fact we must have a relation

$$L(s, \pi \times \sigma) = P(q^{-s}, q^s)L(s, \pi' \times \sigma),$$

where P is another element of $\mathbf{C}[q^{-s}, q^s]$. By [9] this may also be written as

$$(12)\qquad L(s, \pi \times \sigma) = P(q^{-s}, q^s)L(s, \pi_1 \times \sigma)$$

$$= P(q^{-s}, q^s)L(s + (m-1)/2, \pi_0 \times \sigma).$$

Next the representation $\tilde{\xi}$ and the induced representation

$$\xi' = \mathrm{Ind}(G, Q; \tilde{\pi}_m, \ldots, \tilde{\pi}_1)$$

have the same irreducible constituents. Thus

$$\tilde{\pi} = \sigma(\tilde{\pi}_m, \ldots, \tilde{\pi}_1),$$

in other words replacing π by $\tilde{\pi}$ entails replacing π_0 by $\tilde{\pi}_0$. Consequently

$$(13)\quad L(1-s, \tilde{\pi} \times \tilde{\sigma}) = \tilde{P}(q^{-s}, q^s)L(1-s+(m-1)/2, \tilde{\pi}_0 \times \tilde{\sigma}),$$

with $\tilde{P}$ also in $\mathbf{C}[q^{-s}, q^s]$.

On the other hand, since π is a component of (1), we have again by (3.1) the identity

$$(14)\qquad \gamma(s, \pi \times \sigma, \psi) = \prod_{1 \le i \le m} \gamma(s, \pi_i \times \sigma, \psi);$$

or, computing up to units of the ring $\mathbf{C}[q^{-s}, q^s]$,

$$= \prod_{1 \le i \le m} L(1-s, \tilde{\pi}_i \times \tilde{\sigma})/L(s, \pi_i \times \sigma)$$

$$= \prod_{1 \le i \le m} L(1-s+(m-1)/2-i, \tilde{\pi}_0 \times \tilde{\sigma})/L(s+(m-1)/2-i, \pi_0 \times \sigma).$$

But by (8.1), we know that if

$$(15)\qquad L(s, \pi_0 \times \sigma) = \prod (1 - \alpha q^{-s})^{-1},$$

then

$$(16)\qquad L(s, \tilde{\pi}_0 \times \tilde{\sigma}) = \prod (1 - \alpha^{-1} q^{-s})^{-1}.$$

Thus, after a simple computation, we find (up to units),

(17)

$$\gamma(s, \pi \times \sigma, \psi) = L(1 - s + (m-1)/2, \tilde{\pi}_0 \times \tilde{\sigma})/L(s + (m-1)/2, \pi_0 \times \sigma).$$

But by definition we have, again up to units,

$$\gamma(s, \pi \times \sigma, \psi) = L(1 - s, \tilde{\pi} \times \tilde{\sigma})/L(s, \pi \times \sigma). \tag{18}$$

If we compare (17), (18) with (12), (13), we conclude that first (up to units) $P = \tilde{P}$ and then that P divides both polynomials $L(s + (m-1)/2, \pi_0 \times \sigma)^{-1}$ and $L(1 - s + (m-1)/2, \tilde{\pi}_0 \times \tilde{\sigma}_0)^{-1}$. On the other in (15) we have $|\alpha| = 1$, which implies that these two polynomials are relatively prime in $\mathbf{C}[q^{-s}, q^s]$. Thus P is a unit, necessarily equal to 1, and we get indeed

$$L(s, \pi \times \sigma) = L(s, \pi_1 \times \sigma)$$

as claimed.

Having proved (5) for $n = 1$, we prove it now for n arbitrary. Since σ is a component of

$$\mathrm{Ind}(G_t, Q', \sigma_1, \sigma_2, \ldots, \sigma_n), \tag{19}$$

where Q' is of type $(b, b, \ldots, b)$, we have by (3.1) a relation

$$L(s, \pi \times \sigma) = P(q^{-s}, q^s) \prod_{i=1}^{n} L(s, \pi \times \sigma_i)$$

with $P \in \mathbf{C}[q^{-s}, q^s]$. By the previous case this is in fact

$$L(s, \pi \times \sigma) = P(q^{-s}, q^s) \prod_{i=1}^{n} L(s, \pi_1 \times \sigma_i) \tag{20}$$

$$= P(q^{-s}, q^s) \prod_{i=1}^{n} L(s + (m+n)/2 - i, \pi_0 \times \sigma_0).$$

Similarly

(21)

$$L(1-s, \tilde{\pi} \times \tilde{\sigma}) = \tilde{P}(q^{-s}, q^{s}) \prod_{i=1}^{n} L(1-s+(m+n)/2-i, \tilde{\pi}_0 \times \tilde{\sigma}_0).$$

Now again we have, up to units,

$$\begin{aligned}\gamma(s, \pi \times \sigma, \psi) &= \prod_{i=1}^{n} \gamma(s, \pi \times \sigma_i, \psi) \\ &= \prod_{i=1}^{n} L(1-s, \tilde{\pi} \times \tilde{\sigma}_i)/L(s, \pi \times \sigma_i) \\ &= \prod_{i=1}^{n} L(1-s+(m+n)/2-i, \tilde{\pi}_0 \times \tilde{\sigma}_0)/ \\ &\qquad L(s+(m+n)/2-i, \pi_0 \times \sigma_0).\end{aligned}$$

As in the case $n = 1$ we now deduce, from (20) and (21), that $P = \tilde{P}$ (up to units) and then that P divides both the numerator and denominator in the preceding fractions. Now by (8.1) either both of these products are identically one or else $m = n$. In the latter case if they have a common pole we must then have a relation of the form $2n + 1 = i + j$, for some pair i, j satisfying $1 \leq i, j \leq n$. Since this is impossible, in either case these two products are relatively prime. Thus as before $P = 1$ in relation (20) and indeed we obtain

$$L(s, \pi \times \sigma) = \prod_{i=1}^{n} L(s, \pi_1 \times \sigma_i).$$

(8.3). It remains therefore to prove that $L(s, \pi \times \sigma)$ has no pole in the half-plane $\mathrm{Re}(s) > 0$, whenever π and σ are square-integrable. This will follow from the following Proposition.

PROPOSITION. *Suppose π and σ are square-integrable. Then if $r = t$ the integrals $\Psi(s, W, W', \Phi)$ (resp. $\Psi(s, W, W'; 0)$ if $r > t$) converge absolutely for* $\mathrm{Re}(s) > 0$, *uniformly for* $\mathrm{Re}(s)$ *in a compact set.*

Proof. Recall the integration formula

$$(1)\quad \int_{N_r\backslash G_r} F(g)dg$$

$$= \int_{N_{r-1}\backslash G_{r-1}\times Z_r\times K_r} F\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} zk\right] |\det g|^{-1} dgdzdk.$$

Since the elements of $\mathcal{W}(\pi; \psi)$ are K-finite and square-integrable mod N_rZ_r, we deduce that

$$(2)\quad \int_{N_{r-1}\backslash G_{r-1}} \left| W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right]\right|^2 |\det g|^{-1} dg = \|W\|^2 < +\infty.$$

for any $W \in \mathcal{W}(\pi; \psi)$.

If $r = t$ we can write

$$(3)\quad \Psi(s, W, W', \Phi) = \int_{K_r} dk \int_{N_{r-1}\backslash G_{r-1}} WW'\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k\right] |\det g|^{s-1} dg$$

$$\int_{F^\times} \Phi[\eta_r ak] |a|^{rs} \omega_\pi \omega_\sigma(a) d^\times a.$$

Since the integral over $F^\times$ converges for $\mathrm{Re}(s) > 0$, it suffices to show that the integral over $N_{r-1}\backslash G_{r-1}$ also converges to $\mathrm{Re}(s) > 0$. Since W and W' are K-finite, it suffices to show that

$$(4)\quad \int_{N_{r-1}\backslash G_{r-1}} |WW'|\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^{s-1} dg < +\infty$$

for $s > 0$. For this we write (4) as a sum of two integrals, the first over $|\det g| \leq 1$ and the second over $|\det g| \geq 1$. For $s > 0$ the first of these integrals is $\leq \|W\| \cdot \|W'\|$. By (2.5), W and W' are majorized by gauges ξ and ξ'. The second integral is therefore less than or equal to any of the integrals

$$(5)\quad \int \xi\xi'\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^{s-1} dg$$

for $s \gg 0$. The convergence of the integral (5) (for $s \gg 0$) follows in a straight-forward fashion (c.f. (2.7)).

If $r > t$ we use the following integration formula:

$$(6) \quad \int_{N_{r-1}\backslash G_{r-1}} F(g)|\det g|^{-1}dg = \iint F\left[\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} k\right] |\det g|^{-(r-t)}|\det h|^{t-1} dg dh dk,$$

with g in $N_t\backslash G_t$, h in $N_{r-t-1}\backslash G_{r-t-1}$, k in K_r, to show that (2) implies in fact that

$$(7) \qquad \int_{N_t\backslash G_t} |W|^2\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^{-(r-t)} dg < +\infty.$$

This being so, consider the integral

$$\Psi(s, W, W'; 0) = \int_{N_t\backslash G_t} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] W'(g)|\det g|^{s-(r-t)/2} dg.$$

It is easy to see that $W[\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)] \neq 0$ implies that $\eta_t g$ (the last row of g) lies in a fixed compact subset of F^t. In fact if $g = nak$ is the Iwasawa decomposition of g in G_t then $\eta_t g = \eta_t ak$. On the other hand if α is any positive root of G_r then the fact that

$$W\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-t} \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] \neq 0$$

implies

$$\alpha\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] \leq 1.$$

In particular $\|\eta_t a\| \leq 1$ and our assertion follows. Thus it suffices to show that the integral

$$\int_{N_t\backslash G_t} |W|\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |W'|(g)\Phi[\eta_t g] \,|\det g|^{s-(r-t)/2} dg$$

is finite for any $s \geq 0$ and $\Phi \geq 0$ in $\mathcal{S}(F^t)$. But the square of this integral is less than or equal to the product of the integral

$$\int |W|^2\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^{-(r-t)} dg$$

and the integral

$$\int |W'|^2(g)\Phi^2[\eta_t g] \,|\det g|^{2s} dg,$$

both of which are finite, the first by (7) and the second by the previous case. This concludes the proof of the convergence of the integrals for Re(s) > 0. We leave the proof of the normal convergence to the reader. The proof of Theorem (8.2) is now complete.

(8.4). We pass now to the case of a tempered representation. A tempered representation π is a representation of the form

(1) $$\pi = \mathrm{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m),$$

where the π_i are square-integrable. A quasi-tempered representation π is a representation of the form $\pi_0 \otimes \alpha^u$, where π_0 is tempered. A (quasi-) tempered representation is automatically irreducible [H.J. II]. Let π then be a representation of G_r as in (1) and similarly σ be a representation of G_t of the form

(2) $$\sigma = \mathrm{Ind}(G_t, R; \sigma_1, \sigma_2, \ldots, \sigma_n),$$

where the σ_i are square-integrable.

Proposition. *With the above assumptions,*

$$L(s, \pi \times \sigma) = \Pi\, L(s, \pi_i \times \sigma_j),$$

the product taken over $1 \leq i \leq m$, $1 \leq j \leq n$. *In particular* $L(s, \pi \times \sigma)$ *has no pole in the half-plane* $\mathrm{Re}(s) > 0$.

Proof. Exactly as in (8.2), we have

$$L(s, \pi \times \sigma) = P(q^{-s}, q^s) \prod_{i,j} L(s, \pi_i \times \sigma_j) \tag{3}$$

and

$$L(1 - s, \tilde{\pi} \times \tilde{\sigma}) = \tilde{P}(q^{-s}, q^s) \prod_{i,j} L(1 - s, \tilde{\pi}_i \times \tilde{\sigma}_j), \tag{4}$$

where P and $\tilde{P} \in \mathbf{C}[q^{-s}, q^s]$. As before, within unit factors in the ring $\mathbf{C}[q^{-s}, q^s]$, we have on the one hand

$$\gamma(s, \pi \times \sigma, \psi) = L(1 - s, \tilde{\pi} \times \tilde{\sigma})/L(s, \pi \times \sigma) \tag{5}$$

and on the other hand

$$\begin{aligned} \gamma(s, \pi \times \sigma, \psi) &= \prod_{i,j} \gamma(s, \pi_i \times \sigma_j, \psi) \\ &= \prod_{i,j} L(1 - s, \tilde{\pi}_i \times \tilde{\sigma}_j)/L(s, \pi_i \times \sigma_j). \end{aligned} \tag{6}$$

Thus the principal ideals (P) and $(\tilde{P})$ are equal and divide both polynomials $L(s, \pi \times \sigma)^{-1}$ and $L(1 - s, \tilde{\pi} \times \tilde{\sigma})^{-1}$, the first of which has its zeroes in $\mathrm{Re}(s) \leq 0$ and the second in $\mathrm{Re}(s) \geq 1$. Thus P is a unit, necessarily equal to one, and our assertion follows.

9. Computation of the L-factor: induced representations.

(9.1). In this section we compute the L-factor in general. We first prove the existence of certain elements in the Whittaker space of an induced representation. Let therefore π_1 and π_2 be representations of Whittaker type of G_{r_1} and G_{r_2} respectively. Set $r = r_1 + r_2$. Let Q be the upper parabolic subgroup of type (r_1, r_2) in G_r. Let

$$\xi = \mathrm{Ind}(G, Q; \pi_1, \pi_2).$$

PROPOSITION. *Given* $\Phi \in \mathcal{S}(F^{r_2})$, $W_2 \in \mathcal{W}(\pi_2; \psi)$, *there is a* $W \in \mathcal{W}(\xi; \psi)$ *such that*

$$W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r_1} \end{pmatrix}\right] = W_2(g)\Phi[\eta_{r_2}g]\,|\det g|^{r_1/2}$$

for all $g \in G_{r_2}$.

Proof. We proceed as in (5.5). We may assume that π_i acts on $\mathcal{W}(\pi_i; \psi)$. Then the space of ξ is a space of functions on G_r with values in $\mathcal{W}(\pi_1; \psi) \otimes \mathcal{W}(\pi_2; \psi)$. If f is in the space of ξ we denote by $f(g; h_1, h_2)$ the corresponding scalar function on $G_r \times G_{r_1} \times G_{r_2}$. Let W_i be an element of $\mathcal{W}(\pi_i; \psi)$ and Ψ an element of $\mathcal{S}(r_2 \times r_1, F)$. Then, as in (5.5), there is a vector f in the space of ξ such that

$$f\left[w\begin{pmatrix} 1_{r_2} & x \\ 0 & 1_{r_1} \end{pmatrix}; h_1, h_2\right] = \Psi(x)W_1(h_1)W_2(h_2),$$

where

$$w = \begin{pmatrix} 0 & 1_{r_1} \\ 1_{r_2} & 0 \end{pmatrix}.$$

Let W be the corresponding element of $\mathcal{W}(\xi; \psi)$. Then, for all $g \in G_{r_2}$,

$$W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r_1} \end{pmatrix}\right] = \int f\left[w\begin{pmatrix} 1_{r_2} & x \\ 0 & 1_{r_1} \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1_{r_1} \end{pmatrix}; e, e\right]\bar{\theta}\begin{pmatrix} 1_{r_2} & x \\ 0 & 1_{r_1} \end{pmatrix}dx.$$

This is the same as

$$W_1(e)\int \Psi[g^{-1}x]W_2(g)\,|\det g|^{-r_1/2}\bar{\theta}\left[\begin{pmatrix} 1_{r_2} & x \\ 0 & 1_{r_1} \end{pmatrix}\right]dx,$$

or, after a change of variables,

$$W_1(e)W_2(g)\,|\det g|^{r_1/2}\Phi'[-\mu g],$$

where here μ has the form $\mu = \begin{bmatrix} \eta_{r_2} \\ 0 \end{bmatrix}$, with r_1 rows and r_2 columns, and

$$\Phi'(y) = \int \Psi(x)\psi(\mathrm{tr}(yx))dx.$$

Our assertion is therefore clear. □

(9.2). We will also use the following lemma.

LEMMA. *Let π be a representation of Whittaker type of G_r, $a < b < r$ two integers. Given W in $\mathcal{W}(\pi; \psi)$, there is an open compact subgroup Ω_1 of $M(a \times (b-a), F)$, an open compact subgroup Ω_2 of the group $\bar{B}_{b-a}$ of lower triangular matrices in G_{b-a}, and finally an element W^0 of $\mathcal{W}(\pi; \psi)$ such that, for $g \in G_a$, $h \in \bar{B}_{b-a}$,*

$$W^0\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{r-b} \end{pmatrix}\right] = W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-a} \end{pmatrix}\right],$$

if $x \in \Omega_1$, $h \in \Omega_2$ and is zero otherwise. Moreover, for W fixed, we may take Ω_1 and Ω_2 arbitrarily small.

Proof. In fact for $a \leq \mu \leq b-1$, we will prove the existence of a function W_μ in $\mathcal{W}(\pi; \psi)$, an open subgroup Ω_μ^1 of $M(a \times (\mu - a + 1), F)$ and an open subgroup Ω_μ^2 of $\bar{B}_{\mu-a+1}$ such that

$$W_\mu\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{r-\mu-1} \end{pmatrix}\right] = W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-a} \end{pmatrix}\right],$$

for x in Ω_μ^1 and h in Ω_μ^2, and is zero otherwise. We may then take $W^0 = W_{b-1}$.

We define the function W_μ recursively as follows. Set $W_{a-1} = W$, and for μ as above and $g \in G_r$, let

$$(1) \qquad W_\mu(g) = \int W_{\mu-1}\left[g\begin{pmatrix} 1_\mu & 0 & u & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-\mu-2} \end{pmatrix}\right]\phi_\mu(u, v)dudv,$$

where ϕ_μ is a Schwartz-Bruhat function—to be chosen momentarily. We have, in any case,

$$(2)\quad W_\mu\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{r-\mu-1} \end{pmatrix}\right] = W_{\mu-1}\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{r-\mu-1} \end{pmatrix}\right] \hat{\phi}_\mu(x, h),$$

for all $g \in \mathrm{GL}_\mu$, $x \in F^\mu$, $h \in F^\times$. We choose an open compact subgroup ω_μ^1 of F^μ, ω_μ^2 an open compact subgroup of $F^\times$, and ϕ_μ so that the right side of (2) reduces to

$$W_{\mu-1}\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-\mu} \end{pmatrix}\right],$$

if $(x, h) \in \omega_\mu^1 \times \omega_\mu^2$ and is zero otherwise. Thus W_μ is defined. Moveover if $\mu = a$ our assertion is clear. The remainder of the proof follows by an easy ascending induction on μ. We leave the details to the reader. □

(9.3). The key step in the computation of the L-factor is the following lemma. We remark that Lemma (5.5) is a special case.

LEMMA. *Let the notations be as in* (9.1). *Let also σ be a generic representation of G_t. Then*

$$L(s, \pi_2 \times \sigma) = P(q^{-s})L(s, \xi \times \sigma),$$

where P is a polynomial.

Proof. Suppose first that $t = r_2$. Let W_2 be in $\mathcal{W}(\pi_2; \psi)$, W' in $\mathcal{W}(\sigma; \bar{\psi})$, and Φ in $\mathcal{S}(F^{r_2})$. Then choose W in $\mathcal{W}(\xi; \psi)$ as in Proposition (9.1). We get then

$$\Psi(s, W, W'; 0) = \int W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] W'(g)\,|\det g|^{s-(r-t)/2} dg$$

$$= \int W_2(g)W'(g)\Phi[\eta_t g]\,|\det g|^s dg$$

$$= \Psi(s, W_2, W'; \Phi).$$

Thus $L(s, \pi_2 \times \sigma)$ belongs to $\mathbf{C}[q^{-s}, q^s]L(s, \xi \times \sigma)$ which is our assertion. If $t < r_2$, let again W_2 be in $\mathcal{W}(\pi_2; \psi)$ and W' in $\mathcal{W}(\sigma; \bar{\psi})$. Choose Φ in $\mathcal{S}(F^{r_2})$ such that $\Phi[\eta_{r_2}g] = 1$, for all $g \in G_t$ such that

$$W_2\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r-t} \end{pmatrix}\right] \neq 0$$

(c.f. (8.3)). Then, W again being as in Proposition (9.1), we have:

$$\Psi(s, W, W'; 0) = \int_{N_t\backslash G_t} W_2\left[\begin{pmatrix} g & 0 \\ 0 & 1_{r_2-t} \end{pmatrix}\right]$$

$$\cdot W'(g) |\det g|^{s-(r_2-t)/2}\Phi[\eta_{r_2}g]dg$$

$$= \Psi(s, W_2, W'; 0),$$

and the conclusion follows as before.

Suppose now that $r_2 < t < r$. Let again $W_2 \in \mathcal{W}(\pi_2; \psi)$, $W' \in \mathcal{W}(\sigma; \bar{\psi})$ and $\Phi \in \mathcal{S}(F^{r_2})$. Then (Proposition (9.1) and Lemma (9.2)) there is a $W \in \mathcal{W}(\xi; \psi)$ such that for, $g \in G_{r_2}$, $h \in \bar{B}_{t-r_2}$, $x \in M(r_2 \times (t-r_2), F)$,

$$W\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{r-t} \end{pmatrix}\right] = W_2(g)\Phi[\eta_{r_2}g] |\det g|^{r_1/2},$$

if $x \in \Omega_1$, $h \in \Omega_2$ and equal to zero otherwise. Here Ω_1, Ω_2 are compact open subgroups of $M(r_2 \times (t-r_2), F)$ and $\bar{B}_{t-r_2}$ respectively. Then

$$\Psi(s, W, W'; 0) = \int W\left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] W'\left[\begin{pmatrix} g & 0 \\ x & h \end{pmatrix}\right]$$

$$\cdot |\det g|^{s+r_2-(r+t)/2} dg d_r(h) dx,$$

where $g \in N_{r_2}\backslash G_{r_2}$, $h \in \bar{B}_{t-r_2}$. We may choose Ω_1 and Ω_2 so small that

$$W'\left[\begin{pmatrix} g & 0 \\ x & h \end{pmatrix}\right] = W'\left[\begin{pmatrix} g & 0 \\ 0 & 1_{t-r_2} \end{pmatrix}\right]$$

for $x \in \Omega_1$, $h \in \Omega_2$. Then, up to a positive constant factor, we get:

$$\Psi(s, W, W'; 0) = \int_{N_{r_2}\backslash G_{r_2}} W_2(g) W'\left[\begin{pmatrix} g & 0 \\ 0 & 1_{t-r_2} \end{pmatrix}\right] \cdot \Phi[\eta_{r_2} g] \, |\det g|^{s-(t-r_2)/2} dg.$$

As before for an appropriate choice of Φ we get

$$\Psi(s, W, W'; 0) = \Psi(s, W', W_2; 0),$$

and we are done in this case.

Suppose now $t \geq r$. Let $\mu_1, \mu_2, \ldots, \mu_{t-r+1}$ be quasi-characters of $F^\times$. Consider the induced representation

$$\xi' = \mathrm{Ind}(G_{t+1}, Q'; \pi_1', \pi_2),$$

where

$$\pi_1' = \mathrm{Ind}(G_{t+1-r_2}, Q''; \mu_1, \mu_2, \ldots, \mu_{t-r+1}, \pi_1),$$

Q', Q'' being appropriate (upper) parabolic subgroups. We may assume $r_1 > 0$. Then $r_2 < t < t + 1$ and by the previous case $L(s, \pi_2 \times \sigma)^{-1}$ divides $L(s, \xi' \times \sigma)^{-1}$. But we also have

$$\xi' = \mathrm{Ind}(G_{t+1}, Q'''; \mu_1, \mu_2, \ldots, \mu_{t-r+1}, \xi),$$

where Q''' is again an appropriate upper parabolic subgroup. Thus, by Theorem (3.1), $L(s, \xi' \times \sigma)^{-1}$ divides the product

$$(\Pi L(s, \mu_i \times \sigma)^{-1}) L(s, \xi \times \sigma)^{-1}.$$

If we take μ_i highly ramified we have $L(s, \mu_i \times \sigma) = 1$. We conclude that $L(s, \pi_2 \times \sigma)^{-1}$ divides $L(s, \xi \times \sigma)^{-1}$. □

(9.4). The main result of this section will apply to the induced representations of "Langlands' type." Let therefore Q be an upper parabolic subgroup of G_r of type $(r_1, r_2, \ldots, r_m)$. For each i, $1 \leq i \leq m$, let $\pi_{i,0}$ be a tempered representation of G_{r_i}. Let also $(u_1, u_2, \ldots, u_m)$ be a sequence of real numbers satisfying $u_1 > u_2 > \cdots > u_m$. For each i set $\pi_i = \pi_{i,0} \otimes \alpha^{u_i}$. Then the representation we want to consider is the induced representation

$$\xi = \operatorname{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m). \tag{1}$$

We consider similar data for the group G_t: R of type $(t_1, t_2, \ldots, t_n)$, $\sigma_{i,0}$, $(v_1, v_2, \ldots, v_n)$ and $\sigma_i = \sigma_{i,0} \otimes \alpha^{v_i}$. Set

$$\eta = \operatorname{Ind}(G_t, R; \sigma_1, \sigma_2, \ldots, \sigma_n). \tag{2}$$

As we have noted in Section 8 tempered representations are generic. Thus by (2.3) ξ and η are both of Whittaker type.

Proposition. *With the above notations,*

$$L(s, \xi \times \eta) = L(s, \pi_i \times \sigma_j), \tag{3}$$

the product taken for $1 \leq i \leq m$, $1 \leq j \leq n$.

Suppose π (resp. σ) is an arbitrary irreducible, admissible representation of G_r (resp. G_t). Then π (resp. σ) is uniquely representable as a quotient of a representation ξ (resp. η) of Langland's type. In the special case when π is generic, then π and ξ have the same Whittaker model, $\mathcal{W}(\xi; \psi) = \mathcal{W}(\pi; \psi)$. Similarly if σ is generic we have $\mathcal{W}(\eta; \psi) = \mathcal{W}(\sigma; \psi)$. Thus from the definition of the L-factor we get at once that

$$L(s, \xi \times \eta) = L(s, \pi \times \sigma). \tag{4}$$

We also remark that one can actually prove that π is generic if and only if $\pi \simeq \xi$ [J-S III].

In any case one can take (4) as a definition of $L(s, \pi \times \sigma)$ for any pair π, σ of irreducible, admissible representations. Then the results of this section and Section 8 may be used to obtain an explicit formula for the L-factor in all cases (c.f. the introduction and (9.5)).

Proof of the Proposition. We first prove the Proposition when $n =$

1, i.e. when η is irreducible and quasi-tempered. Then we may as well assume it is tempered. What we have to prove then is the relation

$$L(s, \xi \times \eta) = \prod_{i=1}^{m} L(s, \pi_i \times \eta). \tag{5}$$

This is trivial for $m = 1$. So we may assume $m > 1$ and the relation true for $m - 1$. Then we may think of ξ as being the induced representation

$$\xi = \operatorname{Ind}(G_r, Q_1; \pi_1, \xi'), \tag{6}$$

where Q_1 has type $(m_1, r - m_1)$ and ξ' is also an induced representation of Langland's type:

$$\xi' = \operatorname{Ind}(G_{r-m_1}, Q_2; \pi_2, \ldots, \pi_m). \tag{7}$$

Then, by Theorem (3.1),

$$L(s, \xi \times \eta) = P(q^{-s}, q^s) L(s, \pi_1 \times \eta) L(s, \xi' \times \eta), \tag{8}$$

where $P \in \mathbf{C}[q^{-s}, q^s]$. Next by Lemma (9.3)

$$L(s, \xi' \times \eta) = L(s, \xi \times \eta) Q(q^{-s}, q^s), \tag{9}$$

where again $Q \in \mathbf{C}[q^{-s}, q^s]$. On the other hand by the induction hypothesis

$$L(s, \xi' \times \eta) = \prod_{i=2}^{m} L(s, \pi_i \times \eta). \tag{10}$$

Thus we have

$$L(s, \xi \times \eta) = P(q^{-s}, q^s) \prod_{i=1}^{m} L(s + u_i, \pi_{i,0} \times \eta), \tag{11}$$

and

$$\prod_{i=2}^{m} L(s + u_i, \pi_{i,0} \times \eta) = Q(q^{-s}, q^s) L(s, \xi \times \eta). \tag{12}$$

Similarly the representation ξ^ι is of Langland's type:

$$\xi^\iota = \mathrm{Ind}(G_r, Q^\iota; \pi^\iota_m, \pi^\iota_{m-1}, \dots, \pi^\iota_1).$$

Thus

$$(13) \qquad L(1-s, \xi^\iota \times \tilde{\eta}) = \tilde{P}(q^{-s}, q^s) \prod_{i=1}^{m} L(1-s-u_i, \tilde{\pi}_{i,0} \times \tilde{\eta}),$$

and

$$(14) \qquad \prod_{i=1}^{m-1} L(1-s-u_i, \tilde{\pi}_{i,0} \times \tilde{\eta}) = \tilde{Q}(q^{-s}, q^s) L(1-s, \xi^\iota \times \tilde{\eta}).$$

As in Section 8, within a unit factor, we have a relation

$$(15) \quad L(1-s, \xi^\iota \times \tilde{\eta})/L(s, \xi \times \eta) = \prod_{i=1}^{m} L(1-s-u_i, \tilde{\pi}_{i,0} \times \tilde{\eta})/L(s+u_i, \pi_{i,0} \times \eta).$$

From (11), (13) and (15) we deduce at once that $(P) = (\tilde{P})$. On the other hand (11) and (12) imply that P divides $L(s + u_1, \pi_{1,0} \times \eta)^{-1}$. Similarly (13) and (14) imply that $\tilde{P}$ divides $L(1 - s - u_m, \tilde{\pi}_{m,0} \times \eta)^{-1}$. Thus unless P is a unit these two factors have a common pole s_0. By (8.4), we have $\mathrm{Re}(s_0 + u_1) \leq 0$ and $\mathrm{Re}(1 - s_o - u_m) \leq 0$. But this implies $u_1 < u_m$ and we have a contradiction. Hence P is a unit which is our assertion.

Consider finally the case of an arbitrary pair of integers (n, m). We have proved our assertion for the pairs $(n, 1)$ and, by symmetry, for the pairs $(1, m)$. We may therefore proceed by double induction; that is, we assume $n > 1$, $m > 1$ and our assertion true for the pairs $(n, m - 1)$ and $(n - 1, m)$. We then prove it for the pair (n, m). Once more we may think of ξ as being the induced representation (6) and obtain

$$L(s, \xi' \times \eta) = L(s, \xi \times \eta) Q_1(q^{-s}, q^s)$$

with $Q \in \mathbf{C}[q^{-s}, q^s]$. But by the induction hypothesis

$$L(s, \xi' \times \eta) = \prod_{i=2}^{m} \prod_{j=1}^{n} L(s, \pi_i \times \sigma_j),$$

so that

$$\prod_{i=2}^{m}\prod_{j=1}^{n} L(s, \pi_i \times \sigma_j) = Q_1(q^{-s}, q^s)L(s, \xi \times \eta). \tag{16}$$

Interchanging the roles of ξ and η we also get the similar relation

$$\prod_{i=1}^{m}\prod_{j=2}^{n} L(s, \pi_i \times \sigma_j) = Q_2(q^{-s}, q^s)L(s, \xi \times \eta). \tag{17}$$

Replacing (ξ, η) by (ξ^ι, η^ι) in (16), we also get:

$$\prod_{i=1}^{m-1}\prod_{j=1}^{n} L(1 - s, \tilde{\pi}_i \times \tilde{\sigma}_j) = \tilde{Q}_1(q^{-s}, q^s)L(1 - s, \xi^\iota \times \eta^\iota). \tag{18}$$

On the other hand, we have by Theorem (3.1),

$$L(s, \xi \times \eta) = P(q^s, q^{-s}) \prod_{i,j} L(s, \pi_i \times \sigma_j), \tag{19}$$

$$L(1 - s, \xi^\iota \times \eta^\iota) = \tilde{P}(q^s, q^{-s}) \prod_{i,j} L(1 - s, \tilde{\pi}_i \times \tilde{\sigma}_j), \tag{20}$$

and, within a unit factor,

(21)

$$L(1 - s, \xi^\iota \times \eta^\iota)/L(s, \xi \times \eta) = \prod_{i,j} L(1 - s, \tilde{\pi}_i \times \tilde{\sigma}_j)/L(s, \pi_i \times \sigma_j).$$

As before (19), (20), (21) imply that $P = \tilde{P}$ (up to a unit). But then we deduce first from (17) and (19) the relation

$$P(q^{-s}, q^s)Q_2(q^{-s}, q^s) \prod_{i=1}^{m} L(s, \pi_i \times \sigma_1) = 1 \tag{22}$$

and similarly from (18) and (20)

$$P(q^{-s}, q^s)\tilde{Q}_1(q^{-s}, q^s) \prod_{j=1}^{n} L(1 - s, \tilde{\pi}_m \times \tilde{\sigma}_j) = 1. \tag{23}$$

Thus if P is not identically one, there is a pair (i, j) for which $L(s, \pi_i \times \sigma_1)$ and $L(1 - s, \tilde{\pi}_m \times \tilde{\sigma}_j)$ have a common pole, say s_0. Again by (8.4), we must have

$$\operatorname{Re}(s_0) + u_i + v_1 \leq 0, \qquad 1 - \operatorname{Re}(s_0) - u_m - v_j \leq 0.$$

Adding these inequalities we find

$$1 + u_i - u_m + v_1 - v_j \leq 0,$$

a contradiction since in fact $u_i \geq u_m$ and $v_1 \geq v_j$. Thus P is a unit and (19) is precisely the statement we wanted to prove. □

(9.5). We summarize our results on the L-factor. As we have noted in the introduction every representation of $\mathrm{GL}_r(F)$ is, in a formal sense, a sum of square-integrable representations of the various groups $\mathrm{GL}_t(F)$, $t \leq r$. More precisely, as in (0.1), let us denote by $\mathcal{A}_r$ the set of (equivalence classes of) irreducible, admissible representations of $\mathrm{GL}_r(F)$ and by $\mathcal{SQ}_r$ the subset of quasi-square-integrable representations of the same group. Set $\mathcal{A} = \bigcup_r \mathcal{A}_r$ (disjoint union) and similarly $\mathcal{SQ} = \bigcup_r \mathcal{SQ}_r$. Next let Λ denote the set of all (formal) sums λ of the form

$$\lambda = \tau_1 + \tau_2 + \cdots + \tau_p, \tag{1}$$

where the τ_i belong to $\mathcal{SQ}$. If $\tau \in \mathcal{SQ}_r$, we set $\deg(\tau) = r$ and further we set

$$\deg \lambda = \Sigma \deg(\tau_i). \tag{2}$$

To each λ of degree r we associate an element $\pi[\lambda] \in \mathcal{A}_r$ as follows. With λ as in (1) write $\tau_i = \sigma_i \otimes \alpha^{s_i}$, uniquely, with σ_i unitary and s_i real. We may assume $s_1 \geq s_2 \geq \cdots \geq s_p$. Then we let $\pi[\lambda]$ be the unique irreducible quotient of the induced representation

$$\xi = \operatorname{Ind}(G_r, Q; \tau_1, \tau_2, \ldots, \tau_p). \tag{3}$$

As is well known ([B-W], [A.S.]) the map $\lambda \to \pi[\lambda]$ defines a bijection from Λ to $\mathcal{A}$. In particular we obtain by transport of structure an operation of addition, denoted by $\boxplus$, on the set $\mathcal{A}$. Our final Theorem is then as follows:

THEOREM. *With the notation as above, the L-factor $L(s, \pi \times \sigma)$ is biadditive*:

$$L(s, (\pi \boxplus \pi') \times \sigma) = L(s, \pi \times \sigma)L(s, \pi' \times \sigma),$$

$$L(s, \pi \times (\sigma \boxplus \sigma')) = L(s, \pi \times \sigma)L(s, \pi \times \sigma'),$$

for all $\pi, \pi', \sigma, \sigma' \in \mathfrak{A}$.

The proof is an immediate consequence of Propositions (8.4) and (9.4). (Recall that $L(s, \pi \times \sigma) = L(s, \sigma \times \pi)$.) As we remarked in the introduction Theorems (8.2) and (9.5) are consistent with the expected local reciprocity and in fact are consequences of the axioms for ℓ_r given in (0.1).

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