Contemporary Mathematics

Archimedean Rankin-Selberg Integrals

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This paper is dedicated to Stephen Gelbart.

Abstract. The paper gives complete proofs of the properties of the Rankin-Selberg integrals for the group $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$.

Contents

1. Introduction

The goal of these notes is to give a definitive exposition of the local Archimedean theory of the Rankin-Selberg integrals for the group $GL(n)$. Accordingly, the ground field F is either $\mathbb{R}$ or $\mathbb{C}$. The integrals at hand are attached to pairs of

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irreducible representations (π, V) and (π', V') of $GL(n, F)$ and $GL(n', F)$ respectively. More precisely, each integral is attached to a pair of functions W and W' in the Whittaker models of π and π' respectively and, in the case $n = n'$, a Schwartz function in n variables. More generally, it is necessary to consider instead of a pair (W, W') a function in the Whittaker model of the completed tensor product $V \widehat{\otimes} V'$. The integrals depend on a complex parameter s. They converge absolutely for $\Re s >> 0$. The goal is to prove that they extend to holomorphic multiples of the appropriate Langlands L-factor, are bounded at infinity in vertical strips, and satisfy a functional equation where the Langlands ϵ factor appears. This is what is needed to have a complete theory of the converse theorems ([**6**], [**7**], [**8**]). An alternate approach may be found in [**20**].

More is proved. Namely, it is proved that the L-factor itself is a sum of such integrals. At this point in time, this result is not needed. Nonetheless, it has esthetic appeal. Indeed, it shows that the factors L and ϵ are determined by the representations π and π'. Anyway, by using this general result and by following Cogdell and Piatetski-Shapiro ([**8**]), it is shown that for the case $(n, n-1)$ and (n, n) the relevant L-factor is obtained in terms of vectors which are finite under the appropriate maximal compact subgroups. The result is especially simple in the unramified situation, a result proved by Stade ([**22**], [**23**]) with a different proof.

A first version of these notes was published earlier ([**18**]). The present notes are more detailed. Minor mistakes of the previous version have been corrected. More importantly, in contrast to [**18**], the methods are uniform as all the results are derived from an integral representation of the Whittaker functions, the theory of the Tate integral, and the Fourier inversion formula. The estimates for a Whittaker function are derived from coarse estimates which are then improved by applying the same coarse estimates to the derivatives of the Whittaker function, a method first used by Harish-Chandra. This is simpler than giving an explicit description of the Whittaker functions and then deriving estimates, as was done in the previous version. In [**13**], I proposed another approach to the study of the integrals. Again, the approach of the present notes is in fact simpler. Thus I hope that these notes can be indeed regarded as a definitive treatment of the question.

Difficult results on smooth representations and Whittaker vectors due to Wallach ([**26**], Vol. II), Casselman ([**3**]), Casselman and his collaborators ([**4**]) are used in an essential way.

Needless to say, these notes owe much to my former collaborators, Piatetski-Shapiro and Shalika. In particular, the ingenious induction step from $(n, n-1)$ to (n, n) is due to Shalika.

Finally, I would like to thank the referee for reading carefully the manuscript and suggesting improvements to the exposition.

2. The main results

Let F be $\mathbb{R}$ or $\mathbb{C}$. If $F = \mathbb{R}$, we denote by $|x|_F$ the ordinary absolute value. If $F = \mathbb{C}$, we set $|x|_F = x\overline{x}$. We also write $\alpha(x) = \alpha_F(x) = |x|_F$.

In these notes we consider representations (π, V) of $GL(n, F)$. We often write $G_n(F)$ or even G_n for the group $GL(n, F)$. Furthermore, we set $K_n = O(n, \mathbb{R})$ if $F = \mathbb{R}$, and $K_n = U(n, \mathbb{R})$ if $F = \mathbb{C}$. We let $\mathrm{Lie}(G_n(F))$ be the Lie algebra of $G_n(F)$ as a real Lie group and $\mathfrak{U}(G_n(F))$ the enveloping algebra of $\mathrm{Lie}(G_n(F))$. The space V is assumed to be a Frechet space. The representation on V is continuous and C^∞.

Let V_0 be the space of K_n-finite vectors in V so that V_0 is a $(\mathrm{Lie}(G_n(F)), K_n))$-module. We assume that the representation of $(\mathrm{Lie}(G_n(F)), K_n)$ on V_0 is admissible and has a finite composition series. Finally, we assume that the representation is of moderate growth, a notion that we now recall. For $g \in GL(n, \mathbb{C})$ or $g \in GL(n, \mathbb{R})$, we set

$$g^\iota := {}^t g^{-1}\,,\ ||g||_H := \mathrm{Tr}(g\ {}^t\overline{g}) + \mathrm{Tr}(g^{-1}\ \overline{g^\iota})\,. \tag{2.1}$$

Then, for every continuous semi-norm μ on V, there is M and another continuous semi-norm μ' such that, for every $v \in V$, $g \in G_n(F)$,

$$\mu(\pi(g)v) \leq\ ||g||_H^M \mu'(v)\,.$$

It is a fundamental result of Casselman and Wallach that V is determined, up to topological equivalence, by the equivalence class of the representation of the pair $(\mathrm{Lie}(G_n(F)), K_n)$ on V_0. In other words, V is the canonical Casselman-Wallach completion of the Harish-Chandra $(\mathrm{Lie}(G_n(F)), K_n)$-module V_0. It will be convenient to call such a representation a Casselman-Wallach representation.

If (π', V') is similarly a representation of $G_{n'}$ satisfying the same conditions and V_0' is the space of $K_{n'}$-finite vectors in V', then the representation $\pi \otimes \pi'$ of $G_n \otimes G_{n'}$ on the (projective) complete tensor product $V \widehat{\otimes} V'$ is the Casselman-Wallach completion of the $(\mathrm{Lie}(G_n \times G_{n'}), K_n \times K_{n'})$ module $V_0 \otimes V_0'$.

In addition, in these notes, the representations π at hand have a central character $\omega_\pi : F^\times \mapsto \mathbb{C}^\times$ defined by

$$\omega_\pi(z) 1_V = \pi(z 1_n)\,.$$

Let ψ be a non-trivial additive character of F. If V is a real or complex finite dimensional vector space, we will denote by $\mathcal{S}(V)$ the space of complex-valued Schwartz functions on V. Let $\Phi \in \mathcal{S}(V)$ where $V = M(a \times b, F)$, the space of matrices with a rows and b columns. We denote by $\mathcal{F}_\psi(\Phi)$, or simply $\widehat{\Phi}$, the Fourier transform of Φ. Unless otherwise specified, it is the function defined on the **same** space by

$$\mathcal{F}_\psi(\Phi)(X) = \int \Phi(Y)\psi\left(-\mathrm{Tr}(\ {}^t XY)\right) dY\,.$$

The Haar measure is self-dual so that $\mathcal{F}_\psi \circ \mathcal{F}_{\overline{\psi}}$ is the identity.

We let N_n be the group of upper triangular matrices with unit diagonal and we denote by $\theta_{\psi,n}$ or simply θ_ψ the character $\theta_\psi : N_n(F) \to \mathbb{C}^\times$ defined by

$$\theta_\psi(u) = \psi\Big(\sum_i u_{i,i+1}\Big)\,. \tag{2.2}$$

A ψ **form on** V is a continuous linear form λ such that

$$\lambda(\pi(u)v) = \theta_\psi(u)v\,,$$

for each $v \in V$ and each $u \in N_n(F)$. We let A_n be the group of diagonal matrices, B_n the Borel subgroup $B_n = A_n N_n$. We denote by δ_n the module of the subgroup $B_n(F)$. We often write $\overline{N_n}$ for the group ${}^t N_n$.

To formulate our results, we first consider certain induced representations of $GL(n, F)$. Let W_F be the Weil group of F and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_r)$ an r-tuple of irreducible unitary representations of W_F (see the Appendix). Thus the degree of σ_i noted $d_i = \deg(\sigma_i)$ is 1 or 2. Let π_i or π_{σ_i} be the representation of $GL(d_i, F)$

attached to σ_i . Denote by I_i its space. Let $u = (u_1, u_2, \ldots, u_r)$ be an r-tuple of complex numbers. Let P be the **lower** parabolic subgroup of type

$$(d_1, d_2, \ldots, d_r)$$

with Levi-decomposition

$$P = MU$$

in $GL(n, F)$, $n = \sum_i d_i$. Here M is the group of matrices of the form

$$m = \begin{pmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ * & * & * & * \\ 0 & 0 & \ldots & m_r \end{pmatrix}, m_i \in G_{d_i} \,. \tag{2.3}$$

We denote by δ_P the module of the group $P(F)$.

We denote by $(\pi_{\sigma,u}, I_{\sigma,u})$ the representation of $GL(n, F)$ induced by the representation

$$(\pi_1 \otimes \alpha^{u_1}, \pi_2 \otimes \alpha^{u_2}, \ldots, \pi_r \otimes \alpha^{u_r})$$

of P. Thus $I_{\sigma,u}$ may be viewed as a space of functions f on $GL(n, F)$ with values in the projective tensor space $I_1 \widehat{\otimes} I_2 \widehat{\otimes} \cdots \widehat{\otimes} I_r$ such that

$$f(vmg) = \delta_P^{1/2}(m) \\ \times \pi_1(m_1)|\det m_1|^{u_1} \otimes \pi_2(m_2)|\det m_2|^{u_2} \otimes \cdots \otimes \pi_r(m_r)|\det m_r|^{u_r} f(g)$$

for $v \in U(F)$, $m \in M(F)$. The representation $\pi_{\sigma,u}$ is by right shifts.

For each u, there is a non zero continuous linear ψ form λ on $I_{\sigma,u}$ and, within a scalar factor, a unique one. Indeed if σ is irreducible of degree 1, then $\pi_{\sigma,u}$ is a one dimensional character of $G_1(F) = F^\times$ and our assertion is vacuous. If σ is irreducible of degree 2, then $F = \mathbb{R}$ and $\pi_{\sigma,u}$ is a discrete series representation of $G_2(\mathbb{R})$ and our assertion is then well-known ([**21**]). In the general case $\pi_{\sigma,u}$ is an induced representation and our assertion follows from Theorem 15.4.1 in [**26**] II. We often say that $\pi_{\sigma,u}$ is **a generic induced representation**.

For each $f \in I_{\sigma,u}$, we set

$$W_f(g) = \lambda(\pi_{\sigma,u}(g)f) \,.$$

We denote by $\mathcal{W}(\pi_{\sigma,u} : \psi)$ the space spanned by the functions W_f.

For every integer n, we denote by w_n the $n \times n$ permutation matrix whose anti diagonal entries are 1. In particular,

$$w_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

We also set

$$g^\iota = {}^t g^{-1} \,.$$

We set

$$\widetilde{\pi}_i(g) = \pi_i(w_{d_i} g^\iota w_{d_i}) \,.$$

Thus if $d_i = 1$, then π_i is a character of $F^\times$ and $\widetilde{\pi}_i(x) = \pi_i(x)^{-1}$. If $d_i = 2$, then $\widetilde{\pi}_i(g)$ is isomorphic to the representation contragredient to π_i. In particular, it is the representation associated to the representation $\widetilde{\sigma_i}$ of W_F contragredient to σ_i. If f is in $I_{\sigma,u}$, then the function $\widetilde{f}$, defined by

$$\tilde{f}(g) := f(w_n \; g^\iota) \,,$$

belongs to the space of the representation induced by the representation

$$(\widetilde{\pi}_r \otimes \alpha^{-u_r}, \widetilde{\pi}_{r-1} \otimes \alpha^{-u_{r-1}}, \cdots \widetilde{\pi}_1 \otimes \alpha^{-u_1})$$

of the subgroup

$$\widetilde{P} := w_n (P)^\iota w_n \,.$$

Set

$$\begin{aligned} \widetilde{\sigma} &= (\widetilde{\sigma_r}, \widetilde{\sigma_{r-1}}, \dots, \widetilde{\sigma_1}) \\ \widetilde{u} &= (-u_r, -u_{r-1}, \dots, -u_1)\,. \end{aligned}$$

We may identify this induced representation to the space $I_{\widetilde{\sigma},\widetilde{u}}$. If we do so, then $\widetilde{f}$ belongs to $I_{\widetilde{\sigma},\widetilde{u}}$. We define a $\overline{\psi}$ linear form $\widetilde{\lambda}$ on $I_{\widetilde{\sigma},\widetilde{u}}$ by

$$\widetilde{\lambda}(\widetilde{f}) = \lambda(f)\,.$$

We see then that the function

$$\widetilde{W_f}(g) := W_f(w_n g^\iota)$$

verifies

$$\widetilde{W_f}(g) = W_{\widetilde{f}}(g)$$

where

$$W_{\widetilde{f}}(g) = \widetilde{\lambda}(\pi_{\widetilde{\sigma},\widetilde{u}}(g)\widetilde{f})\,.$$

Thus $\widetilde{W_f}$ belongs to $\mathcal{W}(\pi_{\widetilde{\sigma},\widetilde{u}} : \overline{\psi})$.

The semisimple representations attached to $\pi_{\sigma,u}$ and $\pi_{\widetilde{\sigma},\widetilde{u}}$ are contragredient to one another. In general, the representations $\pi_{\sigma,u}$ and $\pi_{\widetilde{\sigma},\widetilde{u}}$ need not be contragredient to one another if they are not irreducible.

Let τ be a semisimple representation of W_F. The factors $L(s,\tau)$, $L(s,\widetilde{\tau})$, $\epsilon(s,\tau,\psi)$ are defined (see the Appendix). As usual, we set

$$\gamma(s,\tau,\psi) = \epsilon(s,\tau,\psi)\ \frac{L(1-s,\widetilde{\tau})}{L(s,\tau)}\,.$$

If τ is of degree 1, these are the Tate factors. In particular, the factor $\epsilon(s,\tau,\psi)$ is defined by the functional equation

$$\int \widehat{\Phi}(x)\tau^{-1}(x)\ |x|_F^{1-s} d^\times x = \gamma(s,\tau,\psi) \int \Phi(x)\tau(x)|x|^s$$

where the Fourier transform $\widehat{\Phi}$, also noted $\mathcal{F}_\psi(\Phi)$, is defined by

$$\widehat{\Phi}(x) = \int \Phi(y)\psi(-yx)dy\,,$$

and dy is the self-dual Haar measure. If we denote by ψ_a the character defined by $\psi_a(x) = \psi(ax)$, we have

$$\gamma(s,\mu,\psi_a) = \mu(a)|a|^{s-1/2}\gamma(s,\mu,\psi)\,.$$

In general,

$$\gamma(s,\tau,\psi_a) = \det(\tau)(a)|a|^{d(s-1/2)}\gamma(s,\tau,\psi)$$

where $\det(\tau)$ is regarded as a character of $F^\times$ and d is the degree of τ.

We let $\mathcal{L}(\tau)$ be the space of meromorphic functions $f(s)$ which are holomorphic multiples of $L(s,\tau)$ and furthermore satisfy the following condition. Let $P(s)$ be a polynomial such that $P(s)L(s,\tau)$ is holomorphic in the strip

$$A \le \Re s \le B\,.$$

Then $P(s)f(s)$ is bounded in the same strip. Then we define a semi-norm on $\mathcal{L}(\tau)$

$$\mu_{P,A,B}(f) = \sup_{A \le \Re s \le B} |P(s)f(s)| \,.$$

The space $\mathcal{L}(\tau)$ is complete for the topology defined by the semi-norms $\mu_{P,A,B}$.

Now consider two pairs (σ, u) and (σ', u'). We set

$$\sigma_u = \bigoplus \sigma_i \otimes \alpha_F^{u_i}$$

and define $\sigma'_{u'}$ similarly.

We choose a ψ linear form λ on $I_{\sigma,u}$ and a $\overline{\psi}$ linear form λ' on $I_{\sigma',u'}$. The integrals we want to consider are as follows. For $f \in I_{\sigma,u}$, $f' \in I_{\sigma',u'}$, set

$$W = W_f \,,\; W' = W_{f'} \,.$$

If $n > n'$, we set

$$\Psi(s, W, W') = \int W \begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-n'}{2}} dg \,. \tag{2.4}$$

In addition, for $0 \le j \le n - n' - 1$, we set

$$\Psi_j(s, W, W') = \int W \begin{pmatrix} g & 0 & 0 \\ X & 1_j & 0 \\ 0 & 0 & 1_{n-n'-j} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-n'}{2}} dX dg \,. \tag{2.5}$$

Here X is integrated over the space $M(m \times j, F)$ of matrices with m rows and j columns. Thus $\Psi_0(s, W, W') = \Psi(s, W, W')$. In each integral, g is integrated over the quotient $N_{n'}(F) \backslash G_{n'}(F)$.

If $n = n'$, we let Φ be a Schwartz function on F^n and we set

$$\Psi(s, W, W', \Phi) = \int W(g) W'(g) \; \Phi[(0, 0, \ldots, 0, 1)g] \; |\det g|^s dg \,. \tag{2.6}$$

Again, g is integrated over the quotient $N_n(F) \backslash G_n(F)$.

In this paper, we prove the following results.

THEOREM 2.1.

(i) *The integrals converge for $\Re s >> 0$.*
(ii) *Each integral extends to a meromorphic function of s which is a holomorphic multiple of $L(s, \sigma_u \otimes \sigma'_{u'})$, bounded at infinity in vertical strips.*
(iii) *The following functional equations are satisfied. If $n = n' + 1$*

$$\Psi(1 - s, \widetilde{W}, \widetilde{W'}) = \omega_{\pi_{\sigma,u}}(-1)^{n-1} \omega_{\pi_{\sigma',u'}}(-1) \gamma(s, \sigma_u \otimes \sigma'_{u'}, \psi) \Psi(s, W, W') \,.$$

If $n > n' + 1$,

$$\Psi_j(1 - s, \rho(w_{n,n'}) \widetilde{W}, \widetilde{W'}) \\ = \omega_{\pi_{\sigma,u}}(-1)^{n'} \omega_{\pi_{\sigma',u'}}(-1) \gamma(s, \sigma_u \otimes \sigma'_{u'}, \psi) \Psi_{n-n'-1-j}(s, W, W') \,.$$

If $n = n'$,

$$\Psi(1 - s, \widetilde{W}, \widetilde{W'}, \widehat{\Phi}) = \omega_{\pi_{\sigma,u}}(-1)^{n-1} \gamma(s, \sigma_u \otimes \sigma'_{u'}, \psi) \Psi(s, W, W', \Phi) \,.$$

Here

$$w_{n,n'} = \begin{pmatrix} 1_{n'} & 0 \\ 0 & w_{n-n'} \end{pmatrix} .$$

Recall that $\omega_{\pi_{\sigma,u}}$ and $\omega_{\pi_{\sigma',u'}}$ are the central characters of $\pi_{\sigma,u}$ and $\pi_{\sigma',u'}$, respectively. Note that $\det\sigma_u = \omega_{\pi_{\sigma,u}}$ and $\det\sigma'_{u'} = \omega_{\pi_{\sigma',u'}}$.

REMARK 2.2. The functional equations are slightly different from the ones in earlier references. This is because the conventions are themselves different.

Following Cogdell and Piatetski-Shapiro, we remark that the assertions of the theorem for a given ψ imply the assertions are true for any ψ. Indeed, consider for instance the case $n = n'$. Set $\pi = \pi_{\sigma,u}$, $\pi' = \pi_{\sigma',u'}$. Let $a \in F^\times$. Set $\psi_a(x) := \psi(ax)$ and

$$m = \mathrm{diag}(a^{n-1}, a^{n-2}, \ldots, a, 1)\,.$$

Then $\det m = a^{\frac{n(n-1)}{2}}$, $\delta_n(m) = |a|^{\frac{(n+1)n(n-1)}{12}}$, and $mw_n = a^{n-1}w_nm^{-1}$ For $W \in \mathcal{W}(\pi : \psi)$, $W' \in \mathcal{W}(\pi' : \overline{\psi})$, set

$$W_m(g) = W(mg)\,,\ W'_m(g) = W'(mg)\,.$$

Then $W_m \in \mathcal{W}(\pi : \psi_a)$, $W'_m \in \mathcal{W}(\pi' : \overline{\psi_a})$. After changing g to $m^{-1}g$, we find

$$\Psi(s, W_m, W'_m, \Phi) = \delta_n(m)|a|^{-s\frac{n(n-1)}{2}}\Psi(s, W, W', \Phi)\,.$$

Thus the assertions about the analytic properties of the integrals are true for ψ_a. We pass to the functional equation. For clarity, we define a priori a factor $\gamma(s, \pi\times\pi', \psi)$ by the functional equation

$$\Psi(1-s, \widetilde{W}, \widetilde{W'}, \mathcal{F}_\psi(\Phi)) = \gamma(s, \pi\times\pi', \psi)\omega_\pi(-1)^{n-1}\Psi(W, W', \Phi)\,.$$

We have to check the relations

$$\gamma(s, \pi\times\pi', \psi_a) = \omega_\pi\omega_{\pi'}(a)^n\ |a|^{n^2(s-1/2)}\gamma(s, \pi\times\pi', \psi)\,,$$

and

$$\gamma(s, \pi\times\pi', \psi) = \gamma(s, \pi'\times\pi, \psi)\,.$$

We stress that

$$\mathcal{F}_{\psi_a}(\Phi)(X) = |a|^{n/2}\Phi(aX)\,.$$

For $n = 1$, from the Tate functional equation, we do get

$$\gamma(s, \chi, \psi_a) = \chi(a)|a|^{s-1/2}\gamma(s, \chi, \psi)\,.$$

For $n > 1$

$$\widetilde{W}_m(g) = W(mw_ng^\iota) = W(a^{n-1}w_nm^{-1}\ g^\iota) = \omega_\pi(a)^{n-1}W(w_nm^{-1}\ g^\iota)\,.$$

It will be convenient to use the notation

$$\epsilon_n = \overbrace{(0, 0, \ldots, 0, 1)}^{n}\,. \tag{2.7}$$

Now

$$\begin{aligned}&\Psi(1-s, \widetilde{W_m}, \widetilde{W'_m}, \mathcal{F}_{\psi_a}(\Phi))\\ &\quad = \omega_\pi\omega_{\pi'}(a)^{n-1}|a|^{\frac{n}{2}}\int W(w_nm^\iota g^\iota)W'(w_nm^\iota g^\iota)\mathcal{F}_\psi(\Phi)(a\epsilon_ng)\ |\det g|^{1-s}dg\,.\end{aligned}$$

After changing g into $a^{-1}g$ and then g into $m^{-1}g$, we find

$$\delta_n(m)\omega_\pi\omega_{\pi'}(a)^n\ |a|^{\frac{n^2+n}{2}s-\frac{n^2}{2}}\Psi(1-s, \widetilde{W}, \widetilde{W'}, \mathcal{F}_\psi(\Phi))\,.$$

Applying the functional equation for ψ, this becomes

$$\delta_n(m)\omega_\pi\omega_{\pi'}(a)^n\ |a|^{\frac{n^2+n}{2}s-\frac{n^2}{2}}\gamma(s,\pi\times\pi',\psi)\omega_\pi(-1)^{n-1}$$
$$\times\int W(g)W'(g)\ \Phi(\epsilon_n g)\ |\det g|^s dg\,.$$

Changing g into mg, we get

$$\omega_\pi\omega_{\pi'}(a)^n\ |a|^{n^2(s-\frac{1}{2})}\gamma(s,\pi\times\pi',\psi)\omega_\pi(-1)^{n-1}\Psi(s,W_m,W'_m,\Phi)\,.$$

Comparing with the functional equation for ψ_a, we do find

$$\gamma(s,\pi\times\pi',\psi_a)=\omega_\pi\omega_{\pi'}(a)^n\ |a|^{n^2(s-1/2)}\gamma(s,\pi\times\pi',\psi)\,.$$

In particular, for $a=-1$, we get

$$\gamma(s,\pi\times\pi',\overline{\psi})=\omega_\pi\omega_{\pi'}(-1)^n\ \gamma(s,\pi\times\pi',\psi)\,.$$

Now suppose $W'\in\mathcal{W}(\pi':\psi)$, $W\in\mathcal{W}(\pi:\overline{\psi})$. Then

$$\Psi(1-s,\widetilde{W'},\widetilde{W},\mathcal{F}_\psi(\Phi))=\omega_\pi\omega_{\pi'}(-1)\Psi(1-s,\widetilde{W},\widetilde{W'},\mathcal{F}_{\overline{\psi}}(\Phi))\,.$$

Applying the functional equation for $\overline{\psi}$, we get

$$\omega_{\pi'}(-1)\gamma(s,\pi\times\pi',\overline{\psi})\omega_\pi(-1)^n\Psi(s,W,W',\Phi)\,.$$

Applying the relation between $\gamma(s,\pi\times\pi',\overline{\psi})$ and $\gamma(s,\pi\times\pi',\psi)$, we find

$$=\omega_{\pi'}(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s,W',W,\Phi)\,.$$

Thus we see that indeed

$$\gamma(s,\pi'\times\pi,\psi)=\gamma(s,\pi\times\pi',\psi)\,.$$

THEOREM 2.3. *Let the notations be as in Theorem 2.1.*

(i) *Suppose $n>n'$. Then each integral $\Psi_j(s,W_f,W_{f'})$ belongs to $\mathcal{L}(\sigma_u\otimes\sigma'_{u'})$ and the map*

$$(f,f')\mapsto\Psi_j(s,W_f,W_{f'})$$

from $I_{\sigma,u}\times I_{\sigma',u'}$ to $\mathcal{L}(\sigma_u\otimes\sigma'_{u'})$ is continuous.

(ii) *Suppose $n=n'$. Then each integral $\Psi(s,W_f,W_{f'},\Phi)$ belongs to $\mathcal{L}(\sigma_u\otimes\sigma'_{u'})$ and the map*

$$(f,f',\Phi)\mapsto\Psi_j(s,W_f,W_{f'},\Phi)$$

from $I_{\sigma,u}\times I_{\sigma',u'}\times\mathcal{S}(F^n)$ to $\mathcal{L}(\sigma_u\otimes\sigma'_{u'})$ is continuous.

We can also consider the projective tensor product of the representations $\pi_{\sigma,u}$ and $\pi_{\sigma',u'}$. It is equivalent to an induced representation of $GL(n,F)\times GL(n',F)$. The linear form $\lambda\otimes\lambda'$ extends to a continuous linear form on the tensor product $I_{\sigma,u}\widehat{\otimes}I_{\sigma',u'}$. For $f\in I_{\sigma,u}\widehat{\otimes}I_{\sigma',u'}$, we can set

$$W(g,g')=\lambda(\pi_{\sigma,u}(g)\otimes\pi_{\sigma',u'}(g')f)$$

and then define integrals containing W. If $n>n'$,

$$\Psi_j(s,W)=\int W\left[\begin{pmatrix} g & 0 & 0\\ X & 1_j & 0\\ 0 & 0 & 1_{n-n'-j}\end{pmatrix},g\right]|\det g|^{s-\frac{n-n'}{2}}dXdg\,.$$

If $n=n'$

$$\Psi(s,W,\Phi)=\int W(g,g)\ \Phi[(0,0,\ldots,0,1)g]\ |\det g|^s dg\,.$$

The assertions of Theorems 2.1 and 2.3 are still true for these more general integrals.

At this point, we recall a result of [**4**]. The authors define a functor $V \mapsto \Psi_\psi(V)$ from the category of the Casselman-Wallach representation to the category of finite dimensional complex vector spaces. The functor is exact and the dual of $\Psi_\psi(V)$ can be functorially identified with the space of (continuous) ψ form on V. As a result, we have the following extension lemma.

LEMMA 2.4. *Let V be a Casselman-Wallach representation and V_1 a closed invariant subspace of V. Any ψ form λ_1 on V_1 extends into a ψ form on V.*

Now let us consider an induced representation $(\pi_{\sigma,u}, I_{\sigma,u})$. We state a useful lemma.

LEMMA 2.5. *Suppose further that*

$$\Re u_1 \leq \Re u_2 \leq \cdots \leq \Re u_r\,.$$

Let $V = V_1/V_2$ be an irreducible subquotient of $I_{\sigma,u}$. Suppose that V is ***generic****, that is, admits a non-zero ψ linear form. Then $V_2 = 0$.*

PROOF. Let λ be a non-zero ψ form on $I_{\sigma,u}$. The map

$$f \mapsto W_f, I_{\sigma,u} \to \mathcal{W}(\pi_{\sigma,u} : \psi)$$

is then injective and thus bijective. The simplest proof of this fact is to adapt the methods of [**17**] where the p-adic case is treated. In particular, the linear form λ cannot vanish identically on a closed invariant subspace of $I_{\sigma,u}$. Let $V = V_1/V_2$ be an irreducible subquotient of $I_{\sigma,u}$. Thus V_1 and V_2 are closed invariant subspaces. Suppose that V admits a non-zero ψ linear form λ_1 which we can view as a linear form on V_1 which vanishes on V_2. By Lemma 2.4, it extends to a ψ linear form on $I_{\sigma,u}$. The extension is a scalar multiple of λ. Thus $V_2 = 0$. □

In particular, consider the Langlands quotient $V = I_{\sigma,u}/V_0$ where V_0 is the maximal invariant subspace $\neq I_{\sigma,u}$. We see that if V is generic then $V_0 = 0$, that is, $I_{\sigma,u}$ is irreducible and generic. In general, it follows that $I_{\sigma,u}$ has a unique minimal invariant subspace which is generic. For an algebraic proof of these results, see [**24**]. See also [**5**] which gives analogous results for general p-adic reductive groups.

We have now a more precise result.

THEOREM 2.6. *Let (σ, u) and (σ', u') be two pairs such that*

$$\Re u_1 \leq \Re u_2 \leq \cdots \leq \Re u_r\,,\ \Re u'_1 \leq \Re u'_2 \leq \cdots \leq \Re u'_{r'}\,.$$

(i) *Suppose $n > n'$. Then for every m in $\mathcal{L}(\sigma_u \otimes \sigma'_{u'})$, there is $f \in I_{\sigma,u}\widehat{\otimes} I_{\sigma',u'}$ such that*

$$m(s) = \Psi(s, W)\,,$$

with

$$W(g, g') = \lambda(\pi_{\sigma,u}(g) \otimes \pi_{\sigma',u'}(g')f)\,.$$

(ii) *Suppose $n = n'$. Then for every m in $\mathcal{L}(\sigma_u \otimes \sigma'_{u'})$, there are elements $f_i \in I_{\sigma,u}\widehat{\otimes} I_{\sigma',u'}$ and Schwartz functions Φ_i such that*

$$m(s) = \sum_i \Psi(s, W_i, \Phi_i)$$

where

$$W_i(g, g') = \lambda(\pi_{\sigma,u}(g) \otimes \pi_{\sigma',u'}(g')f_i)\,.$$

Finally, when $n' = n-1$ or $n' = n$ we have an even more precise result. A **standard** Schwartz function on $M(a \times b, F)$ is a function of the form

$$\Phi(x) = P(x) \exp\left(-\pi \mathrm{Tr}(\ {}^t x.x)\right)\ ,\ F = \mathbb{R}\,,$$

and

$$\Phi(x) = P(x, \overline{x}) \exp\left(-2\pi \mathrm{Tr}(\ {}^t \overline{x}.x)\right)\ ,\ F = \mathbb{C}\,,$$

where P is a polynomial. The character ψ is said to be **standard** if

$$\begin{array}{rcl}\psi(x) & = & e^{\pm 2i\pi x}\ ,\ F = \mathbb{R} \\ \psi(x) & = & e^{\pm 2i\pi (x + \overline{x})}\ ,\ F = \mathbb{C}\,.\end{array}$$

If ψ is standard and Φ is standard, then $\mathcal{F}_\psi(\Phi)$ is standard.

THEOREM 2.7. *Suppose the induced representations $I_{\sigma,u}$ and $I_{\sigma',u'}$ are irreducible.*

(i) *Suppose $n' = n-1$. Then there is $f \in I_{\sigma,u} \otimes I_{\sigma',u'}$, $K_n \times K_{n-1}$ finite, such that*

$$L(s, \sigma_u \otimes \sigma'_{u'}) = \Psi(s, W)\,,$$

with

$$W(g, g') = \lambda(\pi_{\sigma,u}(g) \otimes \pi_{\sigma',u'}(g')f)\,.$$

(ii) *Suppose $n = n'$. Then there are elements $f_i \in I_{\sigma,u} \otimes I_{\sigma',u'}$, $K_n \times K_n$ finite, and standard Schwartz functions Φ_i such that*

$$L(s, \sigma_u \otimes \sigma'_{u'}) = \sum_i \Psi(s, W_i, \Phi_i)$$

where

$$W_i(g, g') = \lambda(\pi_{\sigma,u}(g) \otimes \pi_{\sigma',u'}(g')f_i)\,.$$

REMARK 2.8. If

$$\Re u_1 \leq \Re u_2 \leq \cdots \leq \Re u_r\,,\ \Re u'_1 \leq \Re u'_2 \leq \cdots \leq \Re u'_{r'},$$

the result should be true even if the representations are not both irreducible, but we have not proved this stronger assertion.

3. Majorization of Whittaker functions

3.1. Norms. Let us introduce some convenient notations. If X is a **real or complex** matrix of any size, we set

$$||X||_e := \mathrm{Tr}(X\ {}^t\overline{X})^{1/2}\,.$$

The index e indicates that this is the Euclidean norm. It is useful to keep in mind that

$$(1 + ||X||_e^2 + ||Y||_e^2)^2 \geq (1 + ||X||_e^2)\ (1 + ||Y||_e^2) \geq (1 + ||X||_e^2 + ||Y||_e^2)\,.$$

Thus, for $g \in GL(n, \mathbb{C})$,

$$||g||_H = ||g||_e^2 + ||g^{-1}||_e^2\,.$$

The index H indicates that this is a norm function in the sense of Harish-Chandra. We often drop the index H when this does not create confusion. For $g \in GL(n, \mathbb{C})$ (or $g \in GL(n, \mathbb{R})$) $k_i \in U(n)$ (or $k \in O(n)$),

$$||g||_H = ||g^{-1}||_H = ||g^\iota||_H = ||k_1 g k_2||_H \geq 2n\,.$$

Furthermore, if $g = uak$ with a diagonal, u upper triangular with unit diagonal, $k \in U(n)$, then we set

$$||g||_I = ||a||_H \,.$$

The index I indicates that this definition depends on the Iwasawa decomposition. Thus $||g||_H \geq ||g||_I$. When we integrate over a quotient $N_n \backslash G_n$, we can take $g \in A_n K_n$ and then $||g||_H = ||g||_I$.

If Z is a complex $a \times b$ matrix and $h \in G_a(\mathbb{C})$, then

$$\frac{1}{1 + ||hZ||_e^2} \leq \frac{||h||_H}{1 + ||Z||_e^2} \,. \tag{3.1}$$

Indeed,

$$||Z||_e^2 = ||h^{-1}hZ||_e^2 \leq ||h^{-1}||_e^2 ||hZ||_e^2 \leq ||h||_H \; ||hZ||_e^2 \,.$$

Thus

$$\frac{1}{(1 + ||hZ||_e^2)} \leq \frac{1}{(1 + ||h||_H^{-1} \; ||Z||_e^2)} = \frac{||h||_H}{(||h||_H + ||Z||_e^2)} \leq \frac{||h||_H}{(1 + ||Z||_e^2)} \,.$$

Our assertion follows.

For $Z = 1_n + U \in N_n$, there is a constant C and an integer M such that

$$||Z||_H \leq C(1 + ||U||_e^2)^M \,. \tag{3.2}$$

Indeed,

$$||Z||_e^2 \preceq (1 + ||U||_e^2) \,.$$

Recall that this notation means that there is a constant $D > 0$ such that, for all U,

$$||Z||_e^2 \leq D(1 + ||U||_e^2) \,.$$

Also

$$Z^{-1} = 1 - U + U^2 + \cdots (-1)^{n-1} U^{n-1} \,.$$

Thus

$$||Z^{-1}||_e \leq (1 + ||U||_e + ||U||_e^2 + \cdots ||U||_e^{n-1}) \,.$$

If $||U||_e \geq 1$, then

$$||Z^{-1}||_e^2 \preceq (1 + ||U||_e^2)^M$$

for a suitable M. If $||U||_e < 1$, then

$$||Z^{-1}||_e^2 \preceq 1 \,.$$

Our assertion follows.

We define three functions $\xi_{h,n}$, $\xi_{i,n}$ and $\xi_{s,n}$ on $GL(n, \mathbb{C})$ (or $GL(n, \mathbb{R})$) in the following way. If $g = uak$, $a = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{R}$, u upper triangular with unit diagonal, $k \in U(n)$, then

$$\xi_{h,n}(g) = \prod_{i=1}^{n-1} (1 + (a_i a_{i+1}^{-1})^2) \tag{3.3}$$

$$\xi_{i,n}(g) = \xi_h(g)(1 + (a_n)^2) \tag{3.4}$$

$$\xi_{s,n}(g) = \prod_{i=1}^{n} (1 + a_i^2) \,. \tag{3.5}$$

We will often drop the index n if this does create confusion. The index h stands for homogeneous, the index i for inhomogeneous and the index s for simple. Note that

$$\prod_{i=1}^{n-1}(1+(a_i a_{i+1}^{-1})^2)^i\ (1+a_n^2)^n \geq \prod_{i=1}^{n}(1+a_i^2)\,. \tag{3.6}$$

It follows that

$$\xi_i(g)^n \geq \xi_s(g)$$

and, for a suitable integer m,

$$\xi_h(a)^m(1+a_n^2)^m \succeq (1+||a||_e^2) \tag{3.7}$$

for a diagonal with positive (or simply real) entries. Also we have, for a suitable constant $C>0$,

$$\xi_{h,n}\begin{pmatrix} g & 0 \\ 0 & 1_m \end{pmatrix} = C\xi_{i,n-m}(g)$$

and, for a suitable integer r,

$$\xi_{h,n}\begin{pmatrix} g & 0 \\ 0 & 1_m \end{pmatrix}^r \geq \xi_{s,n-m}(g)\,. \tag{3.8}$$

A direct consequence of (3.1) is the following lemma.

Lemma 3.1. *Let Φ be a Schwartz function on the space of $a\times b$ matrices with entries in F.*

(i) *For every integer N, there is a constant C_N, such that, for every $h \in G_a(F)$,*

$$|\Phi(hZ)| \leq C_N \frac{||h||_H^N}{(1+||Z||_e^2)^N}\,.$$

(ii) *There is an integer N and a constant C such that, for every $h \in G_a(F)$,*

$$\int |\Phi(hZ)|dZ \leq C||h||_H^N\,.$$

We will also use the following elementary lemmas.

Lemma 3.2. *Let Φ be a Schwartz function on the space of $n\times n$ matrices with entries in F. There is an integer M and for each N a constant C such that, for every diagonal matrix with positive entries a,*

$$\int_{N_n} |\Phi|(aZ)dZ \leq C\frac{||a||_H^M}{(1+||a||_e^2)^N}\,.$$

Proof. We write $Z = 1_n + U$ with U upper triangular with 0 diagonal. Then $dZ = dU$ and

$$(1+||a(1+U)||_e^2)^2 = (1+||a||_e^2+||aU||_e^2)^2 \geq (1+||a||_e^2)(1+||aU||_e^2)\,;$$

$$\begin{aligned} |\Phi(a(1_n+U))| &\preceq \frac{1}{(1+||a||_e^2)^N}\ \frac{1}{(1+||aU||_e^2)^M} \\ &\preceq \frac{||a||_H^M}{(1+||a||_e^2)^N(1+||U||_e^2)^M} \end{aligned}$$

for $N >> 0, M >> 0$. The lemma follows. □

LEMMA 3.3.

(i) *Given M, there are constants $A, B, C > 0$ such that the integral*

$$\int_{G_n(F)} \frac{||h||_H^M \ |\det h|^t}{(1+||h||_e^2)^N} d^\times h$$

converges if $N > A, CN > t > B$.

(ii) *Given M, there is B such that the integral*

$$\int_{G_n(F)} ||h||_H^M \Phi(h) \ |\det h|^t d^\times h$$

converges absolutely for all Schwartz functions Φ on $M(n \times n, F)$ and $t > B$.

PROOF. The second assertion follows from the first. We prove the first assertion. We set

$$h = k(a+U)$$

with $k \in K_n$, a diagonal with positive entries, and U upper triangular with zero diagonal. Then

$$dh = dkJ(a)dadU$$

where $J(a)$ is a Jacobian character. The integrand is independent of k so we may integrate over k. Next, for a suitable M_1,

$$\begin{aligned} ||h||_H &= ||a(1+a^{-1}U)||_H \\ &\preceq ||a||_H||(1+a^{-1}U)||_H \\ &\preceq ||a||_H(1+||a^{-1}U||_e^2)^{M_1} \\ &\preceq ||a||_H(1+||a||_H \, ||U||_e^2)^{M_1} \\ &\preceq ||a||_H^{M_1+1}(1+||U||_e^2)^{M_1} \, . \end{aligned}$$

Also

$$\begin{aligned} (1+||a+U||_e^2)^{N_1+N_2} &= (1+||a||_e^2+||U||_e^2)^{N_1+N_2} \\ &\geq (1+||a||_e^2)^{N_1} \ (1+||U||_e^2)^{N_2} \, . \end{aligned}$$

Thus we are reduced to the convergence of a product of two integrals:

$$\int \frac{1}{(1+||U||_e^2)^{N_2-MM_1}} dU \, ,$$

$$\int \frac{||a||_H^{M(M_1+1)} J(a)|\det a|^t}{(1+||a||_e^2)^{N_1}} da \, .$$

The first integral converges for $N_2 >> 0$. For the second integral, we apply the following lemma.

LEMMA 3.4. *Let χ be a positive character of $A_n(\mathbb{R})$ and M be given. There are $A, B, C > 0$ such that the integrals*

$$\int \frac{||a||_H^M \chi(a)|\det a|^t}{(1+||a||_e^2)^N} da \, ,$$

$$\int \frac{||a||_H^M \chi(a)|\det a|^t}{\xi_s(a)^N} da$$

converges for $N > A, CN > t > B$.

PROOF. It suffices to prove our assertion for the second integral. Now $||a||_H^M$ is a sum of positive characters. Thus we may assume $M = 0$. Then the integral is a product

$$\prod_i \int_{\mathbb{R}_+^\times} \frac{|a_i|^{t+t_i}}{(1+a_i^2)^N} d^\times a_i \,.$$

The integral converges for

$$t > \max(-t_i), N > \max(t_i), N > t \,.$$

□

LEMMA 3.5. *Let $M \geq 0$ be an integer and Φ a Schwartz function on F^n. There are $A, B, C > 0$ such that the following integrals converge absolutely for $N > A, NC > t > B$.*

$$\int_{N_n \backslash G_n} \xi_{s,n}(g)^{-N} ||g||_I^M |\det g|^t dg \,, \tag{3.9}$$

$$\int_{N_n \backslash G_n} \xi_{h,n}(g)^{-N} ||g||_I^M \Phi[(0,0,\ldots,0,1)g] |\det g|^t dg \,. \tag{3.10}$$

PROOF. For the first integral, we can write $g = ak$. Then $dg = J(a) da dk$. After integrating over K_n we are reduced to the previous lemma. For the second integral, we again write $g = ak$. Then, for any N,

$$|\Phi[(0,0,\ldots,0,1)g]| \leq C_N (1+a_n^2)^{-N} \,.$$

Now

$$\xi_{h,n}(g)^m (1+a_n^2)^m \geq \xi_{s,n}(g)$$

for a suitable m. Thus we are reduced to the case of the first integral. □

LEMMA 3.6. *If M is given and N is sufficiently large, the integral*

$$\int_{G_n} \xi_{s,n}(h^{-1})^{-N} \; ||h||_H^M \; (1+||h||_e^2)^{-N} d^\times h$$

converges.

PROOF. We write

$$h = k(b+V)$$

where b is diagonal with positive entries and V upper triangular with 0 diagonal. Then $d^\times h = J_1(b) db dV dk$. For some m,

$$(1+||h||_e^2)^m \geq (1+||b||_e^2)(1+||V||_e^2) \,.$$

On the other hand, for a suitable M_1,

$$||h||_H^M \preceq ||b||_H^{M_1} (1+||U||_e^2)^{M_1} \,.$$

Finally,

$$\xi_{s,n'}(h_1^{-1}) = \prod_{i=1}^{n'} (1+b_i^{-2}) \,.$$

The convergence of the integral for $N >> 0$ easily follows. □

3.2. Majorization for one representation. Let $F=\mathbb{R}$ or $F=\mathbb{C}$. Let (π, V) be a smooth representation of $G(F)$ of **moderate growth** on a complex Frechet space V. Let λ be a non zero ψ form on V. To each $v \in V$, we associate the function W_v on $G(F)$ defined by

$$W_v(g) = \lambda(\pi(g)v)\,.$$

We want to obtain a majorization of the functions W_v.

By hypothesis, there is a continuous semi-norm μ on V such that $|\lambda(v)| \leq \mu(v)$. Thus $|W_v(g)| \leq \mu(\pi(g)v)$. There is another continuous ν and an integer M such that $\mu(\pi(g)v) \leq ||g||_H^M \nu(v)$. Thus we have the following coarse majorization.

LEMMA 3.7. *There is M and a continuous semi-norm ν on V such that, for all $v \in V$ and all g,*

$$|W_v(g)| \leq ||g||_H^M \nu(v)\,.$$

We improve on the previous majorization. For $h \in G_n(F)$,

$$(\rho(h)W_v)(g) = W_v(gh) = \lambda(\pi(gh)v) = \lambda(\pi(g)\pi(h)v) = W_{\pi(h)v}(g)\,.$$

Similarly, for $X \in \mathfrak{U}(G)$,

$$(\rho(X)W_v)(g) = W_{d\pi(X)v}(g)\,.$$

Thus

$$|(\rho(X)W_v)(g)| \leq ||g||^M \ \nu(d\pi(X)v)\,.$$

We also note that

$$\begin{aligned} |W_v(uak)| &= |W_v(ak)| \leq ||ak||^M \nu(v) = ||a||^M \nu(v) \\ |(\rho(X)W_v)(uak)| &\leq ||a||^M \nu(d\pi(X)v)\,. \end{aligned}$$

Let X_i be the elements of Lie(N_n) corresponding to the simple roots $\alpha_i(a) = a_i/a_{i+1}$, $1 \leq i \leq n-1$. Thus the only nonzero entry of X_i is the entry in the i-th row and $i+1$-th column which is equal to 1. Then

$$\lambda(d\pi(X_i)v) = mv\,,$$

where $m \in \mathbb{C}^\times$ depends only on the choice of ψ. Moreover,

$$\pi(a)d\pi(X_i)v = d\pi(aX_ia^{-1})\pi(a)v = \alpha_i(a)d\pi(X_i)\pi(a)v\,.$$

Thus

$$\lambda(\pi(a)d\pi(X_i)v) = m\ \alpha_i(a)\lambda(\pi(a)v)\,.$$

More generally, if

$$Y = X_1^{N_1}X_2^{N_2}\cdots X_{n-1}^{N_{n-1}}$$

and $N = \sum_i N_i$, then

$$\lambda(\pi(a)d\pi(Y)v) = m^N \left(\prod \alpha_i(a)^{N_i}\right)\ \lambda(\pi(a)v)\,.$$

Let $\mathfrak{U}_N(G)$ be the subspace of $\mathfrak{U}(G)$ spanned by the products of at most N elements of Lie(G). Let (X_θ) be a basis of $\mathfrak{U}_N(G)$. We may write

$$\mathrm{Ad}(k^{-1})Y = \sum_\theta \xi_\theta(k)X_\theta\,.$$

Then, for any v,

$$\lambda(\pi(a)d\pi(Y)\pi(k)v) = \lambda(\pi(a)\pi(k)d\pi(\mathrm{Ad}k^{-1}Y)v)$$
$$= \sum_\theta \xi_\theta(k)\lambda(\pi(a)\pi(k)d\pi(X_\theta)v)\,.$$

Thus

$$m^N \prod \alpha_i(a)^{N_i}\ \lambda(\pi(a)\pi(k)v) = \sum_\theta \xi_\theta(k)\lambda(\pi(a)\pi(k)d\pi(X_\theta)v)$$

or

$$m^N \prod \alpha_i(a)^{N_i} W_v(ak) = \sum_\theta \xi_\theta(k) W_{d\pi(X_\theta)v}(ak)\,. \tag{3.11}$$

Replacing v by the vector $d\pi(X)v$ with $X \in \mathfrak{U}(G)$, we obtain the formula

$$m^N \prod \alpha_i(a)^{N_i} \rho(X) W_v(ak) = \sum_\theta \xi_\theta(k) W_{d\pi(X_\theta X)v}(ak)\,. \tag{3.12}$$

Since the functions ξ_θ are bounded by a constant, we get

$$\left| m^N \prod \alpha_i(a)^{N_i} \rho(X) W_v(ak) \right| \leq C||a||^M \sum_\theta \nu(d\pi(X_\theta X)v)\,.$$

This gives us the result we need.

Proposition 3.1. *There is an integer $M \geq 0$, and, for every $X \in \mathfrak{U}(G)$ and every integer N, a continuous semi-norm $\nu_{X,N}$ on V, with the following property. For every $g \in G$, $v \in V$,*

$$|\rho(X)W_v(g)|\ \xi_h(g)^N \leq ||g||_I^M \nu_{X,N}(v)\,.$$

We will need the following more general corollary.

Lemma 3.8. *For every integer N, and every $X \in \mathfrak{U}(G_n)$, there are integers M_1 and M_2 and a continuous semi-norm ν such that*

$$|\rho(g_2)\rho(X)W_v(g_1)|\xi_h(g_1)^N \leq ||g_1||_I^{M_1}\ ||g_2||_H^{M_2}\ \nu(v)\,.$$

Proof. Indeed,

$$(\rho(g_2)\rho(X)W_v)(g_1) = (\rho(Adg_2X)\rho(g_2)W_v)(g_1) = (\rho(Adg_2X)W_{\pi(g_2)v})(g_1)\,.$$

Suppose $X \in \mathfrak{U}_N(G)$. Let again X_θ be a basis of the space $\mathfrak{U}_N(G)$. Then

$$Adg_2X = \sum_\theta \xi_\theta(g_2)X_\theta\,.$$

There is M_1 such that, for all θ,

$$|\xi_\theta(g_2)| \leq ||g||^{M_1}\,.$$

Thus we are reduced to estimating $\rho(X)W_{\pi(g_2)v}(g_1)$. By the previous lemma, there is a continuous semi-norm ν such that

$$|\rho(X)W_{\pi(g_2)v}(g_1)|\xi_h(g_1)^N \leq ||g_1||_I^M \nu(\pi(g_2)v)\,.$$

But

$$\nu(\pi(g_2)v) \leq ||g_2||_H^{M_2}\nu'(v)$$

where ν' is another continuous semi-norm. Our assertion follows. □

We can obtain similar majorizations for the function

$$\widetilde{W_v}(g) := W_v(w_n g^\iota) .$$

Indeed, consider the representation π^ι on V defined by

$$\pi^\iota(g) = \pi(g^\iota) .$$

Set

$$\widetilde{\lambda}(v) := \lambda(\pi(w_n)v) .$$

Then

$$\widetilde{\lambda}(\pi^\iota(u)v) = \theta_{\overline{\psi}}(u)\widetilde{\lambda}(v) ,$$

that is, $\widetilde{\lambda}$ is a $\overline{\psi}$ form. Then

$$\widetilde{W_v}(g) = \widetilde{\lambda}(\pi^\iota(g)v) .$$

Replacing π by π^ι, we obtain majorizations for $\widetilde{W_v}$.

3.3. Majorization for a family of representations. Let

$$\mu = (\mu_1, \mu_2, \dots, \mu_n)$$

be an n-tuple of characters of $F^\times$. We assume they are **normalized**, that is, they have a trivial restriction to $\mathbb{R}_+^\times$. Let $u = (u_1, u_2, \dots, u_n) \in \mathbb{C}^n$. We denote by $(\pi_{\mu,u}, I_{\mu,u})$ the representation of $G(F)$ induced by the character

$$\mu_u(a) = \prod_i \mu_i(a_i)\ |a_i|_F^{u_i}$$

of $A_n(F)$, regarded as a character of the group of lower triangular matrices. The space $I_{\mu,u}$ is the space of C^∞ complex-valued functions f on $G(F)$ such that

$$f(vak) = \delta_n^{-1/2}(a)\mu_u(a)f(k)$$

for all $v \in \overline{N}_n$, $a \in A(F)$, $k \in K$. The representation is by right shifts. Alternatively, we may identify the space of functions in $I_{\mu,u}$ to the space of their restrictions to K_n. It is a space I_μ, independent of u. Then we denote by π_u the representation acting on the space I_μ. The topology of I_μ is the one given by the semi-norms

$$\sup_{k \in K_n} |\rho(X)f(k)| ,$$

with $X \in \mathfrak{U}(K)$. We stress that I_μ is regarded as a space of functions on K and only derivatives with respect to elements $X \in \mathfrak{U}(K)$ appear in the definition of the topology. Each representation $\pi_{\mu,u}$ is a C^∞ representation of moderate growth on the space I_μ.

Recall that there is for each u a non-zero ψ form λ_u on I_μ. This form is unique within a scalar factor. Moreover, one can choose the linear form in such a way that the map

$$(u, f) \mapsto \lambda_u(f)$$

is continuous and for each f, the map $u \mapsto \lambda_u(f)$ is holomorphic in u (Theorem 15.4.1 in [**26**] II).

For every $f \in I_\mu$, we set

$$W_{u,f}(g) = \lambda_u(\pi_u(g)f) .$$

We need to obtain majorizations of the functions $W_{u,f}$, similar to the ones of the previous subsection, but uniform with respect to u, for u in a compact set. To do so, we need to show that the representations $\pi_{\mu,u}$ are of moderate growth, **uniformly for u in a compact set**. This is known, but for the sake of completeness we provide complete details. We begin with a series of lemmas on semi-norms.

LEMMA 3.9. *Set*

$$\nu_0(f) = \sup_K |f(k)| \, .$$

Given a compact set $\Omega \subset \mathbb{C}^n$, there is M such that for $u \in \Omega$ and any $f \in I_\mu$

$$\nu_0(\pi_u(g)f) \leq ||g||^M \nu_0(f) \, .$$

PROOF. Indeed, we may write

$$kg = vak_1 \, , v \in \overline{N_n} \, , a \in A(F) \, , \, k, k_1 \in K_n \, .$$

Then

$$f_u(kg) = \mu_u(a)\delta_n^{-1/2}(a)f(k_1) \, .$$

Now for

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_n)$$

and $u \in \Omega$, we have

$$\left|\mu_u(a)\delta_n^{-1/2}\right| \leq \prod (a_i^2 + a_i^{-2})^N$$

for a suitable N. In turn

$$\prod (a_i^2 + a_i^{-2})^N \leq ||a||^M$$

for a suitable M. Moreover

$$||a|| \leq ||va|| = ||kgk_1^{-1}|| = ||g|| \, .$$

Our assertion follows. □

LEMMA 3.10. *Let Ω be a compact set of $\mathbb{C}^n$. For every $X \in \mathfrak{U}(\mathfrak{g})$, there is a continuous semi-norm ν_X on I_μ such that , for every $u \in \Omega$, $f \in I_\mu$,*

$$\nu_0(d\pi_u(X)f) \leq \nu_X(f) \, .$$

PROOF. Say $X \in \mathfrak{U}_N(G)$. Let ϕ be an element of I_μ. Then

$$d\pi_u(X)\phi(k) = \rho(X)\phi(k) = \lambda(\mathrm{Ad}(k)(\check{X}))\phi(k) \, .$$

Let X_θ be a basis of $\mathfrak{U}_N(G)$. Then

$$\mathrm{Ad}(k)(\check{X}) = \sum_\theta \xi_\theta(k) X_\theta$$

where the functions ξ_θ are uniformly bounded on K. Thus it suffices to bound

$$\lambda(X)\phi(k), X \in \mathfrak{U}_N(G) \, .$$

We can write X has a sum of terms of the form

$$YHZ \, , \, Y \in \mathfrak{U}_N(\overline{N}_n), H \in \mathfrak{U}_N(A), Z \in \mathfrak{U}_N(K) \, .$$

Now $\lambda(Y)\phi = 0$ if Y is a product of elements of $\mathrm{Lie}(\overline{N})$. Thus we may as well assume $Y = 1$. Now

$$\lambda(H)\phi = \tau_u(H)\phi$$

where $\tau_u : \mathfrak{U}(A) \to \mathbb{C}$ is an homomorphism depending on u. If $u \in \Omega$, $\tau_u(H)$ is bounded. Thus we are reduced to estimate

$$\lambda(Z)\phi(k)$$

for $Z \in \mathfrak{U}_N(K)$. As before, if Z_θ is a basis of $\mathfrak{U}_N(K)$, this has the form

$$\sum_\theta \xi_\theta(k)\rho(Z_\theta)\phi(k)$$

where the ξ_θ are bounded. This is bounded by a constant times

$$\sum_\theta \sup_K |\rho(Z_\theta)\phi(k)| \,.$$

The lemma follows. □

Lemma 3.11. *Let ν be a continuous semi-norm on I_μ. Let Ω be a compact subset of $\mathbb{C}^n$. Then there is an integer M and a continuous semi-norm $\widetilde{\nu}$ on I_μ such that*

$$\nu(\pi_u(g)f) \leq ||g||^M \widetilde{\nu}(f)$$

for all f, all $u \in \Omega$, and all $g \in G$.

Proof. We may assume

$$\nu(f) = \nu_0(\rho(Y)f)$$

with $Y \in \mathfrak{U}_N(K)$ because the topology of I_μ is defined by these semi-norms. Then

$$\nu(\pi_u(g)f) = \nu_0(d\pi_u(Y)\pi_u(g)f)\,.$$

Let X_θ be a basis of $\mathfrak{U}_N(G)$. Then

$$d\pi_u(Y)\pi_u(g) = \pi_u(g)d\pi_u(\mathrm{Ad}(g^{-1})Y) = \pi_u(g)\sum_\theta \xi_\theta(g)d\pi_u(X_\theta)\,.$$

Each function ξ_θ is bounded by a power of $||g||$. Thus $\nu(\pi_u(g)f)$ is bounded by a power of $||g||$ times

$$\sum_\theta \nu_0(\pi_u(g)d\pi_u(X_\theta)f)\,.$$

By the first lemma, for $u \in \Omega$, this is bounded by

$$||g||^M \sum_\theta \nu_0(d\pi_u(X_\theta)f)\,.$$

Now we apply the previous lemma. □

Lemma 3.12. *Let ν be a continuous semi-norm on I_μ and Ω be a compact subset of $\mathbb{C}^n$. There is an integer M with the following property. For $X \in \mathfrak{U}(G)$, there is a continuous semi-norm $\widetilde{\nu}$ such that, for all $u \in \Omega$ and $f \in I_\mu$,*

$$\nu(\pi_u(g)d\pi_u(X)f) \leq ||g||^M \widetilde{\nu}(f)\,.$$

Proof. By the penultimate lemma,

$$\nu(\pi_u(g)f) \leq ||g||^M \mu(f)$$

for a suitable M and a suitable continuous semi-norm μ. Thus

$$\nu(\pi_u(g)d\pi_u(X)f) \leq ||g||^M \mu(d\pi_u(X)f)\,.$$

To continue we may assume that

$$\mu(f) = \sum_\alpha \nu_0(\rho(Y_\alpha)f)$$

with $Y_\alpha \in \mathfrak{U}(K)$. Then

$$\mu(d\pi_u(X)f) = \sum_\alpha \nu_0(d\pi_u(Y_\alpha X)f)$$

and our assertion follows from the previous lemma. □

Now we obtain coarse majorizations for the Whittaker functions, uniform for u in a compact set.

PROPOSITION 3.2. *Let Ω be a compact set of $\mathbb{C}^n$. There is an integer M with the following property. For every $X \in \mathfrak{U}(G)$, there is a continuous semi-norm ν_X on I_μ such that, for all $u \in \Omega$,*

$$|\rho(X)W_{u,f}(g)| \leq ||g||^M \nu_X(f)\,.$$

PROOF. First, because the map $(u,f) \mapsto \lambda_u(f)$ is continuous, for every $u \in \Omega$, there is $A_u > 0$ and a continuous semi-norm μ_u such that for $||u'-u|| < A_u$, we have $|\lambda_{u'}(f)| \leq \mu_u(f)$. Choose u_i, $1 \leq i \leq r$, such that the balls $||u - u_i|| < A_{u_i}$ cover Ω. Let

$$\nu = \sum_i \mu_i\,.$$

Then

$$|\lambda_u(f)| \leq \nu(f)$$

for $u \in \Omega$. Then

$$\rho(X)W_{u,f}(g) = \lambda_u(\pi_u(g)d\pi_u(X)f)$$

is bounded in absolute value by

$$|\nu(\pi_u(g)d\pi_u(X)f)|\,.$$

Our assertion follows from the previous lemma. □

Now we improve on the majorizations.

PROPOSITION 3.3. *Given a compact set Ω of $\mathbb{C}^n$, there is an integer M, and for every integer N and every element $X \in \mathfrak{U}(G)$, a continuous semi-norm $\nu_{X,N}$ such that, for all $u \in \Omega$, all $g \in G$ and $f \in I_\mu$,*

$$|\rho(X)W_{u,f}(g)\xi_h(g)^N| \leq ||g||^M \nu_{X,N}(f)\,.$$

PROOF. We proceed as in the previous section. With the notations of formula (3.12), we have

$$m^N \prod \alpha_i(a)^{N_i} (\rho(X)W_{u,f})(ak) = \sum_\theta \xi_\theta(k)\left(\rho(X_\theta X)W_{u,f}\right)(ak)\,. \tag{3.13}$$

Since the functions ξ_θ are bounded, our assertion follows at once from this formula and the previous proposition. □

3.4. Majorization for a tempered representation. Now assume that π is a unitary irreducible tempered representation and λ is again a continuous ψ form on π. Thus π is equivalent to an induced representation of the form $\pi_{\sigma,u}$ where σ is a t-uple of irreducible unitary representations of the Weil group and u is purely imaginary. Then we have a more precise majorization. First we recall a result of Wallach. Recall that δ_n is the module of the group $B_n(F)$.

PROPOSITION 3.4. *There is a continuous semi-norm μ and $d \geq 0$ such that, for all vectors v,*

$$|W_v(ak)| \leq \delta_n^{1/2}(a)(1 + ||\log a||_e^2)^d \mu(v)\,.$$

This follows from Theorem 15.2.5 of [**26**] II. The proof is the same as the proof of Lemma 15.7.3 in the same reference.

We improve on this majorization.

PROPOSITION 3.5. *For any integer N and any $X \in \mathfrak{U}(G)$, there is a continuous semi-norm $\nu_{X,N}$ such that, for all vectors v,*

$$|\rho(X)W_v(ak)| \leq \xi_h(a)^{-N}\delta_n^{1/2}(a)(1 + ||\log a||_e^2)^d \nu_{X,N}(v)\,.$$

Indeed, we proceed as before. Our assertion follows from the result just recalled and formula (3.13).

3.5. Majorizations for a tensor product. Now let (π, V) and (π', V') be Casselman Wallach representations of G_n and G'_n, respectively. Let λ be a ψ linear form on V and λ' a $\overline{\psi}$ linear form on V'. On the projective tensor product $V\widehat{\otimes}V'$, consider the linear form $\lambda \otimes \lambda'$. To each $\hat{v} \in V\widehat{\otimes}V'$, we associate the function

$$W_{\hat{v}}(g, g') = \lambda \otimes \lambda'(\pi(g) \otimes \pi'(g')\hat{v})\,.$$

We can obtain majorizations for these functions similar to the ones obtained above. We can argue as before, since our arguments are really valid for any quasi-split group, or simply use an argument of continuity and density. For instance, suppose

$$\begin{aligned} |W_v(g)|\xi_h(g)^N &\leq ||g||^{M'}\mu(v) \\ |W_{v'}(g')| &\leq ||g'||^{M'}\mu'(v') \end{aligned}$$

where μ, μ' are continuous semi-norms on V and V', respectively. Let ν be the largest semi-norm on $V\widehat{\otimes}V'$ such that

$$\nu(v \otimes v') \leq \mu(v)\mu'(v')\,.$$

Then, for every $\hat{v} \in V\widehat{\otimes}V'$,

$$|W_{\hat{v}}(g, g')|\xi_h(g)^N \leq ||g||^M \; ||g'||^{M'} \; \nu(\hat{v})\,.$$

Analogous majorizations are true for a tensor product $I_{\mu,u}\widehat{\otimes}I_{\mu',u'}$. The majorizations are uniform for u, u' in compact sets.

4. (σ, ψ) pairs

The main result of these notes is that certain integrals, depending on a complex parameter s, converge for $\Re s > 0$, have analytic continuation as meromorphic functions of s, with prescribed poles, and satisfy a functional equation. It turns out that these assertions are equivalent to a family of identities relating integrals which converge **absolutely**. This is technically very convenient. In particular, when the data at hand depend on some auxiliary parameters, this allows us to prove our assertions by analytic continuation with respect to the auxiliary parameters. In this section, we develop the tools which allow us to establish this equivalence.

4.1. Spaces of rapidly decreasing functions. We denote by $\mathcal{S}(\mathbb{R}_+^\times)$ the space of C^∞ functions ϕ on $\mathbb{R}_+^\times$ such that for every integers $n \geq 0, m \geq 0$,

$$\sup(t^2+t^{-2})^n \left|\left(t\frac{d}{dt}\right)^m \phi(t)\right| < +\infty \, .$$

We introduce the Mellin transform of such a function:

$$\mathcal{M}\phi(s) := \int_0^{+\infty} \phi(t) t^s \frac{dt}{t} \, .$$

Clearly, the Mellin transform of a function $\phi \in \mathcal{S}(\mathbb{R}_+^\times)$ is entire and bounded in any vertical strip of finite width. The Mellin transform of $t\frac{d\phi}{dt}$ is $s\mathcal{M}\phi(s)$ and the Mellin transform of $t^a\phi(t)$ is $\mathcal{M}\phi(s+a)$. In particular, for any polynomial $P(s)$, the product $P(s)\phi(s)$ is also bounded in any vertical strip of finite width. Conversely if $m(s)$ is an entire function of s such that, for any polynomial P, the product $P(s)m(s)$ is bounded at infinity in vertical strips, then the function defined by

$$\phi(t) := \frac{1}{2i\pi}\int_{a-i\infty}^{a+\infty} m(s) t^{-s} ds$$

is in $\mathcal{S}(\mathbb{R}_+^\times)$ and $\mathcal{M}\phi(s) = m(s)$.

We define similarly the space $\mathcal{S}(F^\times)$. It is the space of C^∞ functions on $F^\times$ such that for any $X \in \mathfrak{U}(F^\times)$ and any m

$$\sup\left|(t^2+t^{-2})^m \rho(X)\phi(t)\right| < +\infty \quad \text{if } F = \mathbb{R} \, ,$$
$$\sup\left|(z\overline{z}+z^{-1}\overline{z}^{-1})^m \rho(X)\phi(z)\right| < \infty \quad \text{if } F = \mathbb{C} \, .$$

If ϕ is in $\mathcal{S}(\mathbb{R}_+^\times)$, the function $x \mapsto \phi(|x|_F)$ is in $\mathcal{S}(F^\times)$. The Mellin transform $\mathcal{M}f$ of a function $f \in \mathcal{S}(F^\times)$ is defined by

$$\mathcal{M}f(s) := \int_{F^\times} f(x) \ |x|_F^s \ d^\times x \, .$$

4.2. Definition of (σ, ψ) pairs. Let σ be a complex, finite dimensional, semi-simple representation of the Weil group W_F of F. Let $\widetilde{\sigma}$ be the contragredient representation. Let ψ be a non-trivial character of F. The factors

$$L(s,\sigma), L(s,\widetilde{\sigma}), \epsilon(s,\sigma,\psi)$$

are defined. Let P be a quadratic polynomial of the form

$$P(s) = As^2 + Bs + C \, , \ A > 0 \, , \ B \in \mathbb{R} \, , \ C \in \mathbb{C} \, .$$

Then there exist two functions $h(t), k(t)$ in $\mathcal{S}(F^\times)$, depending only on $|t|_F$, such that

$$\int_{F^\times} h(t)|t|_F^{-s} d^\times t = \frac{\epsilon(s,\sigma,\psi) \ e^{P(s)}}{L(s,\sigma)} \, ,$$
$$\int_{F^\times} k(t)|t|_F^{-s} d^\times t = \frac{e^{P(1-s)}}{L(s,\widetilde{\sigma})} \, .$$

Indeed, $P(1-s)$ is a polynomial of the same form as P. In a vertical strip

$$\{s = x+iy : -a \leq x \leq a\} \, ,$$

the reciprocals of the L-factors are bounded by an exponential factor $e^{D|y|}$ while $e^{P(s)}, e^{P(1-s)}$ are bounded by a factor e^{-Cy^2} with $C > 0$. Thus the right hand sides are entire and their product by any polynomial are bounded in a vertical strip. We say that (h, k) is a (σ, ψ) **pair**.

This notion has some simple formal properties. If (h, k) is a (σ, ψ) pair then, for every $a \in F^\times$, the functions

$$x \mapsto h(xa)\,,\; x \mapsto |a|k(xa^{-1})$$

form a (σ, ψ) pair. Indeed,

$$\begin{aligned} \int h(xa)|x|^{-s}d^\times x &= \frac{\epsilon(s,\sigma,\psi)\ e^{P(s)+s\log|a|}}{L(s,\sigma)} \\ |a|\int k(xa^{-1})|x|^{-s}d^\times x &= \frac{e^{P(1-s)+(1-s)\log|a|}}{L(s,\widetilde{\sigma})}\,. \end{aligned}$$

Also (k, h) is a $(\widetilde{\sigma}, \overline{\psi})$ pair.

Similarly, let σ_i, $i = 1, 2$, be two representations of the Weil group. Set $\tau = \sigma_1 \oplus \sigma_2$. If (h_i, k_i), $i = 1, 2$, are (σ_i, ψ) pairs, then $(h_1 * h_2, k_1 * k_2)$ is a (τ, ψ) pair. Indeed,

$$\begin{aligned} \int h_i(x)|x|^{-s}d^\times x &= \frac{\epsilon(s,\sigma_i,\psi)\ e^{P_i(s)}}{L(s,\sigma_i)} \\ \int k_i(x)|x|^{-s}d^\times x &= \frac{e^{P_i(1-s)}}{L(s,\widetilde{\sigma_i})} \end{aligned}$$

with

$$P_i(s) = A_i s^2 + B_i s + C_i\,,\; A_i > 0\,.$$

Set

$$Q(s) = P_1(s) + P_2(s) = (A_1 + A_2)s^2 + (B_1 + B_2)s + C_1 + C_2\ \ .$$

Then $A_1 + A_2 > 0$ and

$$\begin{aligned} \int h_1 * h_2(x)|x|^{-s}d^\times x &= \frac{\epsilon(s,\sigma_1,\psi)\epsilon(s,\sigma_2,\psi)\ e^{P_1(s)}e^{P_2(s)}}{L(s,\sigma_1)L(s,\sigma_2)} \\ &= \frac{\epsilon(s,\tau,\psi)e^{Q(s)}}{L(s,\tau)}\,, \\ \int k_1 * k_2(x)|x|^{-s}d^\times x &= \frac{e^{Q(1-s)}}{L(s,\widetilde{\tau})}\,. \end{aligned}$$

4.3. The main lemmas.

PROPOSITION 4.1. *Let σ be a representation of the Weil group. Suppose f, f' are measurable functions on $F^\times$. Suppose that there is N such that, for $s \geq N$,*

$$\int |f(x)|\ |x|^s d^\times x < +\infty\,, \quad \int |f'(x)|\ |x|^s d^\times x < +\infty\,.$$

Suppose further that, for any (σ, ψ) pair (h, k),

$$\int f(x)h(x)d^\times x = \int f'(x)k(x)d^\times x\,.$$

Then the Mellin transform of f, defined a priori for $\Re s >> 0$, extends to a holomorphic multiple of $L(s, \sigma)$, bounded at infinity in any vertical strip. Likewise, the

Mellin transform of f' extends to a holomorphic multiple of $L(s,\tilde{\sigma})$, bounded at infinity in any vertical strip. Finally, the equation

$$\frac{\int f'(x)|x|^{1-s}d^\times x}{L(1-s,\widetilde{\sigma})} = \frac{\epsilon(s,\sigma,\psi)\int f(x)|x|^s d^\times x}{L(s,\sigma)}$$

holds in the sense of analytic continuation.

Proof. We first remark that, for any N,

$$|h(x)| \le C|x|^N\,,\ |k(x)| \le C|x|^N\,.$$

So the integrals of the proposition do converge. We set

$$\theta(a) := \int f(x)h(ax)d^\times x\,.$$

Applying the given identity to the pair

$$(x \mapsto h(ax)\,, x \mapsto |a|k(a^{-1}x)),$$

we get

$$\theta(a) = \int f(x)h(ax)d^\times x = |a|\int f'(x)k(a^{-1}x)d^\times x\,. \tag{4.1}$$

Since $h(x)$ is majorized by a constant times $|x|^M$ for any $M \ge N$, the first expression for $\theta(a)$ is majorized by

$$\int |f(x)|\ |h(ax)|d^\times x \le C|a|^M \int |x|^M|f(x)|d^\times x\,.$$

Thus $\theta(a)$ is rapidly decreasing for $|a| \to 0$. By equation (4.1), it is also rapidly decreasing for $|a| \to \infty$. Thus

$$\int \theta(a)|a|^{-s}d^\times a$$

is convergent for all s and defines an entire function of s, bounded in any vertical strip. For $\Re s >> 0$, we use the first expression for θ to compute this integral. We obtain

$$\int \theta(a)|a|^{-s}d^\times a = \iint f(x)h(xa)|a|^{-s}d^\times x d^\times a$$

or, changing a to ax^{-1},

$$\int f(x)|x|^s d^\times x\ \int h(a)|a|^{-s}d^\times a\,.$$

The absolute convergence of this expression for $\Re s$ large enough justifies this computation. Thus

$$\int \theta(a)|a|^{-s}d^\times a = \int f(x)|x|^s d^\times x\,\frac{\epsilon(s,\sigma,\psi)e^{P(s)}}{L(s,\sigma)}\,.$$

Set

$$m(s) := \int f(x)|x|^s d^\times x\,.$$

Then

$$m(s) = e^{-P(s)}L(s,\sigma)\epsilon(s,\sigma,\psi)^{-1}\int \theta(x)|x|^{-s}d^\times x\,.$$

This shows that $m(s)$, defined a priori for $\Re s >> 0$, extends to a meromorphic function of s which is a holomorphic multiple of $L(s,\sigma)$.

On the other hand, using the second expression for $\theta(a)$, we get

$$\begin{aligned}\int \theta(a)|a|^{-s}d^\times a &= \iint f'(x)k(a^{-1}x)|a|^{1-s}d^\times x d^\times a \\ &= \iint f'(x)k(ax)|a|^{-(1-s)}d^\times x d^\times a \\ &= \int f'(x)|x|^{1-s}d^\times x \int k(a)|a|^{-(1-s)}d^\times a\,.\end{aligned}$$

Again the computation is justified for $\Re s$ small enough. Thus

$$\int \theta(a)|a|^{-s}d^\times a = \int f'(x)|x|^{1-s}d^\times x \, \frac{e^{P(s)}}{L(1-s,\widetilde{\sigma})}\,.$$

We conclude that

$$\int f(x)|x|^s d^\times x \, \frac{\epsilon(s,\sigma,\psi)}{L(s,\sigma)} = \int f'(x)|x|^{1-s}d^\times x \, \frac{1}{L(1-s,\widetilde{\sigma})}\,,$$

in the sense of analytic continuation. Both sides extend to entire functions of s.

Now we prove that $m(s)$ is bounded at infinity in vertical strips. Indeed, consider a half strip

$$S = \{s = x + iy : a \le x \le b, y \ge y_0 \ge 1\}\,.$$

We can choose y_0 so large that $L(s,\sigma)$ has no pole in S. Thus in S

$$|m(s)| \le Ce^{Dy^2}$$

with $D > 0$, or enlarging y_0,

$$|m(s)| \le Ce^{y^3}\,.$$

Now if b is large enough, the integral defining $m(s)$ converges for $\Re s = b$. Thus $|m(s)|$ is bounded on the line $\Re s = b$. Now

$$m(s) = \frac{L(s,\sigma)}{L(1-s,\widetilde{\sigma})\epsilon(s,\sigma,\psi)} \int f'(x)|x|^{1-s}d^\times x\,.$$

We may assume a so small (negative) that the integral on the right converges for $\Re s = a$. We may also assume a so small and y_0 so large that on the half line

$$s = a + iy\,,\ y \ge y_0$$

the ratio

$$\frac{L(s,\sigma)}{L(1-s,\widetilde{\sigma})\epsilon(s,\sigma,\psi)}$$

is bounded (see the lemma below). By the Phragmen-Lindelöf principle, $m(s)$ is bounded in the strip S, as claimed. □

REMARK 4.1. In applications the functions f, f' will be C^∞ and for each $X \in \mathfrak{U}(F^\times)$, there will be $X' \in \mathfrak{U}(F^\times)$ such that the functions $\rho(X)f, \rho(X')f'$ satisfy the assumptions of the proposition. Then for each integer $M \ge 0$,

$$s^M m(s) = \int \rho(X)f(x)|x|^s d^\times x$$

for a suitable $X \in \mathfrak{U}(F^\times)$. By the proposition, $m(s)s^M$ is bounded at infinity on vertical strips.

Lemma 4.2. *Given σ if a is sufficiently small ($a << 0$), there is y_0 such that*

$$\frac{L(s,\sigma)}{L(1-s,\widetilde{\sigma})\epsilon(s,\sigma,\psi)}$$

is bounded on the the half vertical line

$$s = a + iy\,,\; y \geq y_0\,.$$

Proof. It suffices to prove the lemma when σ is irreducible. Say $F = \mathbb{R}$. Recall, for x fixed and $|y| \to +\infty$,

$$|\Gamma(x+iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\frac{\pi}{2}|y|}\,.$$

If σ is a character of $F^\times$, then $\sigma(x) = |x|^{u+iv}(\frac{x}{|x|})^\epsilon$ with u, v real and $\epsilon = 0, 1$. In the definition of $L(s,\sigma)$ the factor $\pi^{-\frac{s+u+iv+\epsilon}{2}}$ has a fixed absolute value for $s = a+iy$, $y \leq y_0$. Likewise for $L(1-s,\widetilde{\sigma})$. Thus we can ignore these exponential factors. Then, apart from exponential factors, the absolute value of the ratio is equal to

$$\left|\frac{\Gamma\left(\frac{\epsilon+u+a+i(v+y)}{2}\right)}{\Gamma\left(\frac{\epsilon+1-u-a-i(v+y)}{2}\right)}\right| \sim \frac{|v+y|^{\frac{\epsilon+u+a-1}{2}}}{|v+y|^{\frac{\epsilon+1-u-a-1}{2}}} = |v+y|^{a+u-\frac{1}{2}}\,.$$

If $a + u - \frac{1}{2} < 0$, this tends to 0 as $y \to +\infty$. Our assertion then follows.

Assume σ is induced by a character Ω of $\mathbb{C}^\times$, say

$$\Omega(z) = (z\overline{z})^{u+iv-\frac{n}{2}}z^n$$

with u, v real and $n \geq 0$ integer. Then, apart from exponential factors the absolute value of the ratio is equal to

$$\left|\frac{\Gamma(n+u+a+i(v+y))}{\Gamma(n+1-u-a-i(v+y))}\right| \sim \frac{|v+y|^{n+u+a-\frac{1}{2}}}{|v+y|^{n+1-u-a-\frac{1}{2}}} = |v+y|^{2u+2a-1}\,.$$

Again, if $2u+2a-1 < 0$, this tends to 0 as $y \to +\infty$ and we obtain our assertion. □

We have also a converse to the previous proposition.

Proposition 4.2. *Suppose that f, f' are measurable functions on $F^\times$. Suppose that the Mellin transforms*

$$\int f(x)|x|^s d^\times x\,,\; \int f'(x)|x|^s d^\times x$$

converge absolutely for $\Re s >> 0$ and extend to holomorphic multiples of $L(s,\sigma)$ and $L(s,\widetilde{\sigma})$, respectively, bounded at infinity in any vertical strip. Finally, suppose that the equation

$$\frac{\int f'(x)|x|^{1-s}d^\times x}{L(1-s,\widetilde{\sigma})} = \frac{\epsilon(s,\sigma,\psi)\int f(x)|x|^s d^\times x}{L(s,\sigma)}$$

holds in the sense of analytic continuation. Then for any (σ,ψ) pair (h,k)

$$\int f(x)h(x)d^\times x = \int f'(x)k(x)d^\times x\,.$$

PROOF. Set

$$\theta(a) = \int f(x)h(ax)d^\times x\,,\ \kappa(a) = |a|\int f'(x)k(a^{-1}x)d^\times x\,.$$

We will show that

$$\theta(a) = \kappa(a)\,.$$

Note that $\theta(a)$ and $\kappa(a)$ depend only on $|a|$. As before

$$|\theta(a)| \le C|a|^N$$

for any large enough N. Thus the Mellin transform

$$\int \theta(a)|a|^{-s}d^\times a$$

is defined by a convergent integral for $\Re s >> 0$. Computing formally at first, we get

$$\begin{aligned}\int \theta(a)|a|^{-s}d^\times a &= \iint f(x)h(ax)|a|^{-s}d^\times a d^\times x \\ &= \int f(x)|x|^s d^\times x \int h(a)|a|^{-s}d^\times a\,.\end{aligned}$$

Again the computation is justified because the final result is absolutely convergent for $\Re s >> 0$. In turn this is

$$\int f(x)|x|^s d^\times x \frac{\epsilon(s,\sigma,\psi)e^{P(s)}}{L(s,\sigma)}\,.$$

By assumption, this extends to an entire function of s. Moreover, since the Mellin transform of f is bounded at infinity in vertical strips, this entire function is bounded in any vertical strip.

Likewise, for $\Re s << 0$,

$$\begin{aligned}\int \kappa(a)|a|^{-s}d^\times a &= \iint f'(x)k(a^{-1}x)|a|^{1-s}d^\times a d^\times x \\ &= \iint f'(x)k(ax)|a|^{-(1-s)}d^\times a d^\times x \\ &= \int f'(x)|x|^{1-s}d^\times x \int k(a)|a|^{-(1-s)}d^\times a\,.\end{aligned}$$

This is equal to

$$\int f'(x)|x|^{1-s}d^\times x \frac{e^{P(s)}}{L(1-s,\widetilde{\sigma})}\,.$$

Again this is an entire function of s bounded at infinity in vertical strips. We conclude that

$$\int \theta(a)|a|^{-s}d^\times a = \int \kappa(a)|a|^{-s}d^\times a$$

in the sense of analytic continuation. Since both sides are bounded in any vertical strip, this is enough to conclude that $\theta(a) = \kappa(a)$. □

4.4. Two variables generalization.

Proposition 4.3. *Let σ_i, $i = 1, 2$, be two representations of the Weil group. Suppose f, f' are measurable functions on $F^\times \times F^\times$. Suppose that the integrals*

$$\iint f(x,y)|x|^{s_1}|y|^{s_2}d^\times x d^\times y\,,\ \iint f'(x,y)|x|^{s_1}|y|^{s_2}d^\times x d^\times y$$

converge absolutely for $\Re s_1 >> 0$, $\Re s_1 >> 0$. Suppose further that for any (σ_1, ψ) pair (h_1, k_1) and any (σ_2, ψ) pair (h_2, k_2)

$$\int f(x,y)h_1(x)h_2(y)d^\times x d^\times y = \int f'(x,y)k_1(x)k_2(y)d^\times x d^\times y\,.$$

Then the integral

$$\int f(x,y)|xy|^s d^\times x d^\times y\,,$$

defined a priori for $\Re s >> 0$, extends to a holomorphic multiple of

$$L(s,\sigma_1)L(s,\sigma_2)\,,$$

bounded at infinity in any vertical strip. Likewise, the integral

$$\int f'(x,y)|xy|^s d^\times x d^\times y$$

extends to a holomorphic multiple of $L(s,\widetilde{\sigma_1})L(s,\widetilde{\sigma_2})$, bounded at infinity in any vertical strip. Finally, the equation

$$\frac{\int f'(x,y)|xy|^{1-s}d^\times x d^\times y}{L(1-s,\widetilde{\sigma_1})L(1-s,\widetilde{\sigma_2})} = \frac{\epsilon(s,\sigma_1,\psi)\epsilon(s,\sigma_1,\psi)\int f(x,y)|xy|^s d^\times x d^\times y}{L(s,\sigma_1)L(s,\sigma_2)}$$

holds in the sense of analytic continuation.

Proof. Clearly,

$$\int f(x,y)|xy|^s d^\times x d^\times y = \int |x|^s \left(\int f(xy^{-1},y)d^\times y\right) d^\times x\,.$$

Likewise for f'. Now $(h_1 * h_2, k_1 * k_2)$ is a $(\sigma_1 \oplus \sigma_2, \psi)$ pair. Conversely, any $(\sigma_1 \oplus \sigma_2, \psi)$ pair is a sum of such convolutions. Thus it suffices to check that

$$\int \left(\int f(xy^{-1},y)d^\times y\right) h_1 * h_2(x)d^\times x = \int \left(\int f'(xy^{-1},y)d^\times y\right) k_1 * k_2(x)d^\times x\,.$$

A simple manipulation gives

$$\int \left(\int f(xy^{-1},y)d^\times y\right) h_1 * h_2(x)d^\times x = \int \left(\iint f(x,y)h_1(xt^{-1})h_2(yt)d^\times x d^\times y\right) d^\times t\,.$$

Since

$$\left(x \mapsto h_1(xt^{-1}), x \mapsto |t|^{-1}k_1(xt)\right)$$
$$\left(y \mapsto h_2(yt), y \mapsto |t|k_1(xt^{-1})\right)$$

are (σ_1, ψ) and (σ_2, ψ) pair respectively, we see this is equal to

$$\int \left(\iint f'(x,y)k_1(xt)k_2(yt^{-1})d^\times x d^\times y\right) d^\times t$$
$$= \int \left(\int f'(xy^{-1},y)d^\times y\right) k_1 * k_2(x)d^\times x\,.$$

Our assertion follows. □

4.5. Holomorphic families of pairs. Let σ_i, $1 \le i \le r$, be r unitary representations of the Weil group of F. Let $u = (u_1, u_2, \dots, u_r)$ be an r-tuple of complex numbers. Set

$$\sigma_u := \sum_{1 \le i \le r} \sigma_i \otimes \alpha_F^{u_i} \,.$$

Fix a quadratic polynomial

$$P(s) = As^2 + Bs + C \,, A > 0 \,.$$

For every u, let (h_u, k_u) be the (σ_u, ψ) pair defined by σ_u and P. We say that (h_u, k_u) is a holomorphic family of (σ_u, ψ) pairs.

LEMMA 4.3. *The functions $h_u(x)$, $k_u(x)$ are continuous functions of (x, u). For each x, they are holomorphic functions of u. If Ω is a compact set of $\mathbb{C}^r$ and $a \in \mathbb{Z}$, there is a constant C such that*

$$|h_u(x)| \; |x|^a \le C \,, \; |k_u(x)| \; |x|^a \le C$$

for u in Ω and $x \in F^\times$.

PROOF. From the Mellin inversion formula

$$h_u(x)|x|^a = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \frac{e^{P(s-a)}}{L(s-a, \sigma_u)} |x|^s ds \,.$$

Suppose u is in a compact set. Then on the line $s = iy$, the integrand is bounded by e^{-Dy^2} with $D > 0$. Our assertion follows. $\square$

More generally, suppose σ'_j, $1 \le j \le r'$, are another r' unitary representations of the Weil group and $v = (v_1, v_2, \dots, v_{r'})$ an r'-tuple of complex numbers. Then we can define a holomorphic family of $(\sigma_u \otimes \sigma'_v, \psi)$ pairs.

4.6. Example: the Tate functional equation. Let Φ be a Schwartz function on F. Denote by $\widehat{\Phi}$ its Fourier transform. Let μ be a normalized character of $F^\times$. Tate's theory asserts that

$$\int \Phi(x)\mu(x)|x|^s d^\times x \,, \; \int \Phi(x)\mu^{-1}(x)|x|^s d^\times x$$

defined a priori for $\Re s >> 0$, extend to holomorphic multiples of $L(s, \mu)$ and $L(s, \mu^{-1})$ respectively, bounded at infinity in vertical strips. Moreover, the functional equation

$$\frac{\int \widehat{\Phi}(x)\mu^{-1}(x)|x|^{1-s} d^\times x}{L(1-s, \mu^{-1})} = \frac{\epsilon(s, \mu, \psi) \; \int \Phi(x)\mu(x)|x|^s d^\times x}{L(s, \mu)}$$

holds in the sense of analytic continuation.

Set $F^0 = \{x \in F : |x| = 1\}$. We can apply the propositions of Section 4.3 to the functions

$$f(x) = \int_{F^0} \Phi(xu)\mu^{-1}(xu)du \,, \; f'(x) = \int_{F^0} \widehat{\Phi}(xu)\mu(xu)du \,.$$

We see that the properties of the Tate integral are equivalent to the assertion that the functional equation

$$\int \widehat{\Phi}(x)\mu^{-1}(x)k(x)d^\times x = \int \Phi(x)h(x)\mu(x)d^\times x \tag{4.2}$$

holds for all (μ,ψ) pairs (h,k). We stress that now both integrals are absolutely convergent.

4.7. Example: generalization of Tate's integral to $GL(n)$. Let

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n)$$

be an n-tuple of normalized characters of $F^\times$ and $u = (u_1, u_2, \ldots, u_n) \in \mathbb{C}^n$. Set

$$\sigma_u := \oplus \mu_i \alpha_F^{u_i} \,.$$

Let $(\pi_{\mu,u}, I_{\mu,u})$ be the corresponding induced representation. We define a continuous invariant pairing on $I_{\mu,u} \times I_{\mu^{-1},-u}$ by

$$\langle \phi_1, \phi_2 \rangle = \int_{K_n} \phi_1(k)\phi_2(k)dk \,.$$

Let ξ be an elementary idempotent for the group K_n. Let $I_{\mu,u}(\xi)$ be the range of the operator $\int \pi_{\mu,u}(k)\xi(k)dk$. Recall that the pairing is perfect when restricted to the product $I_{\mu,u}(\xi) \times I_{\mu^{-1},-u}(\check{\xi})$. Let Φ be in $\mathcal{S}(M_n(F))$. If

$$f(g) = \int_K \phi_1(kg)\phi_2(k)dk,$$

then the integral

$$Z(s,f,\Phi) := \int \Phi(g)f(g)|\det g|^{s+\frac{n-1}{2}}dg$$

has the following properties ([**11**]). It converges for $\Re s >> 0$. It has analytic continuation to a holomorphic multiple of

$$L(s,\sigma_u)\,.$$

It is bounded at infinity in vertical strips. Finally, it satisfies the following functional equation

$$\int \widehat{\Phi}(g)f(g^\iota)|\det g|^{1-s+\frac{n-1}{2}}dg = \gamma(s,\sigma_u,\psi)\int \Phi(g)f(g)|\det g|^{s+\frac{n-1}{2}}dg\,. \tag{4.3}$$

We recall that the Fourier transform $\hat{\Phi}$ of Φ is defined by

$$\hat{\Phi}(X) = \int \Phi(Y)\psi(\mathrm{Tr}(-\,{}^tXY))dY\,,$$

which is not the convention adopted in [**11**]. According to our previous discussion, these assertions are equivalent to the identities

$$\int \widehat{\Phi}(g)f(g^\iota)\kappa(\det g)|\det g|^{\frac{n-1}{2}}dg = \int \Phi(g)f(g)\theta(\det g)|\det g|^{\frac{n-1}{2}}dg\,, \tag{4.4}$$

where (θ,κ) is any (σ_u,ψ) pair.

Remark 4.4. In passing, we remark that if ϕ_1 and ϕ_2 are K_n finite and Φ is a standard Schwartz function, then

$$\int \Phi(g)f(g)|\det g|^{s+\frac{n-1}{2}}dg = L(s,\sigma_u)P(s)$$

where P is a polynomial.

In addition, we remark that both sides in (4.4) are continuous functions of (Φ,ϕ_1,ϕ_2). Using this continuity and an argument of density, we see that to prove (4.4), we may assume that Φ is standard and ϕ_1,ϕ_2 K_n-finite. Applying again the propositions of Section 4.3, we see that to prove that $Z(s, f, \Phi)$ is a holomorphic multiple of $L(s, \sigma_u)$ and (4.3) is satisfied we may assume ϕ_1, ϕ_2 K_n-finite and Φ standard. Both assertions were indeed established in this case in [**11**].

It will be necessary to obtain the functional equations (4.3) and (4.4) for a more general type of coefficients. In a precise way, let λ be a continuous linear form on $I_{\mu,u}$. For ϕ in $I_{\mu,u}$, set

$$f(g) = \lambda(\pi_{\mu,u}(g)\phi)\,.$$

Note that

$$|\lambda(\pi_{\mu,u}(g)\phi)| \leq \nu(\pi_{\mu,u}(g)\phi) \leq ||g||^M \nu_1(\phi)$$

where ν, ν_1 are suitable continuous semi-norms. Thus $|f(g)| \preceq ||g||^M$ for a suitable M.

Proposition 4.4. *With the previous notations, for any* Φ,

$$\int \Phi(g)f(g)\theta(g)|\det g|^{\frac{n-1}{2}}dg = \int \widehat{\Phi}(g)f(g^\iota)\kappa(g)|\det g|^{\frac{n-1}{2}}dg\,.$$

Moreover,

$$\int \widehat{\Phi}(g)f(g^\iota)|\det g|^{1-s+\frac{n-1}{2}}dg = \gamma(s, \sigma_u, \psi)\int \Phi(g)f(g)|\det g|^{s+\frac{n-1}{2}}dg\,,$$

in the sense of analytic continuation.

Proof. Since $|f(g)| \preceq ||g||^M$, the integral

$$\int \Phi(g)f(g)|\det g|^s dg$$

converges absolutely for $\Re s >> 0$ by Lemma 3.3. It suffices to prove the first assertion. By our estimates, both sides of the first equality are continuous functions of Φ, that is, are tempered distributions. Thus it suffices to prove the identity when Φ is a standard Schwartz function. Then there is an elementary idempotent ξ of K_n such that

$$\Phi(g) = \iint \Phi(k_1 g k_2)\xi(k_1)\xi(k_2)dk_1dk_2\,.$$

It follows that

$$\widehat{\Phi}(g) = \iint \Phi(k_1 g k_2)\xi(k_1^\iota)\xi(k_2^\iota)dk_1dk_2\,.$$

Set

$$f_1(g) = \int f(k_1^{-1} g k_2^{-1})\xi(k_1)\xi(k_2)dk_1dk_2\,.$$

Then f_1 has the form

$$f_1(g) = \langle \pi_{\mu,u}\phi_1, \phi_2\rangle$$

where ϕ_1 is a K-finite element of $I_{\mu,u}$ and ϕ_2 a K-finite element of $I_{\mu^{-1},-u}$. Thus the required equality is true for the function f_1. We have

$$\int \Phi(g)f(g)\theta(\det g)|\det g|^{\frac{n-1}{2}}dg = \int \Phi(g)f_1(g)\theta(\det g)|\det g|^{\frac{n-1}{2}}dg$$

$$\int \widehat{\Phi}(g)f(g^\iota)\kappa(g)|\det g|^{\frac{n-1}{2}}dg = \int \widehat{\Phi}(g)f_1(g^\iota)\kappa(g)|\det g|^{\frac{n-1}{2}}dg\,.$$

Our assertion follows. □

Similar arguments of continuity and density will be used extensively below. Often, they will allow us to reduce our assertions to the case of K_n-finite datum.

5. Convergence of the integrals

5.1. Integrals (n, n')**.** Let (π, V) and (π', V') be smooth representations of $GL(n, F)$ and $GL(n', F)$, respectively, of moderate growth. Let λ (resp. λ') be a ψ (resp. $\overline{\psi}$) linear form on V (resp. V').

Suppose $n > n'$. For $v \in V$, $v' \in V'$, set

$$W_v(g) = \lambda(\pi(g)v), W_{v'} = \lambda(\pi(g')v')$$

and consider the integral

$$\Psi(s, W_v, W_{v'}) = \int W_v \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix} W_{v'}(g) |\det g|^{s - \frac{n-m}{2}} dg\,.$$

We claim this integral converges for $\Re s >> 0$. Indeed, for some M and all $N >> 0$,

$$\left| W_v \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right| \leq \xi_{h,n} \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix}^{-N} \left|\left| \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right|\right|_I^M .$$

Now, up to a scalar factor,

$$\xi_{h,n} \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix} = \xi_{i,m}(g) \succeq \xi_{s,m}^r(g)$$

for some $r > 0$. Moreover

$$\left|\left| \begin{pmatrix} g & 0 \\ 0 & 1_{n-m} \end{pmatrix} \right|\right|_I^M \preceq ||g||_I^M ,$$

$$|W_{v'}(g)| \preceq ||g||_I^{M'} .$$

Thus we are reduced to the convergence of the integral

$$\int_{N_m \backslash G_m} \xi_{s,m}(g)^{-N} ||g||_I^M |\det g|^s dg\,.$$

By Lemma 3.5, given M, there are $A, B, C > 0$ such that the integral converges for $N > A, s > B, CN > s$. Our assertion follows.

Now consider the case $n = n'$. Then

$$\Psi(s, W_v, W_v', \Phi) = \int W_v(g) W_{v'}(g) \Phi[(0, 0, \dots, 0, 1)g] |\det g|^s dg\,.$$

Now we are reduced to the convergence of

$$\int_{N_n \backslash G_n} \xi_{h,n}(g)^{-N} \; ||g||^M |\Phi[(0, 0, \dots, 0, 1)g]| \; |\det g|^s dsg\,.$$

By Lemma 3.5, given M, there are $A, B, C > 0$ such that the integral converges for $N > A, s > B, CN > s$. Our assertion follows.

In both cases, the proof gives a result of continuity. For instance, for $n = n'$,

$$\begin{aligned} |\Psi(s, W_v, W_v', \Phi)| &\leq \int |W_v|(g) \; |W_{v'}(g)| \; |\Phi[(0, 0, \dots, 0, 1)g]| \; |\det g|^{\Re s} dg \\ &\leq \mu(v)\mu'(v')\nu(\Phi) \end{aligned} \tag{5.1}$$

where μ, μ', ν are suitable continuous semi-norms. Thus $\Psi(s, W_v, W_v', \Phi)$ depends continuously on (v, v', Φ).

5.2. Integrals involving a unipotent integration. To prove convergence of the integrals $\Psi_j(s,W,W')$, we need a few elementary lemmas.

Consider a matrix $v \in {}^tN_n(F)$. Let us write its rows as

$$(X_1,1,0),(X_2,1,0),(X_3,1,0),\dots(X_n,1)$$

where each X_i is a row matrix of size $i-1$ and 0 represents a string of zeros of variable length. For instance if

$$v = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix},$$

then

$$X_1 = \emptyset, X_2 = x, X_3 = (z,y)\,.$$

LEMMA 5.1. *Consider the Iwasawa decomposition of $v \in {}^tN_n(F)$:*

$$v = ubk\,,$$

$u \in N_n$, $k \in K_n$,

$$b = \mathrm{diag}(b_1,b_2,\dots,b_n)\,,\; b_i > 0\,.$$

Then

$$b_1^2 b_2^2 \cdots b_n^2 = 1\,.$$

For $2 \le i \le n$,

$$b_i^2 b_{i+1}^2 \cdots b_n^2 \ge 1 + ||X_i||_e^2$$

and

$$b_n^2 = 1 + ||X_n||_e^2\,.$$

There exist an integer M and constants $C>0, D>0$ such that, for all i,

$$C\,\frac{1}{\prod_{j=i+1}^n (1+||X_j||_e^2)^M} \le b_i^2 \le D\prod_{j=i}^n (1+||X_j||_e^2)^M\,.$$

PROOF. Here we drop the index e from $||X_i||_e$. Let e_i, $1 \le i \le n$, be the canonical basis of the space of **row** vectors. Then

$$b_i^2 b_{i+1}^2 \cdots b_n^2 = ||(e_i \wedge \cdots \wedge e_n)\; v||^2\,.$$

The entries of $(e_i \wedge \cdots \wedge e_n)\; v$ are polynomials in the entries of the matrices X_j, $i \le j \le n$. Thus

$$b_i^2 b_{i+1}^2 \cdots b_n^2 \le D\prod_{j=i}^n (1+||X_j||^2)^M$$

for a suitable M and D. On the other hand, up to sign, the entries of X_i are among the entries of $(e_i \wedge \cdots \wedge e_n)\; v$. Thus

$$b_i^2 b_{i+1}^2 \cdots b_n^2 = ||(e_i \wedge \cdots \wedge e_n)\; v||^2 \ge 1 + ||X_i||^2 \ge 1\,.$$

Moreover

$$b_n^2 = 1 + ||X_n||^2\,.$$

Now

$$b_i^2 \ge \frac{1}{b_{i+1}^2 \cdots b_n^2} \ge D^{-1} \prod_{j=i+1}^n (1+||X_j||^2)^{-M}\,,$$

$$b_i^2 \le D\,\frac{\prod_{j=i}^n (1+||X_j||^2)^M}{b_{i+1}^2 \cdots b_n^2} \le D\prod_{j=i}^n (1+||X_j||^2)^M\,.$$

The lemma follows. □

An immediate consequence of the lemma is the following observation.

LEMMA 5.2. *There exist an integer r and a constant C such that, for any $a \in A_m(\mathbb{R})$, any $X \in M(n-m\times m, F)$,*

$$\xi_{s,n}^r \begin{pmatrix} a & 0 \\ X & 1_{n-m} \end{pmatrix} \geq C \prod_{i=1}^{m}(1+a_i^2) \prod_{i=m+1}^{n} (1+||X_i||^2)\,.$$

PROOF. Indeed, write the Iwasawa decomposition

$$\begin{pmatrix} 1_m & 0 \\ X & 1_{n-m} \end{pmatrix} = vbk\,.$$

Then

$$\xi_{s,n} \begin{pmatrix} a & 0 \\ X & 1_{n-m} \end{pmatrix} = \prod_{i=1}^{m}(1+a_i^2 b_i^2) \prod_{i=m+1}^{n} (1+b_i^2)\,.$$

Thus for any integer $r \geq 1$,

$$\xi_{s,n}^r \begin{pmatrix} a & 0 \\ X & 1_{n-m} \end{pmatrix} \geq \prod_{i=1}^{m}(1+a_i^2 b_i^2) \prod_{i=m+1}^{n} (1+b_i^2)^r\,.$$

For $1 \leq i \leq m$,

$$b_i^2 \geq C \prod_{j=m+1}^{n} (1+||X_j||^2)^{-M}\,,$$

for some constant C. Since we may decrease C, we may assume $C < 1$. Thus

$$1 + a_i^2 b_i^2 \geq C \left(1 + \frac{a_i^2}{\prod_{j=m+1}^{n}(1+||X_j||^2)^M} \right)\,.$$

On the other hand, for a suitable integer r,

$$\prod_{j=m+1}^{n} (1+b_j^2)^r \geq \prod_{j=m+1}^{n} (1+b_j^2 b_{j+1}^2 \cdots b_n^2) \geq \prod_{j=m+1}^{n} (1+||X_j||^2)\,.$$

The lemma follows. □

Now we establish the convergence of the integrals $\Psi_j(s, W, W')$ for $\Re s >> 0$. We only treat the case $j = n - n' - 1$. The other cases are similar. The integral at hand is

$$\int W_v \begin{bmatrix} ak & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} W_{v'}(ak)\delta_{n'}(a)^{-1} |\det a|^s dadkdX\,,$$

or, after a change of variables,

$$\int W_v \left[\begin{pmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 & 0 \\ 0 & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$
$$\times\, W_{v'}(ak)\delta_{n'}(a)^{-1} |\det a|^s dadkdX\,.$$

The integrand is majorized by a constant times

$$\xi_{h,n}^{-N}\begin{bmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left|\left|\begin{pmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|\right|^{M} ||a||^{M'}\delta_{n'}(a)^{-1}|\det a|^{s}$$

times $\mu(v)\mu'(v')$ where μ, μ' are continuous semi-norms. After integrating over $k \in K_n$, we are reduced to the convergence of

$$\int \xi_{h,n}^{-N}\begin{bmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} ||a||^{M_1}(1+||X||^2)^{M_2}\delta_{n'}(a)^{-1}|\det a|^{s}dadX\,,$$

with M_1, M_2 given and N arbitrarily large. Now, up to a scalar factor,

$$\xi_{h,n}\begin{bmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \xi_{i,n-1}\begin{bmatrix} a & 0 \\ X & 1_{n-n'-1} \end{bmatrix}.$$

Furthermore $\xi_i^n \geq \xi_s$. Thus we are reduced to the convergence of the integral

$$\int \xi_{s,n-1}^{-N}\begin{bmatrix} a & 0 \\ X & 1_{n-n'-1} \end{bmatrix} ||a||^{M_1}(1+||X||^2)^{M_2}\delta_{n'}(a)^{-1}|\det a|^{s}dadX\,.$$

By Lemma 5.2, we are in fact reduced to a product of two integrals

$$\int \prod_{i=1}^{n'}(1+a_i^2)^{-N}||a||^{M_1}\delta_{n'}(a)^{-1}|\det a|^{s}da\,,$$

$$\int \prod_{i=n'+1}^{n-1}(1+||X_i||^2)^{-N}(1+||X||^2)^{M_2}dX\,.$$

The first integral converges for $N > A, s > B, CN > s$ (Lemma 3.4). The second integral converges for $N >> 0$.

The proof gives a result of continuity as in (5.1).

5.3. The tempered case. Let again $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ be an r-tuple of irreducible unitary representations of W_F and u an r-tuple of complex numbers. Let $n = \sum_i \deg(\sigma_i)$. Then if u is purely imaginary, the induced representation $I_{\sigma,u}$ is unitary irreducible and tempered. Consider likewise another pair (σ', u') where $\sigma' = (\sigma'_1, \sigma'_2, \ldots, \sigma'_{r'})$ is an r'-tuple of irreducible unitary representations of W_F and u' an r'-tuple of complex numbers. Let $n' = \sum_{i'} \deg(\sigma'_i)$.

Lemma 5.3. *Suppose $n > n'$. If u and u' are purely imaginary, the integrals $\Psi_k(s, W_f, W_{f'})$ converge absolutely for $\Re s > 0$.*

Proof. We can use the majorizations of Propositions 3.4 and 3.5. Suppose first $k = 0$. Then, for every $N > 0$,

$$\left|W_f\begin{pmatrix} ak & 0 \\ 0 & 1_{n-n'} \end{pmatrix} W'_{f'}(ak)\right| \\ \leq C_N\delta_n^{1/2}\begin{pmatrix} a & 0 \\ 0 & 1_{n-n'} \end{pmatrix}\delta_{n'}^{1/2}(a)\xi_{s,n'}^{-N}(a)(1+||\log a||^2)^{d}\,.$$

We have dropped the index e form $||\log a||_e^2$. Thus we are reduced to the convergence of an integral of the form

$$\int \delta_n^{1/2}\begin{pmatrix} a & 0 \\ 0 & 1_{n-n'}\end{pmatrix}\delta_{n'}^{-1/2}(a)\xi_{s,n'}^{-N}(a)(1+||\log a||^2)^d|\det a|^{s-\frac{n-n'}{2}}da\,.$$

But

$$\delta_n^{1/2}\begin{pmatrix} a & 0 \\ 0 & 1_{n-n'}\end{pmatrix}\delta_{n'}^{-1/2}(a) = |\det a|^{\frac{n-n'}{2}}\,,$$

so we are reduced to the convergence of the integral

$$\int \xi_{s,n'}^{-N}(a)(1+||\log a||)^d|\det a|^s da\,.$$

Now $(1+||\log a||^2)^d$ is a polynomial in the $\log(a_i^2)^2$. Thus, we are reduced to a product of integrals of the form

$$\int \frac{(\log(a_i^2))^{2m}|a_i|^s}{(1+a_i^2)^N}d^\times a_i\,.$$

Such an integral converges for $s>0$, $2N>s$. Our assertion follows.

Now we assume $k>0$. We only treat the case $k=n-n'-1$. We have to show the following integral converges for $s>0$.

$$\int \left| W_f\begin{pmatrix} ak & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1\end{pmatrix}\right| |W'_{f'}(ak)|\ |\det a|^{s-\frac{n-n'}{2}}dadXdk\,,$$

or, after a change of variables

$$\int \left| W_f\left[\begin{pmatrix} a & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} k & 0 & 0 \\ 0 & 1_{n-n'-1} & 0 \\ 0 & 0 & 1\end{pmatrix}\right]\right|$$
$$\times\ |W'_{f'}(ak)|\ |\det a|^{s-\frac{n-n'}{2}}dadXdk\,.$$

Write the Iwasawa decomposition

$$\begin{pmatrix} 1_{n'} & 0 \\ X & 1_{n-n'-1} \\ 0 & 1\end{pmatrix} = vbk_1$$

with

$$b=\operatorname{diag}(b_1,b_2,\dots,b_{n-1})\,.$$

This is majorized by a constant times

$$\int \xi_{i,n-1}^{-N}\begin{pmatrix} a & 0 \\ X & 1_{n-n'-1}\end{pmatrix}(1+||\log a||^2+||\log b||^2)^r\ |\det a|^s dadX\,.$$

Applying Lemma 5.2, we are reduced to the convergence of a sum of products of integrals

$$\int \xi_{s,n'}^{-N}(a)P_1(\log a)|\det a|^s da\,,$$
$$\int P_2(\log b)\prod(1+||X_i||^2)^{-N}dX$$

where $P_1(\log a)$ is a polynomial in the $\log^2 a_i$ and $P_2(\log b)$ a polynomial in the $\log^2(b_i)$. The first integral converges for $N>0$, $s>0$ $2N>s$. By the estimates of Lemma 5.1, the second integral converges for $N>>0$. □

LEMMA 5.4. *If u and u' are purely imaginary, the integrals $\Psi(s, W_f, W_{f'}, \Phi)$ converge absolutely for $\Re s > 0$.*

PROOF. Again we can use the majorizations of Proposition 3.4 and 3.5. In the integral

$$\int |W_f(ak)|\ |W_{f'}(ak)|\ |\Phi[0, 0, \ldots, 0, 1)ak]| \det a|^s dadk\,,$$

we majorize

$$\begin{aligned} |W_f(ak)| &\preceq \xi_{h,n}^{-N}(a)\delta_n^{1/2}(a)(1 + ||\log a||^2)^r\,, \\ |W_{f'}(ak)| &\preceq \delta_n^{1/2}(a)(1 + ||\log a||^2)^r\,, \\ |\Phi[0, 0, \ldots, 0, 1)ak]| &\preceq (1 + a_n^2)^{-N}\,. \end{aligned}$$

Thus we are reduced to the convergence of

$$\int \xi_{i,n}(a)^{-N}(1 + ||\log a||^2)^r |\det a|^s da$$

or

$$\int \xi_{s,n}(a)^{-N}(1 + ||\log a||^2)^r |\det a|^s da\,.$$

As before, this integral converges for $s > 0$, $2N > s$. □

6. Relations between integrals

We will make extensive use of the Dixmier-Malliavin Lemma ([**9**]). For the convenience of the reader, we repeat this lemma in the form we will be using it.

LEMMA 6.1 (Dixmier-Malliavin). *Let G be a connected Lie group. Let (π, V) be a C^∞ representation of G on a Frechet space V. For any vector $v \in V$, there are finitely many vectors v_i and smooth functions of compact support ϕ_i on G such that*

$$v = \sum_i \pi(\phi_i)v_i\,.$$

The lemma will be applied to various subgroups of $G_n(F)$.

6.1. Relation between Ψ_j and Ψ_{j+1}. Consider two induced representations $(\pi, I) = (\pi_{\sigma,u}, I_{\sigma,u})$ and $(\pi', I') = (\pi_{\sigma',u'}, I_{\sigma',u'})$ of $GL(n)$ and $GL(n')$, respectively. Let λ (resp. λ') be a non zero ψ form (resp. $\overline{\psi}$ form) on I (resp. I'). We claim that, for $0 \leq j \leq n - n' - 2$, any integral $\Psi_{j+1}(s, W, W')$, $W \in \mathcal{W}(\pi : \psi)$, $W' \in \mathcal{W}(\pi' : \overline{\psi})$ has the form $\Psi_j(s, W_1, W')$ for a suitable $W_1 \in \mathcal{W}(\pi : \psi)$ and conversely. Moreover, we claim that the the functional equation relating the integrals Ψ_j and Ψ_k, with $k + j = n - n' - 1$, implies the functional equation relating the integrals Ψ_{j+1} and Ψ_{k-1}.

Indeed, let W_0 be an element of $\mathcal{W}(\pi : \psi)$. Let ϕ be a Schwartz function on the space of column matrices with n' entries. Define a function W_1 by

$$W_1(g) := \int W_0 \left[\begin{pmatrix} 1_{n'} & 0 & 0 & Z & 0 \\ 0 & 1_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} g \right] \phi(Z)dZ\,.$$

Here and below $*$ stands for the appropriate integer, in this case the integer $n-(n'+j+2)$. Clearly, the function W_1 belongs to the space $\mathcal{W}(\pi:\psi)$. More precisely, if $W_0=\lambda(\pi(g)v_0)$, then $W_1(g)=\lambda(\pi(g)v_1)$ where v_1 is the vector defined by

$$v_1:=\int \pi\left[\begin{pmatrix} 1_{n'} & 0 & 0 & Z & 0\\ 0 & 1_j & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] v_0\ \phi(Z)dZ\,.$$

In fact, by Lemma 6.1, any vector v can be written as a finite sum

$$v=\sum_i\int \pi\left[\begin{pmatrix} 1_{n'} & 0 & 0 & Z & 0\\ 0 & 1_j & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] v_0\ \phi_i(Z)dZ\,,$$

where the ϕ_i are smooth functions of compact support. Thus any function W is a finite sum of functions of the form W_1.

Let $\widehat{\phi}$ be the Fourier transform of ϕ:

$$\widehat{\phi}(Y)=\int \phi(Z)\psi(-YZ)dZ\,.$$

Here $\widehat{\phi}$ is regarded as a function on the space of row matrices of size n'. Similarly, the function W_2 defined by

$$W_2(g):=\int W_0\left[\begin{pmatrix} 1_{n'} & 0 & 0 & 0 & 0\\ 0 & 1_j & 0 & 0 & 0\\ Y & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} g\right] \widehat{\phi}(-Y)dY$$

belongs to $\mathcal{W}(\pi:\psi)$. Again, we may take $\widehat{\phi}$ to be a smooth function of compact support. Thus any function W is a finite sum of functions of the form W_2.

LEMMA 6.2. *For any $g\in G_{n'}$, $X\in M(j\times n')$ (j rows and n' columns)*

$$\int_{F^{n'}} W_1\left[\begin{pmatrix} g & 0 & 0 & 0 & 0\\ X & 1_j & 0 & 0 & 0\\ Y & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] dY = W_2\left[\begin{pmatrix} g & 0 & 0 & 0 & 0\\ X & 1_j & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right].$$

PROOF. We have

$$
\begin{aligned}
&W_0\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X & 1_j & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\begin{pmatrix} 1_{n'} & 0 & 0 & Z & 0 \\ 0 & 1_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] \\
&= W_0\left[\begin{pmatrix} 1_{n'} & 0 & 0 & gZ & 0 \\ 0 & 1_j & 0 & XZ & 0 \\ 0 & 0 & 1 & YZ & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X & 1_j & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] \\
&= \psi(YZ)W_0\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X & 1_j & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right].
\end{aligned}
$$

Thus the left hand side of the formula of the lemma is equal to

$$\iint W_0\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X & 1_j & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right]\psi(YZ)\phi(Z)dY dZ\,,$$

that is, to

$$\int W_0\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X & 1_j & 0 & 0 & 0 \\ Y & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right]\widehat{\phi}(-Y)dY$$

which is the right hand side of the formula in the lemma. □

It follows from the lemma that, for any W',

$$\Psi_{j+1}(s, W_1, W') = \Psi_j(s, W_2, W')\,.$$

Thus our first claim follows.

Now we claim that

$$\Psi_k(s, \rho(w_{n.n'})\widetilde{W_2}, \widetilde{W'}) = \Psi_{k-1}(s, \rho(w_{n.n'})\widetilde{W_1}, \widetilde{W'})\,.$$

Indeed,

$$\widetilde{W_2}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right]$$

$$= \int \widetilde{W_0}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'} \begin{pmatrix} 1_{n'} & 0 & {}^tY & 0 & 0 \\ 0 & 1_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix}\right] \widehat{\phi(Y)} dY$$

$$= \int \widetilde{W_0}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} \begin{pmatrix} 1_{n'} & 0 & 0 & {}^tY & 0 \\ 0 & 1_{k-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right] \widehat{\phi(Y)} dY\,.$$

Computing as in the proof of the lemma, we find

$$\int \widetilde{W_0}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right] \psi(-Y'\ {}^tY)\widehat{\phi(Y)} dY$$

$$= \widetilde{W_0}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right] \phi(-\,{}^tY')\,.$$

Thus

$$\int \widetilde{W_2}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right] dY'$$

$$= \int \widetilde{W_0}\left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ Y' & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'}\right] \phi(-\,{}^tY') dY' =$$

$$= \int \widetilde{W_0} \left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'} \begin{pmatrix} 1_{n'} & 0 & 0 & 0 & 0 \\ 0 & 1_j & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ Y' & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} \right] \phi(-{}^tY')dY'$$

$$= \widetilde{W_1} \left[\begin{pmatrix} g & 0 & 0 & 0 & 0 \\ X' & 1_{k-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1_* \end{pmatrix} w_{n,n'} \right].$$

Integrating the relation we have just found, we get

$$\Psi_k(s, \rho(w_{n,n'})\widetilde{W_2}, \widetilde{W'}) = \Psi_{k-1}(s, \rho(w_{n,n'})\widetilde{W_1}, \widetilde{W'}).$$

Thus the functional equation for the integrals

$$\Psi_j(s, W, W'), \Psi_k(1-s, \rho(w_{n,n'})\widetilde{W}, \widetilde{W'})$$

implies the functional equations for the integrals

$$\Psi_{j+1}(s, W, W'), \Psi_{k-1}(1-s, \rho(w_{n,n'})\widetilde{W}, \widetilde{W'})$$

and conversely.

We conclude that if we prove that the integrals $\Psi(s, W, W')$ have the required analytic properties, this will imply that all the integrals $\Psi_j(s, W, W')$ have the required analytic properties. Similarly, the functional equation relating the integrals $\Psi_0(s, W, W')$ and $\Psi_{n-n'-1}(1-s, \rho(w_{n,n'})\widehat{W}, \widetilde{W'})$ implies the functional equations relating the integrals $\Psi_j(s, W, W')$ and $\Psi_k(1-s, \rho(w_{n,n'})\widetilde{W}, \widetilde{W'})$, for $j + k = n - n' - 1$.

6.2. Other relations. Consider a Casselman-Wallach representation (ψ, V) of $GL(n)$. Let λ be a ψ form on V. For each $v \in V$, set $W_v(g) = \lambda(\pi(g)v)$.

PROPOSITION 6.1. *Let $r < n$. Given $v \in V$ and a Schwartz function Φ on the space of row vectors of size r there is $v_0 \in V$ such that, for any $g \in G_r$,*

$$W_{v_0} \begin{pmatrix} g & 0 \\ 0 & 1_{n-r} \end{pmatrix} = W_v \begin{pmatrix} g & 0 \\ 0 & 1_{n-r} \end{pmatrix} \Phi[(0, 0, \dots, 1)g].$$

Conversely, given $v \in V$, there are vectors $v_i \in V$ and Schwartz functions Φ_i such that

$$W_v \begin{pmatrix} g & 0 \\ 0 & 1_{n-r} \end{pmatrix} = \sum_i W_{v_i} \begin{pmatrix} g & 0 \\ 0 & 1_{n-r} \end{pmatrix} \Phi_i[(0, 0, \dots, 1)g].$$

PROOF. For the first part, set

$$v_0 = \int \pi \begin{pmatrix} 1_r & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix} v \ \widehat{\Phi}(u) du.$$

Here we regard $\widehat{\Phi}$ as a function on the space of column vectors of size r. Then

$$\begin{aligned} & W_{v_0}\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix} \\ & = \int W_v\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix}\begin{pmatrix} 1_r & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix}\right]\widehat{\Phi}(u)du \\ & = W_v\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix}\right]\int \psi[(0,0,\dots,1)gu]\widehat{\Phi}(u)du \\ & = W_v\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix}\right]\Phi[(0,0,\dots,1)g]\,. \end{aligned}$$

For the second assertion, we proceed similarly. Using Lemma 6.1, we write the given vector v as

$$v = \sum_i \int \pi\begin{pmatrix} 1_r & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix} v_i\Phi_i(u)du$$

with smooth functions of compact support Φ_i. We obtain the desired decomposition:

$$W_v\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix} = \sum_i W_{v_i}\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-r-1} \end{pmatrix}\widehat{\Phi_i}[-(0,0,\dots,1)g]\,.$$

□

Proposition 6.2. *Let (π, V) and λ as in the previous proposition. Let $r < n$ and $t < n-r$. Let $\phi(x,h)$ be a smooth function of compact support on $F^t \times F^\times$. Then given v, there is v_0 such that*

$$W_{v_0}\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{n-r-t} \end{pmatrix} = W_v\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{n-r-t} \end{pmatrix}\phi(x,h)\,.$$

Proof. We may regard ϕ as a Schwartz function on F^{t+1} which vanishes on $F^t \times \{0\}$. Our assertion follows then from the previous proposition. □

7. Integral representations

In this section, we discuss in detail an integral representation of Whittaker functions for the group $GL(n)$. The integral representation is a convergent integral in which appears a Whittaker function for the group $GL(n-1)$ and a Schwartz function on F^n. In [**13**], the point of view is different. The integral representation described here is used inductively to establish an integral representation for Whittaker functions which contains only Schwartz functions.

7.1. Godement sections. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be an n-tuple of normalized characters. Let $u = (u_1, u_2, \ldots, u_n) \in \mathbb{C}^n$. The pair (μ, u) defines a character of A_n or $A_n\, {}^tN_n$. We denote by $I_{\mu,u}$ the induced representation of G. Thus $I_{\mu,u}$ is the space of C^∞ functions $f : G \to \mathbb{C}$ such that

$$f(vag) = f(g) \prod_{i=1}^{n} \mu_i(a)|a_i|^{u_i+i-1-\frac{n-1}{2}}\,,$$

for all $v \in \overline{N_n}$, $g \in G$, and

$$a = \mathrm{diag}(a_1, a_2, \ldots a_n)\,.$$

For fixed μ, $(I_{\mu,u})$ is a holomorphic fiber bundle. A section $f_u(g)$ is a map $\mathbb{C}^r \times G \to \mathbb{C}$ such that, for every u, the function $g \mapsto f_u(g)$ belongs to $I_{\mu,u}$. Such a section is said to be **standard** if, for every $k \in K_n$, $f_u(k)$ is independent of u.

We construct another family of meromorphic sections of $I_{\mu,u}$, the **Godement sections**. As in the case of $GL(2)$ ([**14**]), they are used to establish the analytic properties of our integrals. This type of sections was first introduced in the global theory ([**10**]).

Set $\mu' = (\mu_1, \mu_2, \ldots, \mu_{n-1})$, $u' = (u_1, u_2, \ldots u_{n-1})$. If Φ is a Schwartz function on $M((n-1) \times n, F)$ and ϕ_1 is a standard section of $I_{\mu',u'}$, we set

$$\begin{aligned} f_{\Phi,\phi_1,\mu_n,u_n}(g) :=& \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \\ &\times \int_{G_{n-1}(F)} \Phi[(h,0)g]\phi_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}\, d^\times h\,. \end{aligned} \tag{7.1}$$

It is easily checked that if the integral converges, then it defines an element of $I_{\mu,u}$.

Proposition 7.1.

(i) *The integral (7.1) converges absolutely for*

$$\Re(u_n - u_i) > -1\,,\ 1 \le i \le n-1\,. \tag{7.2}$$

(ii) *It extends to a meromorphic function of u_n which is a holomorphic multiple of*

$$\prod_{1 \le i \le n-1} L(u_n - u_i + 1, \mu_n\mu_i^{-1})\,.$$

(iii) *Let Ω_r be the open set of matrices of rank $n-1$ in $M(n-1 \times n, F)$. If Φ has compact support contained in Ω_r, the integral (7.1) converges for all u_n.*

(iv) *When it is defined, the integral (7.1) represents an element of $I_{\mu,u}$.*

(v) *For a given u, any element of $I_{\mu,u}$ can be written as a finite sum of such integrals, with Φ supported on Ω_r.*

(vi) *Suppose u satisfies (7.2). Then any K_n-finite element of $I_{\mu,u}$ can be written as a finite sum of integrals (7.1) with Φ a standard Schwartz function and ϕ_1 K_{n-1}-finite.*

Proof. Indeed, let us write

$$h = kb$$

where b is lower triangular, with diagonal entries a_i, $1 \le i \le n-1$, and below the diagonal entries $u_{i,j}$. For example for $n = 4$,

$$b = \begin{pmatrix} a_1 & 0 & 0 \\ u_{1,2} & a_2 & 0 \\ u_{1,3} & u_{2,3} & a_3 \end{pmatrix} .$$

The Haar measure dh is the product of the measures $d^\times a_i$, $du_{i,j}$, dk times

$$\prod_{1 \le i \le n-1} |a_i|^{i+1-n} .$$

We first integrate over k keeping in mind that ϕ_1 belongs to an induced representation. We find

$$\int_{G_{n-1}(F)} \Phi[(h,0)g]\phi_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}} dh \prod_{1 \le i \le n-1} |a_i|^{i+1-n}$$
$$= \Phi_1(b) \prod_{i=1}^{n-1} \mu_n \mu_i^{-1}(a_i)\ |a_i|^{u_n - u_i + 1} ,$$

where $\Phi_1(b)$ is a Schwartz function on the vector space of lower triangular matrices (i.e., on $F^{n-1} \times F^{\frac{(n-1)n}{2}}$). Now we set

$$\phi(a_1, a_2, \dots, a_{n-1}) = \int \Phi_1(b) \otimes du_{i,j} .$$

Thus ϕ is a Schwartz function on F^{n-1}. We are reduced to an integral of the form

$$\int \phi(a_1, a_2, \dots, a_{n-1}) \prod_{i=1}^{n-1} \mu_n \mu_i^{-1}(a_i)\ |a_i|^{u_n - u_i + 1} d^\times a_i .$$

The two first assertions follow.

The third assertion is trivial.

It is easily checked that when the integral is absolutely convergent, it represents an element of $I_{\mu,u}$. The same assertion remains true when the integral is defined by analytic continuation.

We prove the fifth assertion. First we recall a well-known result: any element f of $I_{\mu,u}$ can be written in the form

$$f(g) = \int \phi(bg) \prod \mu_i^{-1}(a)|a_i|^{-u_i + \frac{n-1}{2} + 1 - i} d_r b$$

where ϕ is a smooth function of compact support, $d_r b$ a right invariant measure on the group $A_n \overline{N_n}$ and the a_i are the diagonal entries of b. This can derived from Lemma 6.1. Indeed, we may assume

$$f(g) = \int f_1(gx)\phi_1(x)dx$$

with $f_1 \in I_{\mu,u}$ and ϕ_1 a smooth function of compact support. Then we can take

$$\phi(g) = \int f_1(k)\phi_1(g^{-1}k)dk .$$

Let f_1 belong to the space $J^1_{\mu_n,u_n}$ of C^∞ functions f_1 such that

$$f_1 \left[\begin{pmatrix} 1_{n-1} & 0 \\ v & a_n \end{pmatrix} g \right] = f_1(g)\mu_n(a_n)|a_n|^{u_n + \frac{n-1}{2}} ,$$

which are compactly supported modulo the subgroup

$$\left\{ \begin{pmatrix} 1_{n-1} & 0 \\ * & * \end{pmatrix} \right\}.$$

Define

$$f(g) = \int f_1 \left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right] dv$$
$$\times \mu_1^{-1}(a_1)|a_1|^{-u_1+\frac{n-1}{2}} \mu_2^{-1}(a_2)|a_2|^{-u_2-1+\frac{n-1}{2}} \cdots \mu_{n-1}^{-1}(a_n)|a_n|^{-u_n+1-\frac{n-1}{2}}$$
$$\times d^\times a_1 d^\times a_2 \cdots d^\times a_{n-1},$$

with $v \in N_{n-1}$,

$$a = \mathrm{diag}(a_1, a_2, \ldots a_{n-1}).$$

Clearly, $f \in I_{\mu,u}$. It follows from the result that we have recalled that any element f of $I_{\mu,u}$ can be represented in this way for a suitable $f_1 \in J^1_{\mu_n,u_n}$.

The space $J^1_{\mu_n,u_n}$ is invariant on the left under the group of matrices of the form

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},\ h \in G_{n-1}.$$

By Lemma 6.1, any element of $J^1_{\mu_n,u_n}$ is a finite sum of elements of the form

$$\int f_1 \left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right] \phi(h^{-1}) dh$$

with $f_1 \in J^1_{\mu_n,u_n}$ and $\phi \in C_c^\infty(G_{n-1})$. For an element of this form, the corresponding f is given by

$$f(g) = \int_{G_{n-1}} f_1 \left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g \right] \phi_0(h^{-1}) dh$$

where

$$\phi_0(h) := \int \phi(vah) dv\ \mu_1^{-1}(a_1)|a_1|^{-u_1+\frac{n-1}{2}} \mu_2^{-1}(a_2)|a_2|^{-u_2-1+\frac{n-1}{2}} \times \cdots$$
$$\cdots \times \mu_{n-1}^{-1}(a_{n-1})|a_{n-1}|^{-u_{n-1}+1-\frac{n-1}{2}} d^\times a_1 d^\times a_2 \cdots d^\times a_{n-1},$$

with $v \in {}^tN_{n-1}$,

$$a = \mathrm{diag}(a_1, a_2, \ldots a_{n-1}).$$

Now

$$\phi_0(g) = \phi_1(g) |\det g|^{-\frac{1}{2}}$$

where ϕ_1 is in the space $I_{\mu',u'}$ with

$$\mu' = (\mu_1, \mu_2, \ldots, \mu_{n-1}),\ u' = (u_1, u_2, \ldots, u_{n-1}).$$

Let J_0 be the space of C^∞ functions f_0 such that

$$f_0 \left[\begin{pmatrix} 1_{n-1} & 0 \\ v & a_n \end{pmatrix} g \right] = f_0(g)$$

for all $v \in F^{n-1}$, $a_n \in F^\times$ and f_0 has compact support modulo the subgroup

$$R := \left\{ \begin{pmatrix} 1_{n-1} & 0 \\ * & * \end{pmatrix} \right\}.$$

Clearly, we can write

$$f_1(g) = f_0(g)\mu_n(\det g)|\det g|^{\frac{n-1}{2}},$$

with $f_0 \in J_0$. Thus

$$f(g) = \mu_n(\det g)\ |\det g|^{\frac{n-1}{2}} \int f_0\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right] \phi_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}dh\,.$$

We claim there is a function $\Phi \in \mathcal{C}_c^\infty(\Omega_r)$ such that

$$f_0(g) = \Phi[(1_{n-1},0)g]\,.$$

Taking the claim for granted at the moment, we finally get

$$f(g) = \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \int \Phi[(h,0)g]\phi_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}d^\times h\,.$$

We can view Φ as a Schwartz function on $M((n-1)\times n, F)$ which vanishes on the complement of Ω_r.

It remains to establish our claim. Consider the map

$$g \mapsto (1_{n-1},0)g\,.$$

It passes to the quotient and defines a map

$$R\backslash G \to \Omega_r\,.$$

This map is clearly surjective. We claim it is injective. Indeed, let g and g' be two matrices in G such that $(1_{n-1},0)g = (1_{n-1},0)g'$. We may write

$$g = \begin{pmatrix} A \\ X \end{pmatrix},\ g' = \begin{pmatrix} A \\ X' \end{pmatrix}$$

where A has $n-1$ rows of size n and X, X' are row vectors of size n. Since the rows of A and the row X are linearly independent, there is a row vector c of size $n-1$ and a scalar d such that

$$cA + dX = X'\,.$$

Moreover, $d \neq 0$ since the rows of A and the row X' are linearly independent. Hence $rg = g'$ where $r \in R$ is defined by

$$r = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}.$$

Thus the map $R\backslash G \to \Omega_r$ is bijective. Since it is of constant rank, it is a diffeomorphism and our claim follows. We have completely proved the fifth assertion.

Finally, assume u satisfies (7.2). For those values of u_n, the bilinear map

$$(\Phi,\phi_1) \mapsto f_{\Phi,\phi_1,\mu_n,u_n}$$

$$M(n-1\times n, F)\times I_{\mu',u'} \to I_{\mu,u}$$

is continuous. As we have just seen, any element of $I_{\mu,\nu}$ is a sum of functions $f_{\Phi,\phi_1,\mu_n,u_n}$ with $\Phi \in \mathcal{S}(M((n-1)\times n, F))$ (in fact, $\Phi \in \mathcal{C}_c^\infty(\Omega_r)$). It follows that the space spanned by the functions of the form

$$f_{\Phi,\phi_1,\mu_n,u_n}\,,$$

with Φ standard and ϕ_1 K_{n-1}-finite, is dense in $I_{\mu,u}$. Let ξ be an elementary idempotent of K_n. The range $I_{\mu,\sigma}(\xi)$ of the operator

$$\int \rho(k)\xi(k)dk$$

is finite dimensional. The space spanned by the functions $f_{\Phi,\phi_1,\mu_n,u_n}$ with Φ standard such that

$$\Phi(X) = \int \Phi(Xk)\xi(k)\mu_n(\det k)dk$$

is dense in it. Thus it is equal to it. This concludes the proof of the proposition. □

7.2. Integral representation of Whittaker functions. For

$$\Re u_1 < \Re u_2 < \ldots < \Re u_n \,, \tag{7.3}$$

let λ_u be the linear form on $I_{\mu,u}$ defined by the **convergent** integral

$$\lambda_u(f) = \int_{N_n} f(v)\overline{\theta}_{\psi,n}(v)dv\,.$$

By Theorem 15.4.1 in [**26**] II, the linear form extends by analytic continuation into a linear form λ_u on $I_{\mu,u}$, which is defined for all u and never 0. Suppose f_u is a standard section. Then we say that

$$W_u(g) = \lambda_u(\pi_{\mu,u}(g)f_u)$$

is a **standard** family of Whittaker functions.

Now we use Godement sections to define other families of Whittaker functions. We set

$$W_{\Phi,\psi,\phi_1,\mu_n,u_n}(g) = \lambda_u(\rho(g)f_{\Phi,\phi_1,\mu_n,u_n})\,. \tag{7.4}$$

A priori, this is only a meromorphic function of u.

If furthermore u verifies (7.3), then we can write

$$W_{\Phi,\psi,\phi_1,\mu_n,u_n}(g) = \int_{N_n} f_{\Phi,\phi_1,\mu_n,u_n}(vg)\overline{\theta_{\psi,n}}(v)dv\,.$$

We claim that if we replace f_* by its expression as an integral, we obtain a double integral which is absolutely convergent. Indeed, we may assume $\Phi \geq 0$, all μ_i are trivial and all u_i real. We may replace $\theta_{\psi,n}$ by the trivial character. Then the integrand is ≥ 0. The iterated integral is finite. Our claim follows.

It will be convenient to introduce, for u satisfying (7.3), another integral:

$$w_{\Phi,\phi_1,\mu_n,u_n}(g) = \int_{N_{n-1}} f_{\Phi,\phi_1,\mu_n,u_n}\left[\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} g\right]\overline{\theta_{\psi,n-1}}(v)dv\,. \tag{7.5}$$

Again, if we replace f_* by its expression, we obtain a convergent double integral. Thus we can exchange the order of integration. After a change of variables, we obtain

$$w_{\Phi,\phi_1,\mu_n,u_n}(g) = \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \times \int \Phi[(h,0)g]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}d^\times h\,,$$

where we have set

$$W_1(h) := \int \phi_1(vh)\overline{\theta_{n-1,\psi}}(v)dv\,.$$

With this notation, we have, for u satisfying (7.3),

$$W_{\Phi,\phi_1,\mu_n,u_n}(g) = \int_{F^{n-1}} w_{\Phi,\phi_1,\mu_n,u_n}\left[\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix} g\right] \overline{\theta}_{\psi,n}\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix} dv\,.$$

For $g \in G_n$ and $h \in G_{n-1}$, we introduce the notation

$$g.\Phi.h[X] = \Phi[hXg]\,.$$

We obtain in particular a C^∞ left representation of G_n on $\mathcal{S}(M((n-1)\times n, F))$. If Y is in $\mathrm{Lie}(G_n)$, we denote by $Y.\Phi$ the action of Y on Φ. Replacing $w_{\Phi,\phi_1,\mu_n,u_n}$ by its expression, we get

$$\begin{aligned} W_{\Phi,\phi_1,\mu_n,u_n}(g) &= \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \\ &\times \int g.\Phi.h[1_{n-1}, v] W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}\overline{\theta}_{\psi,n}\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix} d^\times h dv\,. \end{aligned}$$

At this point, we introduce the partial Fourier transform $\mathcal{P}(\Phi)$ of a function $\Phi \in \mathcal{S}(M((n-1)\times n, F))$ with respect to the last column. The function $\mathcal{P}(\Phi)$ is thus the function on the same space defined by

$$\mathcal{P}(\Phi)(X_1, X_2, \dots X_{n-1}, X_n) = \int \Phi(X_1, X_2, \dots, X_{n-1}, U)\overline{\psi}(\,{}^tUX_n)dU\,.$$

We denote by e_i, $1 \le i \le n-1$, the canonical basis of F^{n-1}. From now on we view them as **column vectors**. With this notation, we get

$$\begin{aligned} (7.6)\quad W_{\Phi,\phi_1,\mu_n,u_n}(g) &= \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \\ &\times \int \mathcal{P}(g.\Phi.h)[1_{n-1}, e_{n-1}] W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}} d^\times h\,. \end{aligned}$$

More explicitly,

$$\begin{aligned} (7.7)\quad W_{\Phi,\psi,\phi_1,\mu_n,u_n}(g) &= \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \\ &\times \int_{G_{n-1}} \mathcal{P}(g.\Phi)[h, h^\iota e_{n-1}] W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1} d^\times h\,. \end{aligned}$$

At this point, some remarks are in order. A priori, the equality is valid for u satisfying (7.3). The left hand side is a holomorphic function of u. As we are going to see in the next proposition, the integral on the right converges for all u and thus defines an entire function of u. Thus the equality is in fact true for all u. Finally, the equality shows that the left hand side depends only on W_1 (which is a holomorphic function of $u' = (u_1, u_2, \dots, u_{n-1})$. Thus we can also use the notation $W_{\Phi,\psi,W_1,\mu_n,u_n}$ for the left hand side.

Proposition 7.2. *Let $W_1 \in \mathcal{W}(\pi_{\mu',u'} : \psi)$ and Φ a Schwartz function. The integral (7.7) converges absolutely for all u_n. More precisely, suppose that $W_1 = W_{1,u'}$ is a standard family of Whittaker functions. Then the integral converges uniformly for u' in a compact set Ω' of $\mathbb{C}^{n-1}$ and u_n in a compact set Ω of $\mathbb{C}$. Furthermore, given Ω' and Ω and $X \in \mathfrak{U}(G_n)$, there is $M > 0$ and a continuous semi-norm c_X on the space of Schwartz functions such that*

$$|\rho(X)W_{\Phi,\psi,W_1,\mu_n,u_n}(g)| \le c_X(\Phi)||g||^M$$

for all Φ, $u' \in \Omega'$, $u_n \in \Omega$.

PROOF. Let us prove the convergence for $g = e$. We set

$$h = k\beta$$

where k is in K_{n-1} and β is an upper triangular matrix with diagonal part

$$b = \operatorname{diag}(b_1, b_2, \dots, b_{n-1}) .$$

For the purpose of proving convergence, we may replace $W_1(h^{-1})$ by

$$\xi_{h,n-1}(b^{-1})^{-N} ||b||^M ,$$

assume u_n real, μ_n trivial and replace $\mathcal{P}\Phi$ by $\Phi_0 \geq 0$. We find

$$\int \Phi_0[k\beta, \overline{k} b_{n-1}^{-1} e_{n-1}] \; \xi_{h,n-1}(b^{-1})^{-N} |\det b|^{u_n + \frac{n}{2} - 1} ||b||^M d\beta dk .$$

After integrating over k, and the variables above the diagonal, we are reduced to

$$\int \phi(b_1, b_2, \dots, b_{n-1}, b_{n-1}^{-1}) \xi_{h,n-1}(b^{-1})^{-N} ||b||^M |\det b|^{u_n + \frac{n}{2} - 1} J(b) db dk$$

where $J(b)$ is a Jacobian factor and $\phi \geq 0$ is a Schwartz function. Since $||b||^M$ is a sum of positive characters, we are reduced to showing that, given a character $\chi > 0$, the following integral is finite, provided N is large enough,

$$\int \phi(b_1, b_2, \dots, b_{n-1}, b_{n-1}^{-1}) \xi_{h,n-1}(b^{-1})^{-N} \chi(b) db .$$

Now, for $N >> 0$,

$$\phi(b_1, b_2, \dots, b_{n-1}, b_{n-1}^{-1}) \preceq \prod_{i=1}^{n-1} (1 + b_i^2)^{-N} (1 + b_{n-1}^2)^{-N}$$

and there is $m > 0$ such that

$$\xi_{h,n-1}(b^{-1})^m (1 + b_{n-1}^{-2})^m \geq \prod_{i=1}^{n-1} (1 + b_i^{-2}) .$$

Thus we are reduced to an integral of the form

$$\int \prod_{i=1}^{n-1} (1 + b_i^2)^{-N} \prod_{i=1}^{n-1} (1 + b_i^{-2})^{-N} \chi(b) db$$

which converges for $N >> 0$.

Let us prove the estimate for $X = 1$. We write $g = vak$, $v \in N - n$, $a \in A_n$, $k \in K_n$. Since $k.\Phi$ remains in a bounded set, we are reduced to the estimate for $g = a$. Since W_* transforms under a character of the center, we may even assume $a_n = 1$. Following the above computation, we are led to replace ϕ by a character $\eta(a)$ times

$$\phi(a_1 b_1, a_2 b_2, \dots, a_{n-1} b_{n-1}, b_{n-1}^{-1}) .$$

Now

$$\frac{1}{1 + a^2 b^2} \leq \frac{a^2 + a^{-2}}{1 + b^2} .$$

Thus for every N,

$$\phi(a_1 b_1, a_2, b_2, \dots, a_n b_n, b_{n-1}^{-1}) \preceq ||a||^M \prod_{1 \leq i \leq n-1} (1 + b_i^2)^{-N} (1 + b_{n-1}^{-2})^{-N} ,$$

where M depends on N. Our assertion follows.

Finally, to find the estimate with a given X we observe that W_* transforms under a character of the center. Thus we may assume $X \in \mathfrak{U}(SL(n,F))$ and then replace Φ by $X.\Phi$. $\square$

Writing explicitly the definition of $\mathcal{P}(\Phi)$, we get from (7.6)

$$W_{\Phi,\psi,W_1,\mu_n,u_n}(g) = \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}}$$
$$\times \int_{G_{n-1}} \left(\int_{F^{n-1}} g.\Phi.h[1_{n-1},X]\overline{\psi}(\,{}^te_{n-1}X)dX \right) W_1(h^{-1})\mu_n(\det h)^{u_n+\frac{n}{2}}d^\times h\,.$$

The formula is to be understood in terms of **iterated integrals**, as each of the indicated integrals converge absolutely. Furthermore, we can replace h by hv with $v \in N_{n-1}$ and $h \in G_{n-1}/N_{n-1}$. We get then

$$\mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}}$$
$$\times \iint \left(\int g.\Phi.h[v,vX]\overline{\psi}(\,{}^te_{n-1}X)dX \right) \overline{\theta_{\psi,n-1}}(v)dvW_1(h^{-1})\mu_n(\det g)^{u_n+\frac{n}{2}}dh\,.$$

We can change X to $v^{-1}X$ to get

$$\mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}}$$
$$\times \iint \left(\int g.\Phi.h[v,X]\overline{\psi}(\,{}^te_{n-1}X)dX \right) \overline{\theta_{\psi,n-1}}(v)dvW_1(h^{-1})\mu_n(\det h)^{u_n+\frac{n}{2}}dh\,.$$

The outer integral is over G_{n-1}/N_{n-1}.

We can combine the iterated integrals in v and X into a double absolutely convergent integral. We arrive at the following expression

$$W_{\Phi,\psi,W_1,\mu_n,u_n}(g) := \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \tag{7.8}$$
$$\times \int \left(\iint g.\Phi.h[v,X]\overline{\psi}(\,{}^te_{n-1}X)\overline{\theta_{\psi,n-1}}(v)dXdv \right) W_1(h^{-1})\mu_n(\det h)^{u_n+\frac{n}{2}}dh\,.$$

Here $v \in N_{n-1}$, $h \in G_{n-1}/N_{n-1}$ and $X \in F^{n-1}$ (column vectors). We stress the finiteness of the integrals

$$\iint \left|g.\Phi.h[v,X]\overline{\psi}(\,{}^te_{n-1}X)\overline{\theta_{\psi,n-1}}(v)\right| dXdv < +\infty$$

and

$$\int \left| \iint g.\Phi.h[v,X]\overline{\psi}(\,{}^te_{n-1}X)\overline{\theta_{\psi,n-1}}(v)dXdv \right| \left|W_1(h^{-1})\mu_n(\det h)^{u_n+\frac{n}{2}}\right| dh < \infty\,.$$

7.3. A functional equation. We now prove that our integral representation satisfies a functional equation. Recall the notation

$$\widetilde{W}(g) = W(w_n g^\iota)\,.$$

PROPOSITION 7.3.

$$\widetilde{W}_{\Phi,\psi,W_1,\mu_n,u_n}(g) = \mu_n(-1)^{n-1}W_{\widehat{\Phi},\overline{\psi},\widetilde{W_1},\mu_n^{-1},-u_n}\,.$$

PROOF. We need a lemma.

LEMMA 7.1. *For any* $\Phi \in \mathcal{S}(M((n-1)\times n, F))$

$$\iint w_n.\Phi[v,X]\overline{\psi}(\ {}^t e_{n-1}X)\overline{\theta_{\psi,n-1}}(v)dXdv$$
$$= \iint \widehat{\Phi}.w_{n-1}[v,X]\psi({}^t e_{n-1}X)\theta_{\psi,n-1}(v)dXdv\,,$$

where the integrals are for $X \in F^{n-1}$, $v \in N_{n-1}$.

PROOF. We illustrate the case $n = 5$ but the argument is general. Then the formula reduces to the equality of the following two integrals:

$$\int \Phi \begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} & 1 \\ x_{1,2} & x_{2,2} & x_{3,2} & 1 & 0 \\ x_{1,3} & x_{2,3} & 1 & 0 & 0 \\ x_{1,4} & 1 & 0 & 0 & 0 \end{pmatrix} \overline{\psi}(x_{1,4}+x_{2,3}+x_{3,2}+x_{4,1}) \ \otimes dx_{i,j}\,,$$

$$\int \widehat{\Phi} \begin{pmatrix} 0 & 0 & 0 & 1 & y_{5,1} \\ 0 & 0 & 1 & y_{4,2} & y_{5,2} \\ 0 & 1 & y_{3,3} & y_{4,3} & y_{5,3} \\ 1 & y_{2,4} & y_{3,4} & y_{4,4} & y_{5,4} \end{pmatrix} \psi(y_{2,4}+y_{3,3}+y_{4,2}+y_{5,1}) \ \otimes dy_{i,j}\,.$$

The equality follows from the Fourier inversion formula. □

With

$$W = W_{\Phi,\psi,W_1,\mu_n,u_n}\,,$$

we have

$$\widetilde{W}(g) = \mu_n(\det w_n)\mu_n^{-1}(\det g)|\det g|^{-u_n-\frac{n-1}{2}}$$
$$\times \int \left(\iint w_n g^\iota.\Phi.h[v,X]\overline{\psi}(\ {}^t e_{n-1}X)\overline{\theta_{\psi,n-1}}(v)dXdv\right)$$
$$\times W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}d^\times h\,.$$

We apply the previous lemma to the function $g^\iota.\Phi.h$ whose Fourier transform is the function $g.\widehat{\Phi}.h^\iota\ |\det g|^{n-1}\ |\det h|^{-n}$. We get

$$\mu_n(\det w_n)\mu_n^{-1}(\det g)|\det g|^{-u_n+\frac{n-1}{2}}$$
$$\times \int \left(\iint g.\widehat{\Phi}.h^\iota.w_{n-1}[v,X]\psi(\ {}^t e_{n-1}X)\theta_{\psi,n-1}(v)dXdv\right)$$
$$\times W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n-\frac{n}{2}}d^\times h\,.$$

We do a last change of variables setting $h_0 = h^\iota w_{n-1}$. Then

$$W_1(h^{-1}) = \widetilde{W_1}(h_0^{-1})\,,\ \mu_n(\det h) = \mu_n(\det w_{n-1})\mu_n^{-1}(\det h_0)\,.$$

Thus we arrive at

$$\mu_n(\det w_n \det w_{n-1})mu_n^{-1}(\det g)|\det g|^{-u_n+\frac{n-1}{2}}$$
$$\times \int \left(\iint g.\widehat{\Phi}.h_0[v,X]\psi({}^t e_{n-1}X)\theta_{n-1}(v)dXdv\right)$$
$$\times\ \widetilde{W_1}(h_0^{-1})\mu_n^{-1}(\det h_0)|\det h_0|^{-u_n+\frac{n}{2}}dh_0\,.$$

Since $\det w_n \det w_{n-1} = (-1)^{n-1}$, our assertion follows. □

REMARK 7.2. In the above functional equation, the following replacements take place:

$$\begin{array}{rcl}(u_1, u_2, \dots u_{n-1}, u_n) & \mapsto & (-u_{n-1}, -u_{n-2}, \dots, -u_1, -u_n) \\ (\mu_1, \mu_2, \dots \mu_{n-1}, \mu_n) & \mapsto & (\mu_{n-1}^{-1}, \mu_{n-2}^{-1}, \dots, \mu_1^{-1}, \mu_n^{-1}) \\ \psi & \mapsto & \overline{\psi}\,. \end{array}$$

In particular if u satisfies (7.2), in general the n-tuple

$$(-u_{n-1}, -u_{n-2}, \dots, -u_1, -u_n)$$

does not, unless

$$-1 < \Re u_n - \Re u_i < 1\,,\ 1 \leq i \leq n-1\,. \tag{7.9}$$

8. Theorem 2.1: principal series, pairs $(n, n), (n, n-1)$

In this section and the two next sections, we prove Theorem 2.1 for the induced representations $I_{\mu,u}$ (principal series). In this section, we treat the case $n' = n$ and $n' = n-1$. The proof is by induction on n. The case of the pairs $(1, 1)$ or $(1, 0)$ is simply the local theory of Tate's integral. Assuming the theorem for the pair $(n, n-1)$, we prove it for the pair (n, n) by replacing the Whittaker function W on G_n by its integral representation. The integral representation contains a Schwartz function Φ. Formal manipulations transform the integral $\Psi(s, W, W', \Phi_1)$ into the product of an integral for the pair $(n, n-1)$ and an integral $Z(s, f, \Phi_0)$ ([**11**]) for the group G_n. The Schwartz function Φ_0 is built out of Φ and Φ_1. Likewise, assuming the theorem for the pair $(n-1, n-1)$, we prove it for the pair $(n, n-1)$. Again we replace the Whittaker function W on G_n by its integral representation which contains a Schwartz function Φ. Formal manipulations transform the integral $\Psi(s, W, W')$ into the product of an integral for the pair $(n-1, n-1)$ (which contains a Schwartz function Φ_1) and an integral $Z(s, f, \Phi_0)$ ([**11**]) for the group G_{n-1}. The Schwartz function Φ gives rise to the functions Φ_0 and Φ_1.

8.1. Statement of the Theorem. For clarity we state again the functional equations (n, n) and $(n, n-1)$ for the induced representations $I_{\mu,u}$.

For the case (n, n), we consider two pairs (μ, u) and (ν, v) where μ and ν are n-tuple of normalized characters and u, v are in $\mathbb{C}^n$. We let W be in $\mathcal{W}(\pi_{\mu,u} : \psi)$ and $W' \in \mathcal{W}(\pi_{\nu,v} : \overline{\psi})$. Finally, we let Φ_1 be in $\mathcal{S}(F^n)$. Then the integral

$$\Psi(s, W, W', \Phi_1)\,,$$

defined for $\Re s >> 0$, extends to a holomorphic multiple of

$$\prod_{i,j} L(s + u_i + v_j, \mu_i \nu_j)\,,$$

bounded at infinity in vertical strips. Likewise,

$$\Psi(s, \widetilde{W}, \widetilde{W}', \Phi_1)$$

is a holomorphic multiple of

$$\prod_{i,j} L(s - u_i - v_j, \mu_i^{-1} \nu_j^{-1})\,,$$

bounded at infinity in vertical strips. Finally, the functional equation

$$\frac{\Psi(1-s,\widetilde{W},\widetilde{W}',\widehat{\Phi}_1)}{\prod_{i,j} L(s-u_i-v_j,\mu_i^{-1}\nu_j^{-1})}$$
$$=\prod_j \mu_j(-1)^{n-1}\prod_{i,j}\epsilon(s+u_i+u_j,\mu_i\nu_j,\psi)\ \frac{\Psi(s,W,W',\Phi_1)}{\prod_{i,j} L(s+u_i+v_j,\mu_i\nu_j)}$$

holds, in the sense of analytic continuation.

For the case $(n,n-1)$, we consider two pairs (μ,u) and (ν,v) where μ is an n-tuple of normalized characters, ν is a $(n-1)$-tuple, $u\in\mathbb{C}^n$, $v\in\mathbb{C}^{n-1}$. We let $W\in\mathcal{W}(\pi_{\mu,u}:\psi)$ and $W'\in\mathcal{W}(\pi_{\nu,v}:\overline{\psi})$. Then the integral

$$\Psi(s,W,W')\,,$$

defined for $\Re s>>0$, extends to a holomorphic multiple of

$$\prod_{i,j} L(s+u_i+v_j,\mu_i\nu_j)\,,$$

bounded at infinity in vertical strips. Likewise

$$\Psi(s,\widetilde{W},\widetilde{W}')$$

is a holomorphic multiple of

$$\prod_{i,j} L(s-u_i-v_j,\mu_i^{-1}\nu_j^{-1})\,,$$

bounded at infinity in vertical strips. Finally, the functional equation

$$\frac{\Psi(1-s,\widetilde{W},\widetilde{W}')}{\prod_{i,j} L(s-u_i-v_j,\mu_i^{-1}\nu_j^{-1})}$$
$$=\prod_i \mu_i(-1)^{n-1}\prod_j \nu_j(-1)\prod_{i,j}\epsilon(s+u_i+u_j,\mu_i\nu_j)\ \frac{\Psi(s,W,W')}{\prod_{i,j} L(s+u_i+v_j,\mu_i\nu_j)}$$

holds, in the sense of analytic continuation.

As we have seen in Section 2, it suffices to prove the assertions for one choice of ψ. Thus we may assume ψ standard. Set

$$\sigma_u=(\oplus\mu_i\otimes\alpha^{u_i})\otimes(\oplus\nu_j\otimes\alpha^{v_j})\,.$$

As the notation suggests, v will be **constant** in the computation. We let (θ_u,κ_u) be a holomorphic family of (σ_u,ψ) pairs. We define

$$\Psi(\theta_u,W,W',\Phi_1)=\int W(g)W'(g)\theta_u(\det g)\Phi_1(\epsilon_n g)dg\,,$$
$$\Psi(\kappa_u,\widetilde{W},\widetilde{W}',\widehat{\Phi}_1)=\int \widetilde{W}(g)\widetilde{W}'(g)\kappa_u(\det g)\widehat{\Phi}_1(\epsilon_n g)dg\,.$$

These integrals are absolutely convergent. Then the above assertions for (n,n) are equivalent to the functional equations

$$\Psi(\theta_u,W,W',\Phi_1)=\prod_i \mu_i(-1)^{-n-1}\prod_j \nu_j(-1)\Psi(\kappa_u,\widetilde{W},\widetilde{W}',\widehat{\Phi}_1)\,.$$

Now let $W=W_u$ be a standard family of Whittaker functions. Then both sides are entire functions of u. Thus it suffices to prove the assertions for u in a connected open set, for instance, the open set defined by (7.3). Moreover, if we write W as

W_ϕ with $\phi \in I_{\mu,u}$, then both sides are continuous functions of ϕ. Thus we may assume W is K_n-finite. Likewise, we may assume W' is K_n-finite. Furthermore, both sides are continuous functions of Φ_1. Thus we may assume Φ_1 is a standard Schwartz function. In addition, W being K_n-finite is then of the form

$$W = W_{\Phi, W_1, \mu_n, u_n, \psi}\,,$$

with W_1 K_{n-1}-finite and Φ standard. Thus it suffices to prove the assertions of Theorem 2.1 for Φ_1 standard, W' K_n-finite, W of the above form, and any u.

The case $(n, n-1)$ is similar with

$$\Psi(\theta_u, W, W') = \int W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g)\theta_u(\det g)|\,|\det g|^{-\frac{1}{2}} dg\,.$$

8.2. Case (n, n)**.** We assume that we know Theorem 2.1 for the pair $(n, n-1)$. We prove Theorem 2.1 for the pair (n, n). To that end, we consider

$$\Psi(s, W, W', \Phi_1)$$

where $W = W_{\Phi,\psi,W_1,\mu_n,u_n}$ and $W' \in \mathcal{W}(\pi_{\nu,v} : \overline{\psi})$. Assume for now that u is in the set defined by (7.2). Then we can set

$$w = w_{\Phi,\psi,W_1,\mu_n,u_n}$$

and write

$$W(g) = \int w\left[\begin{pmatrix} 1_{n-1} & X \\ 0 & 1 \end{pmatrix} g\right] \overline{\theta_{\psi,n}} \begin{pmatrix} 1_{n-1} & X \\ 0 & 1 \end{pmatrix} dX\,.$$

Indeed, the integrals are absolutely convergent under assumption (7.2).

Then

$$\begin{aligned} \Psi(s, W, W', \Phi_1) &= \int_{N_n\backslash G_n} W(g)W'(g)\Phi_1[\epsilon_n g]\; |\det g|^s dg \\ &= \int_{N_{n-1}\backslash G_n} w(g)W'(g)\Phi_1[\epsilon_n g]|\det g|^s dg\,, \end{aligned}$$

where we embed N_{n-1} into G_n the obvious way:

$$v \mapsto \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}.$$

Replacing w by its integral expression, we get

$$\int_{N_{n-1}\backslash G_n} |\det g|^{s+\frac{n-1}{2}+u_n} \mu_n(\det g)$$
$$\times \left(\int_{G_{n-1}} \Phi[(h,0)g] W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+n/2} dh\right) W'(g)\Phi_1[\epsilon_n g] dg.$$

Now

$$\int_{G_{n-1}} \Phi[(h,0)g]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+n/2}dh\ W'(g)$$

$$= \int_{G_{n-1}/N_{n-1}} \left(\int_{N_{n-1}} \Phi\left[(h,0)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} g\right] W_1(h^{-1})\overline{\theta_{n-1,\psi}}(u)du \right)$$

$$\times\, \mu_n(\det h)|\det h|^{u_n+n/2}dhW'(g)$$

$$= \int_{G_{n-1}/N_{n-1}} \left(\int_{N_{n-1}} \Phi\left[(h,0)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} g\right] W_1(h^{-1})W'\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} g\right) du \right)$$

$$\times\, \mu_n(\det h)|\det h|^{n/2}dh\,.$$

Combining the integral over $N_{n-1}\backslash G_n$ and N_{n-1}, we can write the formula for Ψ as

$$\int_{G_{n-1}/N_{n-1}} \left(\int_{G_n} \Phi[(h,0)g]W'(g)\Phi_1[\epsilon_n g]|\det g|^{s+\frac{n-1}{2}+u_n}\mu_n(\det g)dg \right)$$

$$\times\, W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}}dh\,.$$

We change g to

$$\begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} g.$$

We get

$$\int |\det g|^{s+\frac{n-1}{2}+u_n}\mu_n(\det g)\Phi[(1_{n-1},0)g]\Phi_1[\epsilon_n g]$$

$$\times\, W' \left[\begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} g\right] W_1(h^{-1})|\det h|^{1/2-s}dhdg\,.$$

We set

$$\Phi_0(g) = \Phi[(1_{n-1},0)g]\Phi_1[\epsilon_n g]\,.$$

After changing h to h^{-1}, we arrive at our final expression

$$\Psi(s,W,W',\Phi_1) = \int_{G_n}\int_{N_{n-1}\backslash G_{n-1}} W'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right]\Phi_0(g) \tag{8.1}$$

$$\times\, \mu_n(\det g)|\det g|^{u_n+s+\frac{n-1}{2}}dgW_1(h)|\det h|^{s-\frac{1}{2}}dh\,.$$

We need to justify our computations. We **claim** the following. Suppose that Ω is an open, relatively compact set of $\mathbb{C}^n$. Then there is A such that for $\Re s \geq A$ and $u \in \Omega$, the double integral (8.1) is absolutely convergent. Moreover, the convergence is uniform if we impose $B \geq \Re s \geq A$. If we take Ω contained in (7.2), this will show that our computation is justified. Moreover, by analytic continuation, this will show that if Ω is **any** open, relatively compact set of $\mathbb{C}^n$, there is A such that for $u \in \Omega$ and $\Re s > A$ the integral in (8.1) is absolutely convergent and equal to $\Psi(s,W,W',\Phi_1)$.

It remains to prove our claim. To that end, we may assume $\Phi_0 \geq 0$. We may replace $|W'|$ by

$$\xi_{h,n}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}^{-N}\ ||h||^M\ ||g||^M$$

and $|W_1|$ by $||h||^M$. We are reduced to study the convergence of the following two integrals:

$$\int \xi_{h,n}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}^{-N} ||h||^M |\det h|^{\Re s - \frac{1}{2}} ,$$

$$\int \Phi_0(g)\ ||g||^M\ |\det g|^{\Re s + \Re u_n + \frac{n-1}{2}} dg .$$

By Lemma 3.5, there are A, B, C such that the first integral converges for $N > A, \Re s > B, CN > \Re s$. The convergence of the second integral for $\Re s >> 0$ follows from Lemma 3.3. Our claim is proved. Thus formula (8.1) is true for **any** u.

Similarly, for $\Re s << 0$,

$$\Psi(1-s, \widetilde{W}, \widetilde{W'}, \widehat{\Phi_1}) = \mu_n(-1)^{n-1} \iint \widetilde{W'}\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right] \widehat{\Phi}_0(g)$$
$$\times\, \mu_n^{-1}(\det g) |\det g|^{-u_n + 1 - s + \frac{n-1}{2}} dg \widetilde{W}_1(h) |\det h|^{\frac{1}{2} - s} dh .$$

Indeed, by Proposition 7.3, it suffices to replace W' by $\widetilde{W'}$, Φ_1 by $\widehat{\Phi_1}$, Φ by $\widehat{\Phi}$, W_1 by $\widetilde{W_1}$, μ_n by μ_n^{-1}, u_n by $-u_n$, and insert the factor $\mu_n(-1)^{n-1}$.

To orient the reader, we first establish the functional equation **formally**. Applying the $(n, n-1)$ functional equation to the h-integral, we find that $\Psi(1-s, \widetilde{W}, \widetilde{W'}, \widehat{\Phi_1})$ is equal to

$$\mu_n(-1)^{n-1} \prod_{1 \le i \le n-1} \mu_i(-1) \prod_{1 \le j \le n} \nu_j(-1)^{n-1} \prod_{1 \le i \le n-1, 1 \le j \le n} \gamma(s + u_i + v_j, \mu_i \nu_j, \overline{\psi})$$
$$\times \iint W'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g^\iota\right] \widehat{\Phi}_0(g) \mu_n^{-1}(\det g) |\det g|^{1 - s + \frac{n-1}{2}} dg W_1(h) |\det h|^{s - \frac{1}{2}} dh .$$

Recall

$$\gamma(s + u_i + v_j, \mu_i \nu_j, \overline{\psi}) = \mu_i(-1)\nu_j(-1)\gamma(s + u_i + v_j, \mu_i \nu_j, \psi) .$$

Thus we can rewrite the above expression as

$$\prod_{1 \le i \le n} \mu_i(-1)^{n-1} \prod_{1 \le i \le n-1, 1 \le j \le n} \gamma(s + u_i + v_j, \mu_i \nu_j, \psi)$$
$$\times \iint W'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g^\iota\right] \widehat{\Phi}_0(g) \mu_n^{-1}(\det g) |\det g|^{1 - s + \frac{n-1}{2}} dg W_1(h) |\det h|^{s + \frac{1}{2}} dh .$$

We now apply the functional equation of Proposition 4.4 to the g integral. We get

$$\prod_{1 \le i \le n} \mu_i(-1)^{n-1} \prod_{1 \le j \le n, 1 \le i \le n} \gamma(s + u_i + v_j, \mu_i \nu_j, \psi)$$
$$\times \iint W'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right] \Phi_0(g) \mu_n(\det g) |\det g|^{s + \frac{n-1}{2}} dg W_1(h) |\det h|^{s + \frac{1}{2}} dh .$$

Thus we get the correct functional equation.

Now we make the proof **rigorous**. As we have observed, we may assume that the Schwartz functions are standard and ψ is standard. Then the rest of the proof does not depend on the theory of the (σ_u, ψ) pairs. The function Φ_0 is then K_n-finite on both sides. Thus there is an elementary idempotent ξ on K_n such that

$$\int_{K_n} \Phi_0(k^{-1}X) \mu_n^{-1}(\det k) \xi(k) dk = \Phi_0(X) .$$

We can insert this in the integral for Ψ and change g to kg to obtain

$$\int\iint\left(\int W'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} kg\right]\xi(k)dk\right)$$
$$\times\ \Phi_0(g)\mu_n(\det g)|\det g|^{u_n+s+\frac{n-1}{2}}dgW_1(h)|\det h|^{s-\frac{1}{2}}dh\,.$$

Now let v'_i be a basis of the image of the operator

$$\int_{K_n}\xi(k)\pi_{\nu,v}(k)dk$$

in the space $I_{\nu,v}$. Let W'_i be the corresponding elements of $\mathcal{W}(\pi_{\nu,v}:\overline{\psi})$. Then

$$\int W'(xkg)\xi(k)dk=\sum_i W'_i(x)f'_i(g),$$

where the functions f'_i are matrix coefficients of the representation $I_{\nu,v}$. We see that Ψ decomposes into a sum of products, namely,

$$\sum_i\int W'_i\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right]W_1(h)|\det h|^{s-\frac{1}{2}}dh$$
$$\times\int\Phi_0(g)f'_i(g)\mu_n(\det g)|\det g|^{u_n+s+\frac{n-1}{2}}dg\,.$$

Thus

$$\Psi(s,W,W',\Phi_1)=\sum_i\Psi(s,W'_i,W_1)Z(s,\Phi_0,f'_i\otimes\mu_n)\,. \tag{8.2}$$

By the induction hypothesis, $\Psi(s,W'_i,W_1)$ is a holomorphic multiple of

$$\prod_{1\le i\le n-1,1\le j\le n-1}L(s+u_i+v_j,\mu_i\otimes\nu_j)\,,$$

bounded at infinity in vertical strips. On the other hand, $Z(s,\Phi_0,f'_i\otimes\mu_n)$ is a holomorphic multiple of

$$\prod_{1\le j\le n-1}L(s+u_n+v_j,\mu_n\nu_j)\,,$$

bounded at infinity in vertical strips. Thus the analytic properties of the integral have been established. Likewise for the symmetric integral. It remains to establish the functional equation.

We have

$$\int\widehat{\Phi_0}(k^{-\iota}X)\mu_n^{-1}(\det k)\xi(k)dk=\widehat{\Phi_0}(X)$$

or, changing k to k^ι,

$$\int\widehat{\Phi_0}(k^{-1}X)\mu_n(\det k)\xi^\iota(k)dk=\widehat{\Phi_0}(X)\,.$$

On the other hand,

$$\int\widetilde{W'}[xkg]\xi^\iota(k)dk=\sum_i\widetilde{W'_i}(x)\widetilde{f'_i}(g)\,,$$

where we have set

$$\widetilde{f'_i}(g)=f'_i(g^\iota)\,.$$

Thus

$$\Psi(1-s,\widetilde{W},\widetilde{W}',\widehat{\Phi_0}) = \mu_n(-1)^{n-1}\sum_i \Psi(1-s,\widetilde{W_i'},\widetilde{W_1})Z(1-s,\widetilde{f_i'}\otimes\mu_n^{-1},\widehat{\Phi_0})\,.$$

The stated functional equation follows now from the induction hypothesis and the functional equation of Proposition 4.4.

8.3. Case $(n, n-1)$**.** Now we assume Theorem 2.1 for the pair $(n-1, n-1)$ and we prove Theorem 2.1 for the pair $(n, n-1)$. As before, for now, we only deal with principal series representations.

Here we use the integral representation for

$$W = W_{\Phi,\psi,W_1,\mu_n,u_n}$$

in the following form. We assume as we may that Φ is a product in the following way. If Y is an $n-1\times n-1$ matrix and X a column matrix of size $n-1$, then

$$\Phi(Y,X) = \Phi_1(Y)\Phi_2(X)\,.$$

Then

$$\begin{aligned} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} =& |\det g|^{u_n+\frac{n-1}{2}}\mu_n(\det g) \\ &\times \int \Phi_1[hg]\widehat{\Phi}_2\left[h^\iota e_{n-1}\right]\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}dh\,. \end{aligned}$$

Now we substitute this integral representation of W in the integral $\Psi(s,W,W')$. We get

$$\begin{aligned} \Psi(s,W,W') = &\iint |\det g|^{u_n+s+\frac{n}{2}-1}\mu_n(\det g)|\det h|^{u_n+\frac{n}{2}-1}\mu_n(\det h) \\ &\times \Phi_1[hg]\widehat{\Phi}_2\left[h^\iota e_{n-1}\right]W_1(h^{-1})W'(g)dhdg\,. \end{aligned}$$

We change h to hg^{-1}.

$$\iint |\det g|^s|\det h|^{u_n+\frac{n}{2}-1}\mu_n(\det h)\Phi_1(h)\widehat{\Phi}_2\left[h^\iota\ {}^tge_{n-1}\right]W_1(gh^{-1})W'(g)dhdg\,.$$

Next we change g to gh. We arrive at

$$\begin{aligned} \Psi(s,W,W') = &\int |\det g|^s|\det h|^{u_n+s+\frac{n}{2}-1}\mu_n(\det h) \\ &\times \Phi_1(h)\mathcal{F}_\psi(\Phi_2)\left[\ {}^tge_{n-1}\right]W_1(g)W'(gh)dhdg\,. \end{aligned} \tag{8.3}$$

Here $g\in N_{n-1}\backslash G_{n-1}$, $h\in G_{n-1}$.

To justify our computations, it suffices to prove the last expression is absolutely convergent for $\Re s >> 0$. As before, we majorize

$$\begin{aligned} |W'(gh)| &\leq ||g||_I^M\ ||h||^M\,, \\ |W_1(g)| &\leq \xi_{h,n-1}(g)^{-N}||g||_I^M\,. \end{aligned}$$

We are reduced to prove the absolute convergence of integrals of the form

$$\int_{N_{n-1}\backslash G_{n-1}} \xi_{h,n-1}(g)^{-N}||g||_I^M\ \left|\widehat{\Phi}_2\right|\left[\ {}^tge_{n-1}\right]\ |\det g|^{\Re s}dg\,,$$

$$\int |\det h|^{\Re s}\ ||h||^M\ \Phi_1(h)dh\,.$$

For the first integral, there are A, B, C such that the integral converges for $N > A, \Re s > B, NC > \Re s$ (Lemma 3.5). The second integral converges for $\Re s >> 0$ (Lemma 3.3). Our assertion follows.

For the symmetric integral, we must do the replacements $\psi \mapsto \overline{\psi}$, $\Phi_1 \mapsto \mathcal{F}_\psi(\Phi_1)$, $\Phi_2 \mapsto \mathcal{F}_\psi \Phi_2$ (and other replacements). Since $\mathcal{F}_{\overline{\psi}}\mathcal{F}_\psi(\Phi_2) = \Phi_2$ we get, for $\Re s << 0$,

$$\Psi(1-s, \widetilde{W}, \widetilde{W'}) = \mu_n(-1)^{n-1} \int |\det g|^{1-s} |\det h|^{-u_n+1-s+\frac{n}{2}-1} \mu_n^{-1}(\det h) \times \mathcal{F}_\psi(\Phi_1)(h)\Phi_2\left[\,{}^t g e_{n-1}\right] \widetilde{W}_1(g)\widetilde{W'}(gh) dh dg\,. \tag{8.4}$$

Again, we first prove the functional equation **formally**. By the functional equation for the pair $(n-1, n-1)$ applied to the g-integral, we get that $\Psi(1-s, \widetilde{W}, \widetilde{W'})$ is equal to

$$\mu_n(-1)^{n-1} \prod_{1\le i\le n-1} \mu_i(-1)^{n-2} \prod_{1\le i\le n-1, 1\le j\le n-1} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times \int |\det g|^s |\det h|^{1-s+\frac{n}{2}-1-u_n} \mu_n(\det h)^{-1} W_1(g) W'(g\ h^\iota) \widehat{\Phi}_1(h) \widehat{\Phi}_2[-\ {}^t g e_{n-1}] dh dg\,,$$

because Φ_2 is the Fourier transform of $X \mapsto \widehat{\Phi}_2(-X)$. After changing g into $-g$, we find

$$\prod_{1\le i\le n} \mu_i(-1)^{n-1} \prod_{1\le i\le n-1} \nu_i(-1) \prod_{1\le i\le n-1, 1\le j\le n-1} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times \int |\det g|^s |\det h|^{1-s+\frac{n}{2}-1-u_n} \mu_n(\det h)^{-1} W_1(g) W'(g\ h^\iota) \widehat{\Phi}_1(h) \widehat{\Phi}_2\left[\,{}^t g e_{n-1}\right] dh dg\,.$$

Now we apply the functional equation of Proposition 4.4 to the h integral. We get

$$\prod_{1\le i\le n} \mu_i(-1)^{n-1} \prod_{1\le i\le n-1} \nu_i(-1) \prod_{1\le i\le n, 1\le j\le n-1} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times \int |\det g|^s |\det h|^{s+\frac{n}{2}-1+u_n} \mu_n(\det h) \Phi_1(h) \widehat{\Phi}_2\left[\,{}^t g e_{n-1}\right] W_1(g) W'(gh) dh dg\,.$$

Now we make the proof rigorous. We assume as before that Φ_1 is a standard Schwartz function. Thus there exists an elementary idempotent ξ of K_{n-1} such that

$$\int \Phi_1(k^{-1}X)\mu_n(\det k)^{-1}\xi(k) dk = \Phi_1(X)\,.$$

Substituting this identity into the integral for $\Psi(s, W, W')$ and changing h into kh, we get

$$\int |\det g|^s |\det h|^{u_n+s+\frac{n}{2}-1} \mu_n(\det h)$$
$$\times\ \Phi_1(h)\widehat{\Phi}_2\left[\,{}^t g \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] W_1(g) \left(\int W'(gkh)\xi(k) dk\right) dh dg\,.$$

As before,

$$\int W'(gkh)\xi(k) dk = \sum_i W'_i(g) f_i(h)$$

with $W_i' \in \mathcal{W}(\pi_{\nu,v};\overline{\psi})$ and f_i a matrix coefficient of $\pi_{\nu,v}$. Thus our integral is then a sum of products

$$\Psi(s,W,W') = \sum_i \Psi(s,W_1,W_i',{}^t\widehat{\Phi}_2)\ Z(s+u_n,f_i\otimes\mu_n,\Phi_1)\,.$$

In each term, the first integral is a holomorphic multiple of

$$\prod_{1\le i\le n-1, 1\le j\le n-1} L(s+u_i+v_j,\mu_i\nu_j)\,,$$

bounded at infinity in vertical strips, and the second integral, a holomorphic multiple of

$$\prod_{1\le j\le n-1} L(s+u_n+v_j,\mu_n\nu_j)\,,$$

bounded at infinity in vertical strips. Thus the integral Ψ has the required analytic properties. Likewise for the symmetric integral. The functional equation is proved as before.

8.4. Partial Proof of Theorem 2.7. To prepare for the proof of Theorem 2.7, we prove a partial result.

PROPOSITION 8.1. *Let (μ,u) and (ν,v) be as before. Suppose*

$$u_1\le u_2\le\cdots\le u_n\,,\ v_1\le v_2\le\cdots\le v_{n'}\,.$$

Let $f\in I_{\mu,u}$ be K_n-finite and $f'\in I_{\nu,v}$ be $K_{n'}$-finite.

(i) *Suppose $n'=n-1$. Then*

$$\Psi(s,W_f,W_{f'}) = P(s)\prod L(s+u_i+u_j',\mu_i\otimes\mu_j')$$

where P is a polynomial.

(ii) *Suppose $n'=n$. Then, if Φ is a standard Schwartz function,*

$$\Psi(s,W_f,W_{f'},\Phi) = P(s)\prod L(s+u_i+u_j',\mu_i\otimes\mu_j')$$

where P is a polynomial.

PROOF. We recall that an integral of the form

$$\int f(g)|\det g|^{s+\frac{n-1}{2}}\,dg$$

where f is a K_n-finite coefficient of $I_{\mu,u}$ is a polynomial multiple of $\prod L(s+u_i,\mu_i)$. In particular, our assertion is true for $n=1$. Suppose the assertion of the proposition is true for the pair $(n,n-1)$. To prove it for (n,n), we recall that any K_n-finite element f of $I_{\mu,u}$ is a sum of elements of the form

$$f_{\Phi,\phi_1,\mu_n,u_n}$$

where Φ is standard. So we may as well assume $f=f_{\Phi,\phi_1,\mu_n,u_n}$. Then $W_f = W_{\Phi,\psi,W_1,\mu_n,u_n}$. Then the function Φ_0 in (8.1) is also standard. As we have seen, $\Psi(s,W,W_1,\Phi_1)$ (formula (8.2)) is a sum of product of the form

$$\sum_i \Psi(s,W_i',W_1)\ Z(s+u_n,\Phi_0,f_i'\otimes\mu_n)\,.$$

In each term, the first factor is a polynomial multiple of

$$\prod_{1\le i\le n-1, 1\le j\le n} L(s+u_i+v_j,\mu_i\nu_j)$$

and the second factor a polynomial multiple of

$$\prod_{1\leq j\leq n} L(s+u_n+v_j,\mu_n\nu_j)\,.$$

Our assertion follows. One proves similarly that the assertion of the proposition for $(n-1,n-1)$ implies the assertion for $(n,n-1)$. $\square$

9. Theorem 2.1: principal series, pairs $(n,n-2)$

In this section, we prove Theorem 2.1 for pairs $(n,n-2)$ and principal series representations.

9.1. Review of the integral representations. We keep the notations of the previous section. We set

$$W=W_{\Phi,\psi,W_1,\mu_n,u_n}\,.$$

We first review the integral representation for W. We assume, as we may, that Φ has the form

$$\Phi(X,Y,Z)=\Phi_1(X)\Phi_2(Y)\Phi_3(Z)\,,$$

where Y and Z are column matrices with $n-1$ rows, and X is a matrix with $n-2$ columns and $n-1$ rows. Then, for $g\in G_{n-2}$,

$$W\begin{pmatrix} g&0&0\\ 0&1&0\\ 0&0&1\end{pmatrix}=|\det g|^{u_n+\frac{n-1}{2}}\mu_n(\det g)$$

$$\times\int\Phi_1\left[h\begin{pmatrix} g\\ 0\end{pmatrix}\right]\Phi_2\left[he_{n-1}\right]\widehat{\Phi_3}\left[h^\iota e_{n-1}\right]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}d^\times h\,.$$

The integral is for $h\in G_{n-1}$.

We will write

$$h=h_2\begin{pmatrix} h_1&0\\ 0&1\end{pmatrix}$$

with $h_1\in G_{n-2}$, We have then to take h_2 in a suitable quotient space. We will take

$$h_2=k_2\begin{pmatrix} 1_{n-2}&0\\ 0&a\end{pmatrix}\begin{pmatrix} 1_{n-2}&Y\\ 0&1\end{pmatrix}\,. \tag{9.1}$$

Then

$$d^\times h=d^\times h_1dh_2\,,\ dh_2=dk_2J(a)d^\times adY\,,$$

where J is a suitable Jacobian factor. For comparison with formulas which appear in the functional equations, we remark that we could take

$$h_2=k_2\begin{pmatrix} 1_{n-2}&0\\ 0&a^{-1}\end{pmatrix}\begin{pmatrix} 1_{n-2}&0\\ Y&1\end{pmatrix}\,,$$

with $dh_2=\widetilde{J}(a)dY$, where $\widetilde{J}$ is another Jacobian factor. These two choices of h_2 and dh_2 are exchanged by the automorphism $h\mapsto h^\iota$.

We find then

$$W\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = |\det g|^{u_n+\frac{n-1}{2}}\mu_n(\det g)$$
$$\times \int \Phi_1\left[h_2\begin{pmatrix} h_1 g \\ 0 \end{pmatrix}\right]\Phi_2\left[h_2 e_{n-1}\right]\widehat{\Phi_3}\left[h_2^{\iota}e_{n-1}\right]W_1\left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}h_2^{-1}\right]$$
$$\times\, \mu_n(\det h_1)|\det h_1|^{u_n+\frac{n}{2}-1}d^{\times}h_1\mu_n(\det h_2)|\det h_2|^{u_n+\frac{n}{2}-1}dh_2\,.$$

Recall that we have proved this expression is absolutely convergent for all u_n. Similarly, we have the following lemma.

Lemma 9.1.

$$\int_F W\begin{pmatrix} g & 0 & 0 \\ X & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}dX = |\det g|^{u_n+\frac{n-1}{2}}\mu_n(\det g)$$
$$\times \int \Phi_1\left[h_2\begin{pmatrix} h_1 g \\ X \end{pmatrix}\right]\Phi_2\left[h_2 e_{n-1}\right]\widehat{\Phi_3}\left[h_2^{\iota}e_{n-1}\right]W_1\left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}h_2^{-1}\right]$$
$$\times\, \mu_n(\det h_1)|\det h_1|^{u_n+\frac{n}{2}-1}d^{\times}h_1\mu_n(\det h_2)|\det h_2|^{u_n+\frac{n}{2}-1}dh_2 dX\,,$$

the integral being absolutely convergent.

Proof. We need only prove the absolute convergence of this expression. For $N_1 >> 0$ (see formula (3.1) and Lemma 3.1)

$$\int\left|\Phi_1\left[h_2\begin{pmatrix} h_1 g \\ X \end{pmatrix}\right]\right|dX \preceq \frac{||h_2||_H^{N_1}\ ||g||_H^{N_1}}{(1+||h_1||_e^2)N_1}.$$

Now

$$\left|W_1\left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix}h_2^{-1}\right]\right| \preceq \xi_{i,n-2}(h_1^{-1})^{-N_2}\ ||h_1||^M\ ||h_2||^M$$

for a suitable M and arbitrary N_2 (see Lemma 3.8). Thus, we are reduced to showing that the following two integrals converge absolutely:

$$\int_{G_{n-2}}\xi_{i,n-2}(h_1^{-1})^{-N_2}(1+||h_1||_e^2)^{-N_1}||h_1||^M|\det h_1|^u dh_1\,,$$

where u and M are given and N_1, N_2 are arbitrary, and

$$\int ||h_2||_H^M\ \Phi_2\left[h_2 e_{n-1}\right]\widehat{\Phi_3}\left[h_2^{\iota}e_{n-1}\right]dh_2\,,$$

where M is given.

For the first integral, we write $h_1 = k_1(b+U)$ where b is diagonal with positive entries and U upper triangular with 0 diagonal. Then, for a suitable M_1,

$$||h_1||^M \preceq ||b||_H^{M_1}(1+||U||_e^2)^{M_1}\,.$$

For a suitable m,

$$\xi_{i,n-2}(h_1^{-1})^m = \xi_{i,n-2}(b^{-1})^m \geq \prod_{i=1}^{n-2}(1+b_i^{-2})\,.$$

Also

$$(1+||h_1||_e^2)^2 \geq \prod_{i=1}^{n-2}(1+b_i^2)(1+||U||_e^2)\,.$$

Thus, after a change of notations, we are reduced to the convergence of

$$\int \frac{||b||^M \ |\det b|^u \ J(b)}{\prod_{i=1}^{n-2}(1+b_i^2)^N(1+b_i^{-2})^N} db$$

and

$$\int \frac{1}{(1+||U||_e^2)^N} dU$$

where M, u and $J(b)$ are given and N is arbitrary. It is easy to see that the integrals converge for N large enough.

For the h_2 integral, recall h_2 has the form (9.1). Then

$$||h_2|| \preceq (a^2+a^{-2})^M(1+||Y||_e^2)^M$$

for a suitable M. Moreover,

$$|\Phi_2(h_2 e_{n-1})| = \left|\Phi\left[k_2\begin{pmatrix} Y \\ a \end{pmatrix}\right]\right| \preceq (1+||Y||^2)^{-N_1}(1+a^2)^{-N_2}$$

with N_1 and N_2 arbitrary. Finally,

$$|\Phi_3(h_2^\iota e_{n-1})| = |\Phi_3(k_2^\iota a^{-1} e_{n-1})| \preceq (1+a^{-2})^{-N_3}$$

with N_3 arbitrary. Changing notations we are reduced to the convergence of the integrals

$$\int \frac{dY}{(1+||Y||_e^2)^N},$$

$$\int \frac{(a^2+a^{-2})^M J(a)}{(1+a^2)^N(1+a^{-2})^N} d^\times a,$$

with M and the Jacobian character J given and N arbitrary. Again it is clear that the integrals converge for N large enough. □

In the previous expressions, we change h_1 to $h_1 g^{-1}$. We get

$$(9.2)\quad W\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = |\det g|^{1/2}$$
$$\times \int \Phi_1\left[h_2\begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right]\Phi_2\left[h_2 e_{n-1}\right]\widehat{\Phi_3}\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1}\right]$$
$$\times\, \mu_n(\det h_1 \det h_2)|\det h_1 \det h_2|^{u_n+\frac{n}{2}-1} d^\times h_1 dh_2$$

and

$$(9.3)\quad \int W\begin{pmatrix} g & 0 & 0 \\ X & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dX = |\det g|^{1/2}$$
$$\times \int \Phi_1\left[h_2\begin{pmatrix} h_1 \\ X \end{pmatrix} e_{n-1}\right]\Phi_2\left[h_2 e_{n-1}\right]\widehat{\Phi_3}\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1}\right]$$
$$\times\, \mu_n(\det h_1 \det h_2)|\det h_1 \det h_2|^{u_n+\frac{n}{2}-1} d^\times h_1 dh_2 dX\,.$$

Similarly,

$$[\rho(w_{n,n-2})W]\begin{pmatrix} g & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \mu_n(-1)|\det g|^{u_n+\frac{n-1}{2}}\mu_n(\det g)$$

$$\times\int \Phi_1\left[h\begin{pmatrix} g\\ 0\end{pmatrix}\right]\Phi_3[he_{n-1}]\widehat{\Phi_2}[h^\iota e_{n-1}]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}d^\times h\,,$$

or, introducing h_1 and h_2 as before,

$$(9.4)\quad [\rho(w_{n,n-2})W]\left[\begin{pmatrix} g & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] = |\det g|^{1/2}\mu_n(-1)$$
$$\times\int \Phi_1\left[h_2\begin{pmatrix} h_1\\ 0\end{pmatrix}\right]\Phi_3\left[h_2e_{n-1}\right]\widehat{\Phi_2}\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} gh_1^{-1} & 0\\ 0 & 1\end{pmatrix}h_2^{-1}\right]$$
$$\times\,\mu_n(\det h_1\det h_2)|\det h_1\det h_2|^{u_n+\frac{n}{2}-1}d^\times h_1dh_2\,.$$

9.2. Formal computations. We compute $\Psi(s,\rho(w_{n,n-2})W,W')$ by replacing $\rho(w_{n,n-2})W$ by its integral expression (9.4) and changing g to gh_1. We get

$$(9.5)\quad \Psi(s,\rho(w_{n,n-2})W,W') = \mu_n(-1)\int \mu_n(\det h_2)|\det h_2|^{u_n+\frac{n}{2}-1}$$
$$\times\int \Phi_1\left[h_2\begin{pmatrix} h_1\\ 0\end{pmatrix}\right]\Phi_3\left[h_2e_{n-1}\right]\widehat{\Phi_2}\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}h_2^{-1}\right]W'(gh_1)$$
$$\times\,\mu_n(\det h_1)|\det h_1|^{s+u_n+\frac{n-3}{2}}|\det g|^{s-1/2}d^\times h_1dgdh_2\,.$$

We compute $\Psi_1(s,W,W')$ using (9.3) and changing g to gh_1. We get

$$(9.6)\quad \Psi_1(s,W,W') = \int \mu_n(\det h_2)|\det h_2|^{u_n+\frac{n}{2}-1}$$
$$\times\int \Phi_1\left[h_2\begin{pmatrix} h_1\\ X\end{pmatrix}\right]\Phi_2\left[h_2e_{n-1}\right]\widehat{\Phi_3}\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}h_2^{-1}\right]W'(gh_1)$$
$$\times\,\mu_n(\det h_1)|\det h_1|^{s+u_n+\frac{n-3}{2}}|\det g|^{s-1/2}d^\times h_1dgdXdh_2\,.$$

These expressions converge for $\Re s >> 0$ but we postpone the proof to the next subsection.

Now we prove the functional equation **formally**. We apply formula (9.6) combined with the functional equation of Section 7.3 to get

$$(9.7)\quad \Psi_1(1-s,\widetilde{W},\widetilde{W'}) = \mu_n(-1)^{n-1}\int \mu_n^{-1}(\det h_2)\;|\det h_2|^{-u_n+\frac{n-2}{2}}$$
$$\times\int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1\\ X\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right]\widetilde{W_1}\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}h_2^{-1}\right]\widetilde{W'}(gh_1)$$
$$\times\,\mu_n^{-1}(\det h_1)|\det h_1|^{1-s-u_n+\frac{n-3}{2}}|\det g|^{1-s-1/2}d^\times h_1d^\times gdXdh_2\,.$$

We first apply the functional equation $(n-1, n-2)$ to the g integral. We get

$$\prod_{1\le i\le n} \mu_i(-1)^n \prod_{1\le j\le n-2} \nu_i(-1) \prod_{1\le i\le n-1, 1\le j\le n-2} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times\ \mu_n(-1)\int \mu_n^{-1}(\det h_2)|\det h_2|^{-u_n+\frac{n-2}{2}}$$
$$\int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1 \\ X\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2 e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right] W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-\iota}\right] W'(gh_1^\iota)$$
$$\times\ \mu_n^{-1}(\det h_1)|\det h_1|^{1-s-u_n+\frac{n-3}{2}}|\det g|^{s-1/2} d^\times h_1 dg dX dh_2\,.$$

Finally, we apply the functional equation of Proposition 4.4 to the h_1 integral and the Fourier inversion formula to the X integral. We get

$$\prod_{1\le i\le n} \mu_i(-1)^n \prod_{1\le j\le n-2} \nu_i(-1) \prod_{1\le i\le n, 1\le j\le n-2} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times\ \mu_n(-1)\int \mu_n^{-1}(\det h_2)|\det h_2|^{-u_n-\frac{n-2}{2}}$$
$$\int \Phi_1\left[h_2^\iota\begin{pmatrix} h_1 \\ 0\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2 e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right] W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-\iota}\right] W'(gh_1)$$
$$\times\ \mu_n(\det h_1)|\det h_1|^{s+u_n+\frac{n-3}{2}}|\det g|^{s-1/2} dh_1 dg dh_2\,.$$

After changing h_2 to h_2^ι in the integral, we arrive at the following expression:

$$\prod_{1\le i\le n} \mu_i(-1)^n \prod_{1\le j\le n-2} \nu_i(-1) \prod_{1\le i\le n, 1\le j\le n-2} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$
$$\times\ \mu_n(-1)\int \mu_n(\det h_2)|\det h_2|^{u_n+\frac{n-2}{2}}$$
$$\int \Phi_1\left[h_2\begin{pmatrix} h_1 \\ X\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2^\iota e_{n-1}\right]\Phi_3\left[h_2 e_{n-1}\right] W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-1}\right] W'(gh_1)$$
$$\times\ \mu_n(\det h_1)|\det h_1|^{s+u_n+\frac{n-3}{2}}|\det g|^{s-1/2} d^\times h_1 d^\times g dh_2\,.$$

Comparing with the expression (9.5) for $\Psi(s, \rho(w_{n,n-2})W, W')$, we see that we have "proved" that

$$\Psi_1(1-s, \widetilde{W}, \widetilde{W'}) = \prod_{1\le i\le n} \mu_i(-1)^{n-2} \prod_{1\le j\le n-2} \nu_i(-1) \tag{9.8}$$
$$\times \prod_{1\le i\le n, 1\le j\le n-2} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)\Psi(s, \rho(w_{n,n-2})W, W')\,.$$

9.3. Rigorous proof. To make the proof rigorous, we will appeal to Proposition 4.3. In order to do so, we first establish the convergence of some integrals.

LEMMA 9.2. *Let $v \in \mathbb{C}$. The following three integrals converge absolutely for $\Re s_1 >> 0, \Re s_2 >> 0$.*

$$\int |\det h_2|^v \int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1 \\ X\end{pmatrix}\right] \tag{9.9}$$
$$\times \int \widehat{\Phi_2}\left[h_2 e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right]\widetilde{W_1}\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-1}\right]\widetilde{W'}(gh_1)$$
$$\times\ \mu_n^{-1}(\det h_1)|\det h_1|^{s_1-u_n+\frac{n-3}{2}}|\det g|^{s_2-1/2} d^\times h_1 d^\times g dX dh_2\,,$$

$$\int |\det h_2|^v \int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1 \\ X \end{pmatrix}\right] \tag{9.10}$$
$$\times\ \widehat{\Phi_2}\,[h_2 e_{n-1}]\,\Phi_3\,[h_2^\iota e_{n-1}]\,W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-\iota}\right] W'(gh_1^\iota)$$
$$\times\ \mu_n^{-1}(\det h_1)|\det h_1|^{s_1-u_n+\frac{n-3}{2}}|\det g|^{s_2-1/2}d^\times h_1 d^\times g dX dh_2\,,$$

$$\int |\det h_2|^v \int \Phi_1\left[h_2\begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] \tag{9.11}$$
$$\times\ \Phi_3\,[h_2 e_{n-1}]\,\widehat{\Phi_2}\,[h_2^\iota e_{n-1}]\,W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-1}\right] W'(gh_1)$$
$$\times\ \mu_n(\det h_1)|\det h_1|^{s_1+u_n+\frac{n-3}{2}}|\det g|^{s_2-1/2}d^\times h_1 d^\times g dh_2\,.$$

Proof. Consider the integral (9.9). In the integrand, we use the following majorizations

$$\left|\widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1 \\ X \end{pmatrix}\right]\right| \preceq \frac{||h_2||_H^{N_1}}{(1+||X||_e^2)^{N_1}(1+||h_1||_e^2)^{N_1}}\,,$$

where N_1 is arbitrary;

$$\left|\widetilde{W'}(gh_1)\right| \preceq ||gh_1||^{M_1} \le ||g||^{M_1}\ ||h_1||^{M_1}$$

for a suitable M_1;

$$\left|\widetilde{W_1}\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} h_2^{-1}\right]\right| \preceq \xi_{h,n-1}\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right]^{-N_2} ||g||^{M_2}\ ||h_2||^{M_2}$$

for suitable M_2 and arbitrary N_2.

Accordingly, we are reduced to a product of four integrals.

$$\int \frac{dX}{(1+||X||_e^2)^{N_1}}dX\,,$$
$$\int \frac{||h_1||_H^{M_1}\ |\det h_1|^{s_1}}{(1+||h_1||_e^2)^{N_1}}dh_1\,,$$
$$\int \xi_h\left[\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\right]^{-N_2} ||g||^{M_1+M_2}\ |\det g|^{s_2}dg\,,$$
$$\int \Phi_2(h_2e_{n-1})\Phi_3(h_2^\iota e_{n-1})\ ||h_2||_H^{N_1+M_1}\ |\det g|^{s_2}dgdh_2\,.$$

The first integral converges for $N_1 >> 0$. By Lemma 3.3, there are A, B, C such that the second integral converges for $N_1 > A, s_1 > B, CN_1 > s_1$. Similarly, by Lemma 3.5, there are A', B', C' such that the third integral converges for $N_2 > A', s_2 > B', C'N_2 > s_2$. For the last integral, we write

$$h_2 = k_2\begin{pmatrix} 1_{n-2} & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1_{n-2} & Y \\ 0 & 1\end{pmatrix}.$$

Then

$$dh_2 = J_2(a)d^\times a dY dk_2\,,\ ||h_2|| \preceq (a^2+a^{-2})^{M_1}(1+||Y||_e^2)^{M_3}\,,$$
$$\Phi_2(h_2e_{n-1}) = \Phi\left[k_2\begin{pmatrix} Y \\ a^{-1}\end{pmatrix}\right] \preceq (1+a^{-2})^{-N_3}(1+||Y||_e^2)^{-N_3}\,,$$
$$\Phi_3(h_2^\iota e_{n-1}) = \Phi(k_2^\iota a e_{n-1}) \preceq (1+a^2)^{-N_3}\,.$$

Here M_1 is a suitable constant and N_3 is arbitrary. After a change of notations, we are reduced to a product of integrals

$$\int \frac{dY}{(1+||Y||_e^2)^{N_3}}\,,$$

$$\int (a^2+a^{-2})^M \quad (1+a^2)^{-N_3}(1+a^2)^{-N_3} J(a)||a||^M d^\times a\,.$$

Here M, J are given and N_3 is arbitrary. These integrals converge for $N_3 >> 0$. We are done with integral (9.9).

The convergence of the integral (9.10) is similar because the factor containing W' admits the same majorization as before, namely,

$$|W'(gh_1^\iota)| \preceq ||g||^M \;\; ||h_1^\iota||^M = ||g||^M \;\; ||h_1||^M\,.$$

The convergence of the integral (9.11) is also similar but somewhat simpler because there is no X integration. This time, we have

$$\left|\Phi_1\left[h_2\begin{pmatrix} h_1 \\ 0\end{pmatrix}\right]\right| \preceq ||h_2||^N \;\; (1+||h_1||^2)^{-N}$$

with N arbitrary and

$$|W'(gh_1)| \preceq ||g||^M \;\; ||h_1||^M$$

and the other majorizations are as before. This concludes the proof of the convergence of the integrals (9.9) to (9.11). □

This already shows that formula (9.5) for $\Psi(s,\rho(w_{n,n-2})W,W')$ and formula (9.6) for $\Psi_1(s,W,W')$ are absolutely convergent for $\Re s >> 0$, as was claimed.

Let (θ_1,κ_1) be a ψ pair for

$$\left(\bigoplus_{i=1}^{n-1}\mu_i\otimes\alpha^{u_i}\right)\otimes\left(\bigoplus_{j=1}^{n-2}\nu_j\otimes\alpha^{v_j}\right)$$

and (θ_2,κ_2) a ψ pair for

$$\mu_n\otimes\alpha^{u_n}\otimes\left(\bigoplus_{j=1}^{n-2}\nu_j\otimes\alpha^{v_j}\right)\,.$$

The previous formal computation leading to the functional equation (9.8) is replaced by the following sequence of computations.

$$\begin{aligned}
&\int \mu_n^{-1}(\det h_2)|\det h_2|^{-u_n+\frac{n-2}{2}}\\
&\quad\times\int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1\\ X\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right]\widetilde{W_1}\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}h_2^{-1}\right]\widetilde{W'}(gh_1)\\
&\quad\times\;\mu_n^{-1}(\det h_1)|\det h_1|^{-u_n+\frac{n-3}{2}}\kappa_1(\det h_1)\kappa_2(\det g)|\det g|^{-1/2}d^\times h_1 dg dX dh_2\\
&=\prod_{i=1}^{n-1}\mu_i(-1)^{n-2}\prod_{j=1}^{n-2}\nu_j(-1)\int\mu_n^{-1}(\det h_2)|\det h_2|^{-u_n+\frac{n-2}{2}}\\
&\quad\times\int \widehat{\Phi_1}\left[h_2\begin{pmatrix} h_1\\ X\end{pmatrix}\right]\widehat{\Phi_2}\left[h_2e_{n-1}\right]\Phi_3\left[h_2^\iota e_{n-1}\right]W_1\left[\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}h_2^{-\iota}\right]W'(gh_1^\iota)\\
&\quad\times\;\mu_n^{-1}(\det h_1)|\det h_1|^{-u_n+\frac{n-3}{2}}\kappa_1(\det h_1)\theta_2(\det g)|\det g|^{-1/2}d^\times h_1 dg dX dh_2 =
\end{aligned}$$

$$= \prod_{i=1}^{n-1} \mu_i(-1)^{n-2} \prod_{j=1}^{n-2} \nu_j(-1) \int \mu_n^{-1}(\det h_2)|\det h_2|^{-u_n - \frac{n-2}{2}}$$
$$\times \int \Phi_1 \left[h_2^\iota \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right] \widehat{\Phi_2}\,[h_2 e_{n-1}]\, \Phi_3\,[h_2^\iota e_{n-1}]\, W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h_2^{-\iota} \right] W'(gh_1)$$
$$\times \mu_n(\det h_1)|\det h_1|^{u_n + \frac{n-3}{2}} \theta_1(\det h_1)\theta_2(\det g)|\det g|^{-1/2} d^\times h_1 dg dh_2$$
$$= \prod_{i=1}^{n-1} \mu_i(-1)^{n-2} \prod_{j=1}^{n-2} \nu_j(-1) \int \mu_n(\det h_2)|\det h_2|^{u_n + \frac{n-2}{2}}$$
$$\times \int \Phi_1 \left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right] \widehat{\Phi_2}\,[h_2^\iota e_{n-1}]\, \Phi_3\,[h_2 e_{n-1}]\, W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right] W'(gh_1)$$
$$\times \mu_n(\det h_1)\ |\det h_1|^{u_n + \frac{n-3}{2}} \theta_1(\det h_1)\theta_2(\det g)|\det g|^{-1/2} dh_1 dg dh_2 .$$

Indeed, all the integrals converge absolutely by the previous lemma. The first equality is a consequence of the functional equation $(n-1, n-2)$ written in terms of pairs:

$$\int \widetilde{W_1} \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right] \widetilde{W'}(gh_1)\ \kappa_2(\det g)|\det g|^{-1/2} dg$$
$$= \prod_{i=1}^{n-1} \mu_i(-1)^{n-2} \prod_{j=1}^{n-2} \nu_j(-1) \int W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} h_2^{-\iota} \right]$$
$$\times W'(gh_1)\theta_2(\det g)\ |\det g|^{-1/2} dg\,.$$

The second equality is a consequence of Proposition 4.4 and the Fourier inversion formula:

$$\int \widehat{\Phi_1} \left[h_2 \begin{pmatrix} h_1 \\ X \end{pmatrix} \right] W'(gh_1^\iota)\mu_n^{-1}(\det h_1)|\det h_1|^{-u_n + \frac{n-3}{2}} \kappa_1(\det h_1) dh_1 dX$$
$$= |\det h_2|^{-(n-2)} \int \Phi_1 \left[h_2^\iota \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right]$$
$$\times W'(gh_1^\iota)\mu_n(\det h_1)\ |\det h_1|^{u_n + \frac{n-3}{2}} \theta_1(\det h_1) dh_1\,.$$

The last equality is obtained by changing h_2 to h_2^ι.

We now apply the equality we have just obtained and Proposition 4.3 to obtain our conclusion.

10. Theorem 2.1: principal series, pairs (n, n')

We now prove Theorem 2.1 for all pairs (n, n') and principal series representations. We prove our assertion by induction on the integer $a = |n - n'|$. We have already established our assertions for $a = 0, 1, 2$. We now assume $a > 2$ and our assertion true for $a - 1$. Again, we assume $n > n'$ so that here $n - n' > 2$.

10.1. Review of the integral representation. We first review the integral representation for

$$W = W_{\Phi, \psi, W_1, \mu_n, u_n}\,.$$

Recall that if Φ is a Schwartz function on the space of matrices with $n-1$ rows and n columns, we define the Fourier transform of Φ by

$$\widehat{\Phi}(X) = \int \Phi[Y]\overline{\psi}(\mathrm{tr}(Y\ {}^tX))dY\,.$$

It is a function defined on the same space. We also define the partial Fourier transform $\mathcal{P}(\Phi)$ with respect to the last column:

$$\mathcal{P}(\Phi)[U,X] = \int \Phi[U,Y]\overline{\psi}({}^tXY)dY\,.$$

Then

$$W(g) = \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \times \int_{G_{n-1}(F)} \mathcal{P}(g.\Phi)[h,\ h^\iota e_{n-1}]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}d^\times h\,.$$

For $g \in G_{n-1}(F)$, we find

$$W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \mu_n(\det g)|\det g|^{u_n+\frac{n-1}{2}} \times \int_{G_{n-1}(F)} \mathcal{P}(\Phi)[hg, h^\iota e_{n-1}]W_1(h^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}d^\times h\,.$$

Now assume that ${}^tge_{n-1} = e_{n-1}$. Changing h to hg^{-1}, we find

$$W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = |\det g|^{\frac{1}{2}} \times \int_{G_{n-1}(F)} \mathcal{P}(\Phi)[h,\ h^\iota e_{n-1}]W_1(gh^{-1})\mu_n(\det h)|\det h|^{u_n+\frac{n}{2}-1}d^\times h\,. \tag{10.1}$$

We can use this formula to evaluate

$$W\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix}$$

with $g \in G_{n'}(F)$. We write

$$h = k\begin{pmatrix} 1_{n'} & 0 \\ 0 & g_2 \end{pmatrix}\begin{pmatrix} 1_{n'} & Y \\ 0 & 1_{n-n'-1} \end{pmatrix}\begin{pmatrix} h_1 & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix}$$

with $h_1 \in G_{n'}$, Y a matrix with n' rows and $n-n'-1$ columns, $k \in K_{n-1}$, $g_2 \in G_{n-1-n'}$. Then

$$d^\times h = dkd^\times g_2|\det g_2|^{-n'}d^\times h_1\,.$$

We further write

$$g_2 = k_2aZ$$

with a a diagonal matrix in $G_{n-n'-1}$ with positive entries and $Z \in N_{n-n'-1}$, $k_2 \in K_{n-n'-1}$. Then

$$d^\times g_2 = dk_2\delta_{n-n'-1}(a)dadZ\,.$$

Altogether we may as well write

$$h = h_2\begin{pmatrix} h_1 & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix}$$

with $h_1 \in G_{n'}$ and

$$h_2 = k_2 \begin{pmatrix} 1_{n'} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{n'} & Y \\ 0 & Z \end{pmatrix}, \tag{10.2}$$

where $k_2 \in K_{n-n'-1}$, a is a diagonal matrix in $G_{n-n'-1}$ with positive entries, Y is a matrix with n' rows and $n - n' - 1$ columns, and $Z \in N_{n-n'-1}$. Then

$$d^\times h = d^\times h_1 dh_2\,,\; dh_2 = dk_2 \delta_{n-1-n'}(a) |\det a|^{-n'} dY dZ da\,.$$

Recall that $d^\times h_1$ is a Haar measure on $G_{n'}$.

We find then

$$W \begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} = |\det g|^{\frac{1}{2}} \tag{10.3}$$

$$\times \int \mathcal{P}(\Phi) \left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^\iota e_{n-1} \right] W_1 \left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right]$$
$$\times \;\mu_n(\det h_1 \det h_2) |\det h_1 \det h_2|^{u_n + \frac{n}{2} - 1} d^\times h_1 dh_2\,.$$

This integral is absolutely convergent. We need a more general formula.

LEMMA 10.1.

$$\int W \begin{pmatrix} g & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} dX = |\det g|^{\frac{1}{2}}$$
$$\times \int \mathcal{P}(\Phi) \left[h_2 \begin{pmatrix} h_1 \\ X \end{pmatrix}, h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^\iota e_{n-1} \right] W_1 \left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right]$$
$$\times \;\mu_n(\det h_1 \det h_2) |\det h_1 \det h_2|^{u_n + \frac{n}{2} - 1} d^\times h_1 dh_2 dX\,,$$

the integral being absolutely convergent.

PROOF. We first compute formally. To evaluate

$$W \begin{pmatrix} g & 0 & 0 \\ X & 1_{n-n'-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we apply the previous formula with Φ replaced by the function

$$\begin{pmatrix} 1_{n'} & 0 \\ X & 1_{n-n'-1} \end{pmatrix} . \Phi\,.$$

To arrive at the stated formula, we integrate over X. To justify our formal computation, we only need to prove the absolute convergence of our expression for $g = e$ and Φ a product. Thus the contribution of Φ has the form

$$\Phi_1 \left[h_2 \begin{pmatrix} h_1 \\ X \end{pmatrix} \right] \Phi_2 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'} \end{pmatrix} \right] \Phi_3 (h_2^\iota e_{n-1})\,,$$

for suitable Schwartz functions Φ_i. The proof is similar to the proof of Lemma 9.1. First, by Lemma 3.1, for $N_1 >> 0$,

$$\int \left| \Phi_1 \left[h_2 \begin{pmatrix} h_1 \\ X \end{pmatrix} \right] \right| dX \preceq \frac{||h_2||_H^{N_1}}{(1 + ||h_1||_e^2)^{N_1}}\,.$$

Now

$$\left| W_1 \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right] \right| \preceq \xi_{h,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right]^{-N_2} ||h_1||^M \ ||h_2||^M$$

for a suitable M and arbitrary N_2. Thus we are reduced to showing that the following integral converges absolutely.

$$\int ||h_2||_H^{M+N_1} \Phi_2 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'} \end{pmatrix} \right] \Phi_3(h_2^\iota e_{n-1}) \\ \times \ \xi_{h,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right]^{-N_2} ||h_1||_H^M (1 + ||h_1||_e^2)^{-N_1} dh_1 dh_2.$$

Here M is given and N_1, N_2 are arbitrary. We may as well assume Φ_2, Φ_3 positive and K_{n-1} invariant.

Now we write

$$h_2 = k_2 \begin{pmatrix} 1_{n'} & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1_{n'} & Y \\ 0 & Z \end{pmatrix}$$

where a is a diagonal matrix with positive entries, $Z = 1_{n-n'-1} + U$ is in $N_{n-n'-1}$. Then

$$\Phi_3(h_2^\iota e_{n-1}) \preceq (1 + a_{n-n'-1}^2)^{-N_3}$$

with $N_3 >> 0$ and

$$\xi_{h,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & 1 \end{pmatrix} h_2^{-1} \right] = \xi_{h,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & a \end{pmatrix} \right].$$

Now there is $m > 0$ such that

$$\xi_{h,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & a \end{pmatrix} \right]^m (1 + a_{n-n'-1}^2)^m \geq \xi_{s,n-1} \left[\begin{pmatrix} h_1^{-1} & 0 \\ 0 & a \end{pmatrix} \right] \\ = \xi_{s,n'}(h_1^{-1}) \prod_{i=1}^{n-n'-1} (1 + a_i^2).$$

Thus we are reduced to the convergence of the following integrals:

$$\int ||h_2||_H^{M+N_1} \prod_{i=1}^{n-n'-1} (1 + a_i^2)^{-N_2} \Phi_2 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix} \right] dh_2 ,$$

$$\int \xi_{s,n'}(h_1^{-1})^{-N_2} \ ||h_1||_H^M \ (1 + ||h_1||_e^2)^{-N_1} d^\times h_1 .$$

Here M is given and N_1, N_2 are arbitrary.

For the first integral, we observe that

$$||h_2||_H \preceq ||a||_H^{M_1} (1 + ||Y||_e^2)^{M_1} (1 + ||U||_e^2)^{M_1},$$

$$\Phi_2 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix} \right] \preceq \prod_{i=1}^{n-n'-1} (1 + a_i^{-2})^{-N_3} \ ||a||_H^{N_4} \ (1 + ||Y||_e^2)^{-N_4} (1 + ||U||_e^2)^{-N_4}$$

for suitable M_1 and $N_3 >> 0$, $N_4 >> 0$. The convergence of the integral follows for suitable N_4 and N_2, N_3 large with respect to N_1.

For the second integral, we apply Lemma 3.6. □

10.2. Alternate expression. There is an alternate expression for the integral representation. We only give it when Φ is a product of the following form:

$$\Phi[X_0, X_1, X_2, \dots, X_{n-n'}] = \Phi_0(X_0) \prod_{i=1}^{n-n'} \Phi_i(X_i)\,.$$

Here the matrices have $n-1$ rows, X_0 has n' columns and the other matrices have only 1 column. Under this extra assumption, the original formula takes the form

$$\begin{aligned} (10.4)\quad W\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} &= |\det g|^{\frac{1}{2}} \int \Phi_0\left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] \\ &\times \prod_{i=1}^{n-n'-1} \Phi_i(h_2 e_{n'+i})\, \widehat{\Phi_{n-n'}}(h_2^{\iota} e_{n-1}) W_1\left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1}\right] \\ &\times\ \mu_n(\det h_1 \det h_2) |\det h_1 \det h_2|^{u_n + \frac{n}{2} - 1} d^{\times} h_1 dh_2\,. \end{aligned}$$

The alternate formula has the following form:

$$\begin{aligned} (10.5)\quad W\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} &= |\det g|^{\frac{1}{2}} \int \Phi_0\left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] \\ &\times\ \Phi_1(h_2 e_{n'+1}) \prod_{i=1}^{n-n'-1} \widehat{\Phi_{i+1}}(h_2^{\iota} e_{n'+i}) W_1\left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1}\right] \\ &\times\ \mu_n(\det h_1 \det h_2) |\det h_1|^{u_n + \frac{n}{2} - 1} |\det h_2|^{u_n + n' + 1 - \frac{n}{2}} d^{\times} h_1 dh_2\,. \end{aligned}$$

In this new formula, h_2 is taken modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & Y_1 \\ 0 & 1_{n-n'-1} & \end{pmatrix}, \quad g \in G_{n'}\,,$$

where Y_1 is a matrix with n' rows and $n-n'-2$ columns. In a more precise way, in this new formula, we may take

$$h_2 = k_2 \begin{pmatrix} 1_{n'} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{n'} & Y_0 \ \ 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} \begin{pmatrix} 1_{n'} & 0 \\ 0 & Z \end{pmatrix},$$

where Y_0 is a column matrix with n' rows and $Z \in N_{n-n'-1}$. Then

$$dh_2 = dk_2 \delta_{n-n'-1}(a) |\det a|^{-n'} da dY_0 dZ\,.$$

To see that the alternate formula is correct, we start with the original formula. We write

$$h_2 = k_2 \begin{pmatrix} 1_{n'} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{n'} & Y_0 \ \ 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} \begin{pmatrix} 1_{n'} & 0 \ \ Y_1 \\ 0 & Z \end{pmatrix}$$

where Y_0 is a column matrix with n' rows, Y_1 has $n-n'-2$ columns and n' rows and $Z \in N_{n-n'-1}$. We then apply the following lemma.

LEMMA 10.2.

$$\begin{aligned} &\int \prod_{i=2}^{n-n'-1} \Phi_i\left[h_2 \begin{pmatrix} 1_{n'} & 0 \ \ Y_1 \\ 0 & Z \end{pmatrix} e_{n'+i}\right] dY_1 \overline{\theta_\psi}(Z) dZ \\ &\qquad = |\det h_2|^{-(n-n'-2)} \int \prod_{i=2}^{n-n'-1} \widehat{\Phi_i}\left[h_2^{\iota} \begin{pmatrix} 1_{n'} & 0 \\ 0 & Z \end{pmatrix}^{\iota} e_{n'+i-1}\right] \overline{\theta}_\psi(Z) dZ. \end{aligned}$$

PROOF. To prove the lemma, we may assume $h_2 = 1$. The lemma follows then from the Fourier inversion formula. We illustrate the case $n = 6, n' = 2$ but the argument is general.

$$\int \Phi_2 \left[\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \\ 0 \end{pmatrix} \right] \Phi_3 \left[\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \\ 1 \end{pmatrix} \right] \overline{\psi}(z_1 + t_2) dx_1 dx_2 dy_1 dy_2 dz_1 dz_2 dt_2$$

$$= \int \widehat{\Phi_2} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ u \\ v \end{pmatrix} \right] \widehat{\Phi_3} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ w \end{pmatrix} \right] \psi(u + w) du dv dw.$$

□

We also record the corresponding formula for $\rho(w_{n,n'})W$. The original formula is

$$(\rho(w_{n,n'})W) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \mu_n(\det w_{n-n'}) \mu_n(\det g) |\det g|^{u_n + \frac{n-1}{2}}$$
$$\times \int_{G_{n-1}(F)} \mathcal{P}(w_{n,n'}\Phi)[hg,\ h^\iota e_{n-1}] W_1(gh^{-1}) \mu_n(\det h) |\det h|^{u_n + \frac{n}{2} - 1} d^\times h\,.$$

The alternate formula for $\rho(w_{n,n'})W$ is

$$(10.6)\quad (\rho(w_{n,n'})W) \begin{pmatrix} g & 0 \\ 0 & 1_{n-n'} \end{pmatrix} = \mu_n(\det w_{n-n'}) |\det g|^{\frac{1}{2}} \int \Phi_0 \left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right]$$
$$\times \Phi_{n-n'}(h_2 e_{n'+1}) \prod_{i=1}^{n-n'-1} \widehat{\Phi_{n-n'-i}}(h_2^\iota e_{n'+i})\ W_1 \left[\begin{pmatrix} g h_1^{-1} & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right]$$
$$\times \mu_n(\det h_1 \det h_2) |\det h_1|^{u_n + \frac{n}{2} - 1} |\det h_2|^{u_n + n' + 1 - \frac{n}{2}} d^\times h_1 dh_2\,.$$

Before proceeding, we remark that it is convenient to choose our variables in such a way that $|\det h_2| = 1$. Indeed, in the original formula, we can write

$$h = h_2 \begin{pmatrix} h_1 & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix}$$

$$h_2 = k_2 \begin{pmatrix} (\det a)^{-r} 1_{n'} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{n'} & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} 1_{n'} & Y \\ 0 & 1_{n-n'-1} \end{pmatrix},$$

with $r = \frac{1}{n'}$. Then $|\det h_2| = 1$ and

$$d^\times h = dh_2 d^\times h_1\,,\ dh_2 = \delta_{n-n'-1}(a) dk_2 da dZ dY\,.$$

Recall that $G_n^0 = \{g \in G_n(F) : |\det g| = 1\}$. In other words, now h_2 is integrated on the quotient of G_{n-1}^0 by the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix}, \quad g \in G_{n'}^0\,.$$

A similar remark applies to the alternate expression. Then h_2 is in the quotient of G^0_{n-1} by the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & Y_1 \\ 0 & 1_{n-n'-1} \end{pmatrix}, \quad g \in G^0_{n'},$$

where Y_1 is a matrix with n' rows and $n-n'-2$ columns. In a more precise way, in the alternate formula, we may take

$$h_2 = k_2 \begin{pmatrix} \det a^{-r} 1_{n'} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1_{n'} & Y_0 & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} \begin{pmatrix} 1_{n'} & 0 \\ 0 & Z \end{pmatrix}.$$

Then

$$dh_2 = \delta_{n-n'-1}(a) dk_2 da dZ dY_0 .$$

10.3. Formal computations. We now prove the functional equation **formally**.

We compute $\Psi(s, \rho(w_{n,n'})W, W')$ by replacing $\rho(w_{n,n'})W$ by its alternate integral expression and changing g into gh_1. We find the following result.

Lemma 10.3.

$$\begin{aligned}\Psi(s, \rho(w_{n,n'})W, W') &= \mu_n(\det w_{n-n'}) \\ &\times \int \Phi_0 \left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right] \Phi_{n-n'}(h_2 e_{n'+1}) \prod_{i=1}^{n-n'-1} \widehat{\Phi_{n-n'-i}}(h_2^\iota e_{n'+i}) \\ &\times \; W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] W'(gh_1) |\det g|^{s - \frac{n-1-n'}{2}} dg \\ &\times \; \mu_n(\det h_1) |\det h_1|^{u_n + s + \frac{n'-1}{2}} \mu_n(\det h_2) d^\times h_1 dh_2 ,\end{aligned}$$

where $h_2 \in G^0_{n-n'}$ is integrated modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & U \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-n'-2} \end{pmatrix}, \quad g \in G^0_{n'} .$$

We compute $\Psi_{n-n'-1}(s, W, W')$ by replacing W by the formula of Lemma 10.1 and changing g to gh_1. We get

$$\begin{aligned}\Psi_{n-n'-1}(s, W, W') = \int \left(\int \mathcal{P}(\Phi) \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix}, h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^\iota e_{n-1} \right] dY \right) \\ \times W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] W'(gh_1) |\det g|^{s - \frac{n-1-n'}{2}} dg \\ \times \mu_n(\det h_1) |\det h_1|^{u_n + s + \frac{n'-1}{2}} \mu_n(\det h_2) d^\times h_1 dh_2 ,\end{aligned}$$

where $h_2 \in G^0_{n-1}$ is integrated modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & 1_{n-1-n'} \end{pmatrix}, \quad g \in G^0_{n'} .$$

This can also be written in the following way.

LEMMA 10.4.

$$\Psi_{n-n'-1}(s,W,W') = \int \left(\int \mathcal{P}(\Phi)\left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix}, h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^\iota e_{n-1} \right] dY \right)$$
$$\times W_1 \left[\begin{pmatrix} g & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} h_2^{-1} \right] W'(gh_1) |\det g|^{s-\frac{n-1-n'}{2}} dg$$
$$\times \mu_n(\det h_1) |\det h_1|^{u_n+s+\frac{n'-1}{2}} \mu_n(\det h_2) dh_1 dh_2 dU\,,$$

where $h_2 \in G^0_{n-1}$ is integrated modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g \in G^0_{n'}\,.$$

PROOF. Indeed, it suffices to integrate in stages and to change variables as follows

$$Y \mapsto Y + \begin{pmatrix} U \\ 0 \end{pmatrix} h_1\,.$$

□

Now we start the formal computation. Taking into account the previous lemma and Proposition 7.3, we get

$$\Psi_{n-n'-1}(1-s,\widetilde{W},\widetilde{W'}) = \mu_n(-1)^{n-1} \int \left(\int \widehat{\Phi_0} \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix} \right] dY \right) \tag{10.7}$$
$$\times \prod_{i=1}^{n-n'-1} \widehat{\Phi}_i(h_2 e_{n'+i}) \Phi_{n-n'}(h_2^\iota e_{n-1}) \widetilde{W}_1 \left[\begin{pmatrix} g & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} h_2^{-1} \right] \widetilde{W'}(gh_1)$$
$$\times |\det g|^{1-s-\frac{n-1-n'}{2}} dg \mu_n^{-1}(\det h_1) |\det h_1|^{-u_n+1-s+\frac{n'-1}{2}} dh_1 \mu_n^{-1}(\det h_2) dh_2 dU\,.$$

We apply the $(n-1,n')$ functional equation to the g-integral. We get

$$\mu_n(-1)^{n-1} \prod_{1\le i\le n-1} \mu_i(-1)^{n'} \prod_{1\le j\le n'} \nu_j(-1) \tag{10.8}$$
$$\times \prod_{1\le i\le n-1, 1\le j\le n'} \gamma(s+u_i+v_j,\mu_i\nu_j,\psi) \int \left(\int \widehat{\Phi_0} \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix} \right] dY \right))$$
$$\times \prod_{i=1}^{n-n'-1} \widehat{\Phi}_i(h_2 e_{n'+i}) \Phi_{n-n'}(h_2^\iota e_{n-1}) W_1 \left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} w_{n-1,n'} h_2^{-\iota} \right] W'(gh_1^\iota)$$
$$\times |\det g|^{s-\frac{n-1-n'}{2}} dg \mu_n^{-1}(\det h_1) |\det h_1|^{-u_n+1-s+\frac{n'-1}{2}} dh_1 \mu_n^{-1}(\det h_2) dh_2\,.)$$

Recall that h_2 is taken modulo the unimodular subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g \in G^0_{n'}\,.$$

We change h_2 into $h_2 w_{n-1,n'}$ and then h_2 into h_2^ι. Now h_2 is taken modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & U \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-n'-2} \end{pmatrix}, \quad g \in G^0_{n'} .$$

We get then

$$\mu_n(-1)^{n-1}\mu_n(\det w_{n-n'-1}) \prod_{1\le i\le n-1} \mu_i(-1)^{n'} \prod_{1\le j\le n'} \nu_j(-1) \tag{10.9}$$

$$\times \prod_{1\le i\le n-1, 1\le j\le n'} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi) \int \left(\int \widehat{\Phi_0} \left[h_2^\iota \begin{pmatrix} h_1 \\ Y \end{pmatrix} \right] dY \right)$$
$$\times \Phi_{n-n'}(h_2 e_{n'+1}) \prod_{i=1}^{n-n'-1} \widehat{\Phi_{n-n'-i}}(h_2^\iota e_{n'+i}) W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] W'(g h_1^\iota)$$
$$\times |\det g|^{s-\frac{n-1-n'}{2}} dg \mu_n^{-1}(\det h_1) |\det h_1|^{-u_n+1-s+\frac{n'-1}{2}} dh_1 \mu_n(\det h_2) dh_2 .$$

Next we apply the functional equation of Proposition 4.4 to the h_1 integral and the Fourier inversion formula. We get

$$\mu_n(-1)^{n-1}\mu_n(\det w_{n-n'-1}) \prod_{1\le i\le n-1} \mu_i(-1)^{n'} \prod_{1\le j\le n'} \nu_j(-1) \tag{10.10}$$

$$\times \prod_{1\le i\le n, 1\le j\le n'} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi) \int \Phi_0 \left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right]$$
$$\times \Phi_{n-n'}(h_2 e_{n'+1}) \prod_{i=1}^{n-n'-1} \widehat{\Phi_{n-n'-i}}(h_2^\iota e_{n'+i}) W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] W'(g h_1)$$
$$\times |\det g|^{s-\frac{n-1-n'}{2}} dg \mu_n(\det h_1) |\det h_1|^{u_n+s+\frac{n'-1}{2}} dh_1 \mu_n(\det h_2) dh_2 .$$

Now

$$\mu_n(-1)^{n-1}\mu_n(\det w_{n-n'-1}) = \mu_n(\det w_{n-n'})\mu_n(-1)^{n'} .$$

Thus the expression we get is the one we wrote down for $\Psi(s, \rho(w_{n,n-n'})W, W')$ (Lemma 10.3) times

$$\prod_{1\le i\le n, 1\le j\le n'} \gamma(s+u_i+v_j, \mu_i\nu_j, \psi)$$

and

$$\prod_{i=1}^{n} \mu_i(-1)^{n'} \prod_{j=1}^{n'} \nu_j(-1) .$$

So we are done.

10.4. Rigorous proof. Let (θ_1, κ_1) be a ψ pair for

$$\left(\bigoplus_{i=1}^{n-1} \mu_i \otimes \alpha^{u_i}\right) \otimes \left(\bigoplus_{j=1}^{n'} \nu_j \otimes \alpha^{v_j}\right)$$

and (θ_2, κ_2) a ψ pair for

$$\mu_n \otimes \alpha^{u_n} \otimes \left(\bigoplus_{j=1}^{n'} \nu_j \otimes \alpha^{v_j}\right).$$

As before, the correct proof is based on the sequence of equalities obtained by replacing in the previous sequence $|\det g|^{1-s}$ by $\kappa_1(\det g)$, $|\det g|^s$ by $\theta_1(\det g)$, $|\det h_1|^{1-s}$ by $\kappa_2(\det h_1)$, and $|\det h_1|^s$ by $\theta_2(\det h_1)$. We have to show that our computation and our use of the pairs is legitimate. As before, this reduces to checking the convergence of three integrals. We now establish the convergence of these integrals. The rest of the proof is the same as before and is omitted.

Lemma 10.5. *The integral*

$$\int \left(\int \Phi \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix}, h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^{\iota} e_{n-1} \right] dY \right)$$
$$\times W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] W'(gh_1) |\det g|^{s_2} dg |\det h_1|^{s_1} d^{\times} h_1 dh_2 ,$$

where $h_1 \in G_{n'}$, $g \in N_{n'} \backslash G_{n'}$, $h_2 \in G^0_{n-1}$ is taken modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix}, \quad g \in G^0_{n'} ,$$

converges absolutely for $\Re s_1 >> 0$, $\Re s_2 >> 0$.

Proof. For simplicity we assume that Φ is a product (it is in the applications). We may further assume that it is ≥ 0 and K_{n-1} invariant. Thus the contribution of the Schwartz functions is

$$\Phi_0 \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix} \right] \Phi_1 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix} \right] \Phi_2 \left[h_2^{\iota} e_{n-1} \right] .$$

Now

$$\int \Phi_0 \left[h_2 \begin{pmatrix} h_1 \\ Y \end{pmatrix} \right] dY \preceq \frac{||h_2||^N}{(1 + ||h_1||_e^2)^N} ,$$

$$|W'(gh_1)| \preceq ||g||^M ||h_1||^M ,$$

$$\left| W_1 \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right] \right| \preceq \xi_{h,n-1} \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right]^{-N} ||g||^M \; ||h_2||^M$$

for some M and all N. After a change of notations, we are reduced to the convergence of two integrals.

The first integral is

$$\int \frac{||h_1||^M |\det h_1|^{s_1}}{(1 + ||h_1||_e^2)^N} d^{\times} h_1 .$$

For given M, there are A, B, C such that the integral converges for $N > A$, $s_1 > B$, $CN > s_1$ (Lemma 3.3).

Now we change notations again. We write M for $M+N$. The second integral is then

$$\int ||h_2||^M \Phi_1 \left[h_2 \begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix} \right] \Phi_2 \left[h_2^{\iota} e_{n-1} \right]$$
$$\times \xi_{h,n-1} \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1} \right]^{-N} ||g||^M |\det g|^{s_2} dg dh_2 \,.$$

We write

$$h_2 = k_2 \begin{pmatrix} \det a^{\frac{1}{n'}} 1_{n'} & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1_{n'} & Y_0 \\ 0 & Z \end{pmatrix}$$

where a is a diagonal matrix with positive entries, $Z \in N_{n-n'-1}$, $Z = 1_{n-n'-1} + U$ with U upper triangular and 0 diagonal Then

$$dh_2 = dk_2 J_1(a) da dY_0 dU \,,$$
$$||h_2||_H^M \preceq ||a||_H^{M_1} (1 + ||Y_0||_e^2)^{M_1} (1 + ||U||_e^2)^{M_1} \,,$$

for a suitable M_1. The contribution of Φ_1, Φ_2 is

$$\Phi_1 \begin{pmatrix} \det a^{\frac{1}{n'}} \; Y_0 \\ a^{-1} + a^{-1} U \end{pmatrix} \Phi_2(a_{n-n'-1} e_{n-1})$$
$$\preceq \frac{||a||^{M_2}}{(1 + ||Y_0||_e^2)^{N_2} (1 + ||U||_e^2)^{N_2} (1 + a_{n-n'-1}^2)^N}$$

with N_2 arbitrary, M_2 depends on N_2, and N arbitrary. Now $\xi_{h,n-1}$ does not depend on U, Y_0, k_2. We are left with the product of two integrals

$$\int \frac{dY_0 dU dk_2}{(1 + ||Y_0||_2^2)^{N_2 - M_1} (1 + ||U||_e^2)^{N_2 - M_1}} \,,$$
$$\int \frac{||a||^{M_1 + M_2} J_1(a)}{(1 + a_{n-n'-1}^2)^N} \xi_{h,n-1} \left[\begin{pmatrix} \det a^{-\frac{1}{n'}} g & 0 \\ 0 & a \end{pmatrix} \right]^{-N} ||g||^M \; |\det g|^{s_2} dg da \,.$$

The first integral converges for $N_2 >> 0$. In the second integral, we change g to $g \det a^{\frac{1}{n'}}$. We have

$$||a||^{M_1 + M_2} J_1(a) \; ||\det a^{\frac{1}{n'}} g||^M \preceq ||a||^{M_3} \; ||g||^M \,.$$

We are reduced to

$$\int \frac{||a||^{M_3} \; |\det a|^{s_2}}{(1 + a_{n-n'-1}^2)^N} \xi_{h,n-1} \left[\begin{pmatrix} g & 0 \\ 0 & a \end{pmatrix} \right]^{-N} ||g||^M \; |\det g|^{s_2} dg \,.$$

We use again the fact that

$$(1 + a_{n-n'-1}^2)^m \xi_{h,n-1} \left[\begin{pmatrix} g & 0 \\ 0 & a \end{pmatrix} \right]^m \succeq \xi_{s,n-1} \left[\begin{pmatrix} g & 0 \\ 0 & a \end{pmatrix} \right]$$
$$= \xi_{s,n'}(g) \prod_{i=1}^{n-n'-1} (1 + a_i^2)$$

to arrive at a product

$$\int \frac{||a||^{M_3} \; |\det a|^{s_2}}{\prod_{i=1}^{n-n'-1} (1 + a_i^2)^N} da \int_{N_{n'} \backslash G_{n'}} ||g||_I^M \; |\det g|^{s_2} \xi_{s,n'}(g)^{-N} dg \,.$$

There are A, B, C such that the integrals converges for $N > A$, $s_2 > B$, $CN > s_2$ (Lemma 3.4 and Lemma 3.5). □

The first step in the rigorous proof comes from (10.7) with $\kappa_1(\det g)$ replacing $|\det g|^{1-s}$ and $\kappa_2(\det h_1)$ replacing $|\det h_1|^{1-s}$. Correspondingly, we need to verify the convergence of the following integral.

LEMMA 10.6. *The integral*

$$\iint \Phi\left[h_2\begin{pmatrix} h_1 \\ Y \end{pmatrix}, h_2\begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}, h_2^{\iota} e_{n-1}\right]$$
$$\times\, W_1\left[\begin{pmatrix} g & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} h_2^{-1}\right] W'(gh_1)|\det g|^{s_2} dg |\det h_1|^{s_1} d^{\times}h_1 dh_2 dU dY\,,$$

where $h_2 \in G^0_{n-1}$ is integrated modulo the subgroup of matrices of the form

$$\begin{pmatrix} h_1 & 0 & 0 \\ U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_1 \in G^0_{n'}\,, \tag{10.11}$$

converges absolutely for $\Re s_1 >> 0$, $\Re s_2 >> 0$.

PROOF. Indeed, we recall that the present integral is obtained from the previous one by a simple change of variables. Namely, we replace h_2 by

$$h_2\begin{pmatrix} 1_{n'} & 0 & 0 \\ -U & 1_{n-n'-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that h_2 is in G^0_{n-1} modulo the subgroup of matrices of the form (10.11) and then we replace Y by

$$Y + \begin{pmatrix} U \\ 0 \end{pmatrix} h_1\,.$$

□

The second step in the rigorous proof comes from (10.9). Correspondingly, we need to establish the convergence of the following integral.

LEMMA 10.7. *The integral*

$$\int\left(\int \Phi\left[h_2^{\iota}\begin{pmatrix} h_1 \\ Y \end{pmatrix}, h_2 e_{n'+1}, h_2^{\iota}\begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}\right] dY\right)$$
$$\times\, W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1}\right] W'(gh_1^{\iota})|\det g|^{s_2} dg |\det h_1|^{s_1} d^{\times}h_1 dh_2\,,$$

where $h_1 \in G'_n$, $g \in N'_n\backslash G'_n$, $h_2 \in G^0_{n-1}$ is taken modulo the subgroup of matrices of the form

$$\begin{pmatrix} g & 0 & U \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-n'-2} \end{pmatrix}, \quad g \in G^0_{n'}$$

converges absolutely for $\Re s_1 >> 0$, $\Re s_2 >> 0$.

PROOF. As before, we may assume $\Phi \geq 0$, K_{n-1}-invariant and a product. Then the contribution of Φ takes the form

$$\Phi_1\left[h_2^{\iota}\begin{pmatrix} h_1 \\ Y \end{pmatrix}\right] \Phi_2(h_2 e_{n'+1})\, \Phi_3\left[h_2^{\iota}\begin{pmatrix} 0 \\ 1_{n-n'-1} \end{pmatrix}\right].$$

Moreover,

$$\left|W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1}\end{pmatrix} h_2^{-1}\right] W'(gh_1^\iota)\right|$$
$$\preceq \xi_{h,n-1}\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1}\end{pmatrix} h_2^{-1}\right]^{-N} ||g||_H^M\ ||h_2||_H^M\ ||h_1||_H^M\,,$$

for suitable M and $N >> 0$.

As before,

$$\int \Phi_1\left[h_2^\iota\begin{pmatrix} h_1 \\ Y\end{pmatrix}\right] dY \preceq \frac{||h_2||_H^N}{(1+||h_1||_e^2)^N}$$

for $N >> 0$. We are reduced again to a product of two integrals. The first one is

$$\int \frac{||h_1||^M\ |\det h_1|^{s_1}}{(1+||h_1||_e^2)^N} dh_1\,.$$

It converges for $N > A$, $s_1 > B$, $CN > s_1$.

The second integral is, after a change of notations,

$$\iint ||h_2||^M \Phi_2(h_2 e_{n'+1})\Phi_3\left[h_2^\iota\begin{pmatrix} 0 \\ 1_{n-n'-1}\end{pmatrix}\right]$$
$$\times\ ||g||^M \xi_h\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1}\end{pmatrix} h_2^{-1}\right]^{-N_1} |\det g|^{s_2} dg dh_2\,.$$

Here Φ_2 is a Schwartz function on the space of column matrices with $n-1$ rows and Φ_3 a Schwartz function on the space of matrices with $n-n'-1$ columns and $n-1$ rows. The variables are as follows: $g \in N_{n'}\backslash G_{n'}$ and h_2 in a quotient of G_{n-1}^0. More precisely,

$$h_2 = k_2\begin{pmatrix} \det a^{\frac{1}{n'}}\ 1_{n'} & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1_{n'} & \begin{pmatrix} Y_0 & 0\end{pmatrix} \\ 0 & Z^{-1}\end{pmatrix}.$$

Here a is a diagonal matrix of size $n-n'-1$ with positive entries, Y_0 a column with n' rows and $Z \in N_{n-n'-1}$, ${}^tZ = 1_{n-n'-1} + U$, where U is lower triangular with 0 diagonal. Then

$$dh_2 = dk_2 J(a) da dY_0 dU\,,$$

$$h_2^\iota = k_2^\iota\begin{pmatrix} \det a^{-\frac{1}{n'}}\ 1_{n'} & 0 \\ * & a+aU\end{pmatrix},$$

$$||h_2||_H^M \preceq ||a||_H^{M_1}(1+||Y_0||_e)^{M_1}(1+||U||_e)^{M_1}\,.$$

Thus the contribution of Φ_2, Φ_3 has the form

$$\Phi_2\left[\begin{pmatrix} \det a^{\frac{1}{n'}}\ Y_0 \\ a_1^{-1} \\ 0 \\ * \\ 0\end{pmatrix}\right]\Phi_3\left[\begin{pmatrix} 0 \\ a+aU\end{pmatrix}\right]$$
$$\preceq \frac{||a||_H^{M_2}}{(1+||Y_0||_e^2)^{N_2}(1+||U||_e^2)^{N_2}(1+||a||_e^2)^{N_1}}\,.$$

where N_2 is arbitrary, M_2 depends on N_2 and N_1 is arbitrary. On the other hand, $\xi_{h,n-1}$ does not depend on U, Y_0. We are left with a product of two integrals:

$$\int \frac{dU dY_0 dk_2}{(1+||Y_0||_e^2)^{N_2-M_1}(1+||U||_e^2)^{N_2-M_1}} ,$$

$$\int \frac{||a||^{M_1+M_2}}{(1+||a||_e^2)^{N_1}} \xi_{h,n-1}\left[\begin{pmatrix} \det a^{-\frac{1}{n'}}\, g & 0 \\ 0 & a \end{pmatrix}\right]^{-N} |\det g|^{s_2} ||g||^M dg J_1(a) da .$$

The first integral converges, provided N_2 is large enough. We treat the second integral as the analogous integral in Lemma 10.5. □

The last step in the rigorous proof comes from (10.10). Correspondingly, we need to establish the convergence of the following integral.

LEMMA 10.8. *The integral*

$$\int \Phi\left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, h_2 e_{n'+1}, h_2^{\iota} e_{n-1}\right]$$
$$\times\ W_1\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-n'-1} \end{pmatrix} h_2^{-1}\right] W'(gh_1) |\det g|^{s_2} dg |\det h_1|^{s_1} d^{\times} h_1 dh_2 ,$$

where $h_2 \in G_{n-1}^0$ is taken modulo the subgroup of matrices of the form

$$\begin{pmatrix} h_1 & 0 & U \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-n'-2} \end{pmatrix}, \quad h_1 \in G_{n'}^0 .$$

converges absolutely for $\Re s_1 >> 0$ $\Re s_2 >> 0$.

PROOF. We may again assume $\Phi \geq 0$, K_{n-1} invariant and a product. Then the contribution of Φ is

$$\Phi_1\left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] \Phi_2(h_2 e_{n'+1}) \Phi_3\left[h_2^{\iota} e_{n-1}\right] .$$

The proof is similar to the proof of the previous lemma. Here, there is no integration over Y. We have

$$\Phi_1\left[h_2 \begin{pmatrix} h_1 \\ 0 \end{pmatrix}\right] \preceq \frac{||h_2||_H^N}{(1+||h_1||_e^2)^N} ,$$
$$|W'(gh_1)| \preceq ||g||^M\ ||h_1||^M .$$

The other majorizations and the rest of the proof are the same as before. □

11. Theorem 2.1 for general representations

We have proved our assertions for the induced representations of the principal series. Thus if $F = \mathbb{C}$ we are done. We assume $F = \mathbb{R}$. Consider two pairs (σ, u) and (σ', u'). Thus σ is an r-tuple of unitary irreducible representations σ_i, $1 \leq i \leq r$ of degree $d_i = 1, 2$. Let $n = \sum_i d_i$. Let $\pi_i = \pi_{\sigma_i}$ be the corresponding irreducible representation of $GL(d_i, \mathbb{R})$. Thus if $d_i = 2$, then π_i is a subrepresentation of a principal series representation $I_{\nu_{1,i},\nu_{2,i}}$, with $\nu_{1,i}, \nu_{2,i}$ not normalized (see the Appendix). If $d_i = 1$, then π_i is a character of $\mathbb{R}^{\times}$ that we also write as $\nu_{1,i}$. Let μ be the n-tuple formed by the $\nu_{i,j}$ and v the n-tuple formed by the complex numbers u_i, repeated d_i times. For instance if $r = 3$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, then

$$\mu = (\nu_{1,1}, \nu_{1,2}, \nu_{2,2}, \nu_{3,3}) ,\ v = (u_1, u_2, u_2, u_3) .$$

Then $I_{\sigma,u}$ is a sub representation of $I_{\mu,v}$. Let λ be a non-zero ψ form on $I_{\mu,v}$. Since $I_{\sigma,u}$ admits a non-zero ψ form, the restriction of λ to $I_{\sigma,u}$ is non-zero (Lemma 2.4). Define similarly (μ',v') and let λ' be a ψ form on $I_{\mu',v'}$. It follows that the results of the previous sections apply to the integrals $\Psi_k(s,W_f,W_{f'})$ or $\Psi_k(s,W_f,W_{f'},\Phi)$, with $f\in I_{\sigma,u}$, $f'\in I_{\sigma',u'}$. In particular, these integrals converge for $\Re s >> 0$ and are holomorphic multiples of

$$\prod L(s+v_i+v'_j,\mu_i\mu'_{i'})\,.$$

For clarity, let us repeat what we want to prove. Consider first the case $n>n'$.

Proposition 11.1. *Suppose $n'<n$. Then the integrals*

$$\Psi_k(s,W_f,W_{f'})$$

are holomorphic multiple of

$$L(s,\sigma_u\otimes\sigma'_{u'})\,.$$

They satisfy the functional equation

$$\Psi_{n-n'-1-k}(1-s,\rho(w_{n,n'})\widetilde{W_f},\widetilde{W_{f'}})\frac{1}{L(1-s,\widetilde{\sigma}_{-u}\otimes\widetilde{\sigma}'_{-u'})}$$
$$=\omega_{\pi_{\sigma,u}}(-1)^{n'}\omega_{\pi_{\sigma',u'}}(-1)\epsilon(s,\sigma_u\otimes\sigma'_{u'},\psi)\Psi_k(s,W_f,W'_{f'})\frac{1}{L(s,\sigma_u\otimes\sigma'_{u'})}\,.$$

Proof. We claim that, for given u,u',

$$L(s,\sigma_u\otimes\sigma_{u'})=P(s)\prod L(s+v_i+v'_i,\mu_{j,i}\otimes\mu'_{j',i'})\,,$$

where P is a polynomial, and

$$\gamma(s,\sigma_u\otimes\sigma'_{u'},\psi)=\prod\gamma(s+v_i+v_{i'},\mu_{j,i}\otimes\mu_{j',i'})\,.$$

Indeed, it suffices to prove this assertion when σ and σ' are irreducible. This is checked in the Appendix. Thus we already know that $\Psi_k(s,W_f,W_{f'})$ is a meromorphic multiple of $L(s,\sigma_u\otimes\sigma'_{u'})$ and we know the functional equation of the proposition. It remains only to show that in fact $\Psi_k(s,W_f,W_{f'})$ is a holomorphic multiple of $L(s,\sigma_u\otimes\sigma'_{u'})$.

If u and u' are purely imaginary, then in the functional equation, by Lemma 5.3, the left hand side is holomorphic for $\Re s>0$ and the left hand side is holomorphic for $\Re(1-s)>0$, that is, $\Re s<1$. Thus both sides are actually holomorphic functions of s. Thus we have obtained our assertion for u and u' imaginary. Let $(\theta_{u,u'},\kappa_{u,u'})$ be an analytic family of $(\sigma_u\otimes\sigma'_{u'},\psi)$ pair. As explained before, our assertions are equivalent to the identity

$$\Psi_{n-n'-1-k}(\kappa_{u,u'},\rho(w_{n,n'})\widetilde{W_f},\widetilde{W'_{f'}})=\omega_{\pi_{\sigma,u}}(-1)^{n'}\omega_{\pi_{\sigma',u'}}(-1)\Psi_k(\theta_{u,u'},W_f,W'_{f'})\,.$$

We have thus obtained this identity for (u,u') imaginary. Since both sides are holomorphic functions of (u,u'), the identity is true for all (u,u') and we are done. □

The case $n=n'$ is treated similarly using Lemma 5.4.

12. Proof of Theorems 2.3 and 2.6: preliminaries

We first change our notations somewhat. Let σ be a semisimple representation of W_F. We can write

$$\sigma = \bigoplus_{1\leq i\leq r} \sigma_i \otimes \alpha_F^{u_i}$$

where the σ_i are normalized irreducible representations and

$$\Re u_1 \leq \Re u_2 \leq \cdots \leq \Re u_r\,.$$

This decomposition is not unique but the equivalence class of the induced representation $(\pi_{(\sigma_i),(u_i)}, I_{\pi(\sigma_i),(u_i)})$ depends only on σ. We will denote it (π_σ, I_σ). We will call the real parts of the u_i's **the exponents** of σ. The exponents of $\widetilde{\sigma}$ are the opposites of the exponents of σ. We will write $\sigma \preceq \sigma'$ if the largest exponent of σ is less than or equal to the smallest exponent of σ'. If s_0 is a pole of $L(s,\sigma)$, then there is an exponent u of σ such that

$$\Re s_0 + u \leq 0\,.$$

Let τ be another representation of W_F. We will denote by $\mathcal{I}(\sigma,\tau)$ the space spanned by the integrals $\Psi(s, W_v)$ (or $\Psi(s, W_v, \Phi)$) for $v \in I_\sigma \widehat{\otimes} I_\tau$. We will prove first that $\mathcal{I}(\sigma,\tau) \subseteq \mathcal{L}(\sigma\otimes\tau)$. Then we will prove that the two spaces are in fact equal.

12.1. The spaces $\mathcal{L}(\sigma)$. Let σ be a semisimple representation of the Weil group W_F. Recall that we denote by $\mathcal{L}(\sigma)$ the space of meromorphic functions $F(s)$ of the form

$$F(s) = L(s,\sigma)h(s)\,,$$

where h is an entire function, such that, for any $n \in \mathbb{N}$ and any vertical strip $a \leq \Re s \leq b$, the product $s^n F(s)$ is bounded at infinity in the strip. For $\sigma = 0$, the zero representation, $\mathcal{L}(0)$, is the space of entire functions $F(s)$, such that for any n and any vertical strip, the product $s^n F(s)$ is bounded at infinity in the strip.

In this subsection, we establish some simple properties of these spaces.

Lemma 12.1. *Let σ_1 be a subrepresentation of σ. Then $\mathcal{L}(\sigma_1) \subseteq \mathcal{L}(\sigma)$. In particular, $\mathcal{L}(0) \subseteq \mathcal{L}(\sigma)$.*

Proof. Indeed, $\sigma = \sigma_1 \oplus \sigma_2$. If F is in $\mathcal{L}(\sigma_1)$, then

$$F(s) = h(s)L(s,\sigma_1)$$

with h entire. We can write

$$F(s) = k(s)L(s,\sigma)\,,\; k(s) = \frac{h(s)}{L(s,\sigma_2)},$$

and k is entire. Hence $F \in \mathcal{L}(\sigma)$. □

Proposition 12.1. *Let σ be given. Let $P(\sigma)$ be the set of poles of $L(s,\sigma)$. For every $s_0 \in P(\sigma)$, let n_{s_0} be its multiplicity. Suppose we are given, for every $s_0 \in P(\sigma)$, a polar part*

$$\mathcal{P}(s_0) = \frac{A_{n_{s_0}}}{(s-s_0)^{n_{s_0}}} + \frac{A_{n_{s_0}-1}}{(s-s_0)^{n_{s_0}-1}} + \cdots + \frac{A_1}{s-s_0}\,.$$

Then there is an element $F \in \mathcal{L}(\sigma)$ having at each $s_0 \in P(\sigma)$ the polar part $\mathcal{P}(s_0)$.

Proof. Indeed, suppose first σ is irreducible. Then $\mathcal{L}(\sigma) = \mathcal{L}(\Omega)$ where Ω is a character of $F^\times$, $F = \mathbb{R}$ or $F = \mathbb{C}$. Then the poles of $L(s,\Omega)$ are simple. For any $\Phi \in \mathcal{S}(\mathbb{F})$, the analytic continuation of the integral

$$\int \Phi(x)\Omega(x)|x|_{\mathbb{C}}^s d^\times x$$

belongs to $\mathcal{L}(\Omega)$. Its polar parts at the poles of $L(s,\Omega)$ depend only on the derivatives of Φ at 0, as can be seen by integrating by parts. By Borel's Lemma, these derivatives are arbitrary. Our assertion follows in that case.

Now we can proceed by induction on the number m of irreducible components of σ. Thus we assume $m \geq 2$ and our assertion is proved for $m-1$. We write

$$\sigma = \sigma_1 \oplus \sigma_2$$

where σ_1 is irreducible. For each $s_0 \in P(\sigma_1)$, let n_0 be its multiplicity in $L(s,\sigma)$. At s_0 the Laurent expansion of $L(s,\sigma_2)$ has the form

$$\frac{k_{n_0-1}}{(s-s_0)^{n_0-1}} + \cdots$$

with $k_{n_0-1} \neq 0$ (and $n_0 - 1 \geq 0$). By the previous case, we can find an element $F \in \mathcal{L}(\sigma_1)$ such that, for any $s_0 \in P(s_0)$, the residue of F at s_0 is $A_{n_0} k_{n_0-1}^{-1}$. Then the leading term of the polar part of $F(s)L(s,\sigma_2)$ at s_0 is

$$\frac{A_{n_{s_0}}}{(s-s_0)^{n_0}}\,.$$

Now $F(s)L(s,\sigma_2)$ is in $\mathcal{L}(\sigma)$. Thus we are reduced to the case where, for every $s_0 \in P(\sigma_1)$, the given part $\mathcal{P}(s_0)$ has the form

$$\mathcal{P}(s_0) = \frac{A_{n_{s_0}-1}}{(s-s_0)^{n_{s_0}-1}} + \cdots + \frac{A_1}{s-s_0}\,.$$

But an element F of $\mathcal{L}(\sigma)$ whose polar parts at any $s_0 \in P(\sigma_1)$ has this property is in fact in $\mathcal{L}(\sigma_2)$. We then apply the induction hypothesis to σ_2 and the inclusion $\mathcal{L}(\sigma_2) \subseteq \mathcal{L}(\sigma)$ to reach our conclusion. □

Proposition 12.2. *Suppose that*

$$\sigma = \sigma_1 \oplus \sigma_2\,.$$

Then

$$\mathcal{L}(\sigma) = \mathcal{L}(\sigma_1)L(s,\sigma_2) + \mathcal{L}(\sigma_2)\,.$$

Proof. If σ_1 is irreducible, this follows from the proof of the previous proposition. We prove our assertion by induction on the number m of irreducible components of σ_1. Thus we may assume $m \geq 2$ and our assertion established for $m-1$. We write

$$\sigma_1 = \tau_1 \oplus \tau_2$$

where τ_1 is irreducible. Then

$$\mathcal{L}(\sigma_1 \oplus \sigma_2) = \mathcal{L}(\tau_1)L(s,\tau_2 \oplus \sigma_2) + \mathcal{L}(\tau_2 \oplus \sigma_2)\,.$$

By the induction hypothesis, this is also:

$$= \mathcal{L}(\tau_1)L(s,\tau_2)L(s,\sigma_2) + \mathcal{L}(\tau_2)L(s,\sigma_2) + \mathcal{L}(\sigma_2)$$
$$= (\mathcal{L}(\tau_1)L(s,\tau_2) + \mathcal{L}(\tau_2))L(s,\sigma_2) + \mathcal{L}(\sigma_2).$$

Since τ_1 is irreducible, this is equal to

$$\mathcal{L}(\tau_1 \oplus \tau_2) L(s, \sigma_2) + \mathcal{L}(\sigma_2) .$$

The proposition follows. □

PROPOSITION 12.3. *Suppose that*

$$\sigma = \sigma_1 \oplus \sigma_2 \, , \, \sigma_1 \preceq \sigma_2 \, .$$

Then

$$\mathcal{L}(\sigma) = \mathcal{L}(\sigma_1) + \frac{L(s, \sigma_1)}{L(1-s, \widetilde{\sigma_1})} \mathcal{L}(\sigma_2) .$$

PROOF. Any element of

$$\frac{L(s, \sigma_1)}{L(1-s, \widetilde{\sigma_1})} \mathcal{L}(\sigma_2)$$

is indeed a holomorphic multiple of

$$L(s, \sigma_1) L(s, \sigma_2) = L(s, \sigma) .$$

Moreover, it follows from the Stirling formula that its product by a power of s is bounded at infinity in a vertical strip. Thus it is indeed in $\mathcal{L}(\sigma)$. Moreover, as we have seen, $\mathcal{L}(\sigma_1) \subseteq \mathcal{L}(\sigma)$.

Now we claim that a pole s_0 of $L(s, \sigma_2)$ cannot be a pole of $L(1-s, \widetilde{\sigma_1})$. Indeed if it so, then there is an exponent u of σ_1 and an exponent v of σ_2 such that

$$\Re s_0 + v \leq 0 \;\text{ and }\; 1 - \Re s_0 - u \leq 0 .$$

Adding these inequalities, we get

$$1 + v - u \leq 0$$

which is a contradiction since $v - u \geq 0$.

Let s_0 be a pole of $L(s, \sigma_2)$ and n_2 its order. Let $n_1 \geq 0$ be the order of s_0 as a pole of $L(s, \sigma_1)$. Since s_0 is not a pole of $L(1-s, \widetilde{\sigma_1})$, the Laurent expansion of

$$\frac{L(s, \sigma_1)}{L(1-s, \widetilde{\sigma_1})}$$

at s_0 has the form

$$\frac{A_{n_1}}{(s-s_0)^{n_1}} + \cdots$$

with $A_{n_1} \neq 0$. On the other hand, the polar part of $f \in \mathcal{L}(\sigma_2)$ at s_0 has the form

$$\sum_{i=1}^{n_2} \frac{B_i}{(s-s_0)^i} \, ,$$

where the B_i are arbitrary. Thus the polar part of the product

$$f(s) \frac{L(s, \sigma_1)}{L(1-s, \widetilde{\sigma_1})}$$

has the form

$$\sum_{i=1}^{n_1+n_2} \frac{C_i}{(s-s_0)^i}$$

where the C_i are arbitrary for $n_1+1 \leq i \leq n_1+n_2$. Hence if F is given in $\mathcal{L}(\sigma)$ we may choose $f \in \mathcal{L}(\sigma_2)$ such that at any pole s_0 of $L(s,\sigma_2)$ the difference

$$F(s) - f(s)\frac{L(s,\sigma_1)}{L(1-s,\widetilde{\sigma_1})}$$

has a pole of order at most n_1, where n_1 is the order s_1 as a pole of $L(s,\sigma_1)$. This difference is in $\mathcal{L}(\sigma_1)$ and we are done. □

We need a strengthening of this proposition.

PROPOSITION 12.4. *Suppose that*

$$\sigma = \sigma_1 \oplus \sigma_2$$

where σ_1 *is irreducible and* $\sigma_1 \preceq \sigma_2$. *Suppose that*

$$\tau = \tau_1 \oplus \tau_2$$

where τ_2 *is irreducible and* $\tau_1 \preceq \tau_2$. *Then*

$$\mathcal{L}(\sigma\otimes\tau) = \mathcal{L}(\sigma_1\otimes\tau) + \frac{L(s,\sigma_1\otimes\tau)}{L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})}\mathcal{L}(\sigma_2\otimes\tau) + \mathcal{L}(\sigma\otimes\tau_1)\,.$$

PROOF. As before, each term in the right hand side is contained in $\mathcal{L}(\sigma\otimes\tau)$. Let $u_1 \leq u_2 \cdots \leq u_r$ be the exponents of σ, u_1 being the exponent of σ_1. Likewise, let $v_1 \leq v_2 \leq \cdots \leq v_s$ be the exponents of τ, v_s being the exponent of τ_2. We first observe that $L(s,\tau_2\otimes\sigma)$ and $L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})$ do not have a common pole. Indeed if s_0 is such a pole, then

$$\Re s_0 + v_s + u_j \leq 0 \text{ and } 1 - \Re s_0 - u_1 - v_i \leq 0$$

for some i and j. Adding the two inequalities, we find

$$1 + v_s - v_i + u_j - u_1 \leq 0\,.$$

Since $v_s - v_i \geq 0$, $u_j - u_1 \geq 0$, this is a contradiction. We have to find an element of the right hand side which at any pole s_0 of $L(s,\sigma\otimes\tau)$ has a given polar part. If s_0 is a pole of $L(s,\sigma_1\otimes\tau)$ but not $L(s,\sigma_2\otimes\tau)$, this is possible because of the term $\mathcal{L}(\sigma_1\otimes\tau)$. If s_0 is a pole of $L(s,\sigma_2\otimes\tau)$ but not a pole of $L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})$, one can use the term $\frac{L(s,\sigma_1\otimes\tau)}{L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})}\mathcal{L}(\sigma_2\otimes\tau)$ as before. If s_0 is a pole of $L(s,\sigma_2\otimes\tau)$ and a pole $L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})$, then s_0 is not a pole of $L(s,\sigma\otimes\tau_2)$. Thus it is in fact a pole of $L(s,\sigma\otimes\tau_1)$. One can use the term $\mathcal{L}(\sigma\otimes\tau_1)$ to complete the argument. □

We have also the following lemma.

LEMMA 12.2. *For any* σ,

$$\mathcal{L}(\sigma) = \mathcal{L}(0)L(s,\sigma) + \mathcal{L}(0)\,.$$

PROOF. As before, the right hand side is contained in the left hand side. Let F be an element of $\mathcal{L}(\sigma)$. Let $\mathcal{P}(s_0)$ be its polar part at a point $s_0 \in P(\sigma)$. Thus

$$\mathcal{P}(s_0) = \frac{A_n}{(s-s_0)^n} + \frac{A_{n-1}}{(s-s_0)^{n-1}} + \cdots + \frac{A_1}{(s-s_0)}\,.$$

Since e^{s^2} never vanishes, its Taylor expansion at s_0 has the form

$$k_0 + k_1(s-s_0) + \cdots + k_{n-1}(s-s_0)^{n-1}$$

with $k_0 \neq 0$. The equation

$$\left(\frac{B_n}{(s-s_0)^n}+\frac{B_{n-1}}{(s-s_0)^{n-1}}+\cdots+\frac{B_1}{(s-s_0)}\right)$$
$$\times\left(k_0+k_1(s-s_0)+\cdots+k_{n-1}(s-s_0)^{n-1}\right)$$
$$=\frac{A_n}{(s-s_0)^n}+\frac{A_{n-1}}{(s-s_0)^{n-1}}+\cdots+\frac{A_1}{(s-s_0)}$$

gives a triangular linear system of equations for the B_i, hence can be solved uniquely. Call $\mathcal{P}_1(s_0)$ the polar part given by the B_i. There is a function F_1 in $\mathcal{L}(\sigma)$ with polar part $\mathcal{P}_1(s_0)$ at each $s_0 \in P(\sigma)$. Then

$$F_2(s)=F(s)-F_1(s)e^{s^2}$$

has no poles thus is in $\mathcal{L}(0)$. On the other hand,

$$F_1(s)=L(s,\sigma)h(s)$$

where $h(s)$ is entire. Recall that for x fixed and $|y| \to +\infty$

$$|\Gamma(x+iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\frac{\pi}{2}|y|}\,.$$

It follows that in a vertical strip $|h(s)|$ is bounded by $e^{C|y|}$, for some C. Thus the product $h(s)e^{s^2}$ is rapidly decreasing in any vertical strip and thus is in $\mathcal{L}(0)$. □

12.2. Proof of Theorem 2.3. Let $(\pi_{\sigma,u}, I_{\sigma,u})$ and $(\pi_{\sigma',u'}, I_{\sigma',u'})$ be generic induced representations. Suppose $n > n'$. For clarity, we state again the result we want to prove.

PROPOSITION 12.5.

(i) *For every $f \in I_{\sigma,u}$, $f' \in I_{\sigma',u'}$, the function $s \mapsto \Psi(s, W_f, W_{f'})$ belongs to $\mathcal{L}(\sigma_u \otimes \sigma'_{u'})$.*

(ii) *The bilinear map*

$$(f,f') \mapsto \Psi(s,W_f,W_{f'})$$

$I_{\sigma,u} \times I_{\sigma',u'} \to \mathcal{L}(\sigma_u \otimes \sigma'_{u'})$ *is continuous.*

PROOF. We set $\pi = \pi_{\sigma,u}$ and $\pi' = \pi_{\sigma',u'}$. We first prove that if $P(s)$ is any polynomial then

$$P(s)\Psi_{n-n'-1}(s,W_f,W_{f'})=\sum_{i,j}\Psi_{n-n'-1}(s,W_{d\pi(X_i)f},W_{d\pi'(X'_j)f'})$$

where $X_i \in \mathfrak{U}(G_n)$, $X_j \in \mathfrak{U}(G_{n'})$. Indeed, it suffices to prove this for a polynomial of degree 1. Say $F=\mathbb{R}$. Let $U \in \mathrm{Lie}(G_{n'})$ with $\mathrm{Tr}(U)=1$. Thus $\det\exp tU = e^t$.

It easy to see that the integral

$$\int W_f\left[\begin{pmatrix} ge^{tU} & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\right]dX$$

converges, uniformly for g and t in compact sets. In fact, it is equal to

$$e^{(n-n'-1)t}\int W_f\left[\begin{pmatrix} g & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} e^{tU} & 0 & 0\\ 0 & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\right]dX.$$

The integral

$$(12.1)\qquad \iint W_f\left[\begin{pmatrix} ge^{tU} & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\right] W_{f'}(ge^{tU})|\det ge^{tU}|^{s-\frac{n-n'-1}{2}}dgdX$$

is independent of t. We compute its derivative and write that the derivative is 0. Let us set

$$V=\begin{pmatrix} U & 0\\ 0 & 1_{n-n'}\end{pmatrix}.$$

Then the derivative of (12.1) is equal to

$$\begin{aligned}&\int W_{d\pi(V)f}\left[\begin{pmatrix} ge^{tU} & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\right] dX\ \ W_{f'}(ge^{tU})|\det ge^{tU}|^{s-\frac{n-n'-1}{2}}\\ &\quad+\int W\left[\begin{pmatrix} ge^{tU} & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}\right] dX\ W_{d\pi'(U)f'}(ge^{tU})\\ &\quad+\left(s-\frac{n-n'-1}{2}+(n-n'-1)\right)\int W\left[\begin{pmatrix} g & 0 & 0\\ X & 1_{n-n'-1} & 0\\ 0 & 0 & 1\end{pmatrix}e^{tU}\right]dX\\ &\quad\times\ W'(ge^{tU})|\det ge^{tU}|^{s-\frac{n-n'-1}{2}}.\end{aligned}$$

Moreover, if t is in a compact set and s is fixed, then each term is bounded by $\xi_i(g)^{-N}\ ||g||_H^M$ with M fixed and $N>>0$. Thus we can integrate with respect to g, provided $\Re s>>0$. Hence we can differentiate (12.1) under the integral sign. Writing the derivative at $t=0$, we find

$$\begin{aligned}&\Psi_{n-n'-1}(s,W_{d\pi(V)f},W_{f'})+\Psi_{n-n'-1}(s,W_f,W_{d\pi'(U)f'})\\ &\qquad\qquad+\left(s+\frac{n-n'-1}{2}\right)\Psi_{n-n'-1}(s,W_f,W_{f'})=0\,.\end{aligned}$$

Our assertion follows.

A similar, easier to prove, assertion is valid for the integral $\Psi(s,W_f,W_{f'})$.

Since any integral Ψ is bounded at infinity in any vertical strip, we see that any product $P(s)\Psi(s,W_f,W_{f'})$ where P is a polynomial is bounded at infinity in vertical strip. The first assertion is proved.

For the second assertion, we recall that if a is sufficiently large and $a<b$ then for $a\le\Re s\le b$ the majorization

$$|\Psi(s,W_f,W_{f'})|\le\mu(f)\mu(f')$$
$$|\Psi_{n-n'-1}(s,W_f,W_{f'})|\le\mu(f)\mu(f')$$

for suitable continuous semi-norms μ,μ'. Now $\widetilde{W_f}=W_{\widetilde f}$ and likewise for f'. Thus for $a\le\Re s\le b$ we get

$$|\Psi_{n-n'-1}(s,\widetilde{W_f},\widetilde{W_{f'}})|\le\widetilde\mu(\widetilde f)\widetilde{\mu}'(\widetilde{f'})$$

for a large enough and suitable semi-norms on the space $I_{\widetilde\sigma,\widetilde u}$, $I_{\widetilde{\sigma'},\widetilde{u'}}$. However, $f\mapsto\widetilde\mu(\widetilde f)$, $f'\mapsto\widetilde{\mu}'(\widetilde{f'})$ are continuous semi-norms. So, finally we can assume that we have also

$$|\Psi_{n-n'-1}(s,\widetilde{W_f},\widetilde{W_{f'}})|\le\mu(f)\mu(f')\,.$$

Combining with our earlier observations, we conclude that given a polynomial P, for a large enough, there are continuous semi-norms μ, μ' such that, for $a \leq \Re s \leq b$,

$$|P(s)\Psi(s, W_f, W_{f'})| \leq \mu(f)\mu(f')$$

and for $1 - b \leq \Re s \leq 1 - a$

$$|P(s)\Psi_{n-n'-1}(s, \rho(w_{n,n'})\widetilde{W_f}, \widetilde{W_{f'}})| \leq \mu(f)\mu(f') .$$

Now consider the functional equation

$$P(s)\Psi(s, W_f, W_{f'}) \\ = \frac{L(s, \sigma_u \otimes \sigma'_{u'})(\det \sigma_u)^{n'-1} \ \det \sigma'_{u'}}{L(1-s, \widetilde{\sigma_u} \otimes \widetilde{\sigma'_{u'}})\epsilon(s, \sigma_u \otimes \sigma'_{u'}, \psi)} P(s)\Psi_{n-n'-1}(s, \rho(w_{n,n'})\widetilde{W_f}, \widetilde{W_{f'}}).$$

Now if a is large enough and y_0 is large enough, the ratio

$$\left| \frac{L(s, \sigma_u \otimes \sigma'_{u'})}{L(1-s, \widetilde{\sigma_u} \otimes \widetilde{\sigma'_{u'}})\epsilon(s, \sigma_u \otimes \sigma'_{u'}, \psi)} \right|$$

is bounded for $1 - b \leq \Re s \leq 1 - a$ and $|\Im s| \geq y_0$. Suppose in addition that $P(s)L(s, \sigma_u \otimes \sigma'_{u'})$ is holomorphic for $1 - b \leq \Re s \leq b$. By the maximum principle, we have then

$$|P(s)\Psi(s, W_f, W_{f'})| \leq (C+1)\mu(f)\mu(f')$$

for $1 - b \leq \Re s \leq a$. This proves the continuity in assertion (ii). □

Let again $\pi = \pi_{\sigma,u}$ and $\pi' = \pi_{\sigma',u'}$ be generic induced representations. Suppose $n = n'$. Again, we state the result we want to prove.

PROPOSITION 12.6.

(i) *For every $f \in I_{\sigma,u}$, $f' \in I_{\sigma',u'}$, every $\Phi \in \mathcal{S}(F^n)$, the function $s \mapsto \Psi(s, W_f, W_{f'}, \Phi)$ belongs to $\mathcal{L}(\sigma_u \otimes \sigma'_{u'})$.*

(ii) *The trilinear map*

$$(f, f', \Phi) \mapsto \Psi(s, W_f, W_{f'}, \Phi)$$

$I_{\sigma,u} \times I_{\sigma',u'} \times \mathcal{S}(F^n) \to \mathcal{L}(\sigma_u \otimes \sigma'_{u'})$ *is continuous.*

The proof is similar.

12.3. Extension of Theorem 2.1 to the tensor product. Let us keep to the notations of the previous subsection. To every $f \in I_{\sigma,u}\widehat{\otimes} I_{\sigma',u'}$ we associate a function W_f on $G_n \times G_{n'}$. As explained before, we can consider more general integrals involving the functions W_f. For instance, assume $n' = n - 1$. Then we set

$$\Psi(s, W_f) = \int W_f \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g' \right] |\det g|^{s - \frac{1}{2}} dg .$$

We have also the integral

$$\Psi(s, \widetilde{W_f})$$

where

$$\widetilde{W_f}(g, g') = W_f(w_n g^\iota, w_{n'} g'^\iota) .$$

The integrals converge for $\Re s >> 0$. Let (θ, κ) be a $(\sigma_u \otimes \sigma'_{u'}, \psi)$ pair.

Consider the identity

$$\int W_f\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g'\right]\theta(\det g)|\det g|^{-\frac{1}{2}}dg$$
$$= \int \widetilde{W_f}\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, g'\right]\kappa(\det g)|\det g|^{-\frac{1}{2}}dg\,.$$

Both sides converge and are continuous functions of f. The identity is true when f is a pure tensor, or a sum of pure tensors. By continuity, it is true for all f. It follows that the assertions of Theorem 2.1 are true for the integrals $\Psi(s, W_f), \Psi(s, \widetilde{W_f})$. Then, as in the previous subsection, one proves that $\Psi(s, W_f) \in \mathcal{L}(\sigma_u \otimes \sigma'_{u'})$ and the map $f \mapsto \Psi(s, W_f)$ is continuous.

12.4. Proof of Theorem 2.6 for irreducible representations of the Weil group. In this subsection, we prove Theorem 2.6 for two given irreducible representations of the Weil group that we shall denote by σ and τ. We first consider the case when they are both of degree 1. In this case, our assertion reduces to the following elementary lemma.

LEMMA 12.3. *Suppose ω is a normalized character of $F^\times$. If F is in $\mathcal{L}(\omega)$, then there is a Schwartz function Φ on F such that*

$$\int \Phi(x)|x|^s\omega(x)d^\times x = F(s)\,.$$

PROOF OF THE LEMMA: In any case, for any Φ, the analytic continuation of

$$\int \Phi(x)|x|^s\omega(x)d^\times x$$

is in $\mathcal{L}(\omega)$ and the residue at any pole s_0 of $L(s,\omega)$ is arbitrary. By linearity, we are reduced to the case where $F(s)$ is in fact entire. In this case, there is a function f on $R_+^\times$ such that

$$F(s) = \int_0^\infty f(t)t^s\frac{dt}{t}\,.$$

The function is $O(t^n)$ for any $n \in \mathbb{Z}$ and for any m, the derivative $\frac{d^m f}{dt^m}$ has the same properties. Now define a function Φ on F by

$$\Phi(x)\omega(x) = f(|x|_F)\,.$$

The function Φ is a Schwartz function with the required properties. □

We now prove the theorem when one representation has dimension 2 and the other has dimension 1. Then the theorem reduces to the following lemma.

LEMMA 12.4. *Let Ω be a normalized character of $\mathbb{C}^\times$. Let σ be the representation of $W_\mathbb{R}$ induced by Ω. Let (π_σ, I_σ) be the corresponding irreducible representation of $GL(2,\mathbb{R})$. Let $F \in \mathcal{L}(\sigma) = \mathcal{L}(\Omega)$. There is $W \in \mathcal{W}(\pi_\sigma : \psi)$ such that*

$$\int W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}|a|^{s-\frac{1}{2}}d^\times a = F(s)\,.$$

PROOF. We recall the construction of $\pi = \pi_\sigma$ (see [**14**] for instance). We first construct a representation π_+ of $G_+ = \{g \in GL(2,\mathbb{R}) : \det g > 0\}$. The representation π is induced by π_+. Let $\mathcal{S}(\mathbb{C},\Omega)$ be the space of Schwartz functions on $\mathbb{C}$ such that

$$\Phi(zh) = \Omega(h)^{-1}\Phi(z)$$

for all h such that $h\overline{h} = 1$. Then, for $a = h\overline{h}$,

$$\begin{aligned}
\pi_+ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Phi(z) &= \Phi(zh)\Omega(h)(h\overline{h})^{1/2} \\
\pi_+ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Phi(z) &= \Phi(z)\psi(xz\overline{z}) \\
\pi_+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi(z) &= \gamma\widehat{\Phi}(\overline{z})
\end{aligned}$$

where γ is a suitable constant. The operators are unitary for the L^2 norm

$$||\Phi||_2^2 := \int |\Phi(z)|^2 dz\,.$$

Thus, we obtain a unitary representation on the space $L^2(\mathbb{C}, \Omega)$ of square integrable functions such that

$$\Phi(zh) = \Omega(h)^{-1}\Phi(z)$$

for all h such that $h\overline{h} = 1$. The unitary representation is topologically irreducible. In fact, its restriction to the space of triangular matrices in G_+ is already irreducible. Let π_- be the representation obtained by replacing ψ by $\overline{\psi}$. Then π is the direct sum of $\pi_+ \oplus \pi_-$.

We take for granted that $\mathcal{S}(\mathbb{C}, \Omega)$ is the space of smooth vectors in $L^2(\mathbb{C}, \Omega)$. Then the linear form

$$\lambda(\Phi) = \Phi(1)$$

is a Whittaker linear form on $\mathcal{S}(\mathbb{C}, \Omega)$. We extend it by 0 on π_-. For any $\Phi \in \mathcal{S}(\mathbb{C}, \Omega)$, the corresponding function W_Φ is defined by

$$W_\Phi(g) = \pi_+(g)\Phi(e)$$

if $\det g > 0$ and $W_\Phi(g) = 0$ if $\det g < 0$. We have

$$\int W_\Phi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-\frac{1}{2}} d^\times a = \int \Phi(z)\Omega(z)\ (z\overline{z})^s d^\times z\,.$$

By the previous lemma, we can choose $\Phi_1 \in \mathcal{S}(\mathbb{C})$ such that

$$\int \Phi_1(z)\Omega(z)\ (z\overline{z})^s d^\times z = F(s)\,.$$

If we set

$$\Phi(z) = \int_{h\overline{h}=1} \Phi_1(zh)\Omega(h) dh,$$

the function W_Φ has the srequired property. □

Now we prove the lemma when σ and τ are both of dimension 2. We may assume that σ and τ are induced by normalized characters of $\mathbb{C}^\times$. We may also assume that ψ is standard.

Proposition 12.7. *Given $F(s)$ in $\mathcal{L}(\sigma \otimes \tau)$, there are finitely many vectors v_i in $I_\sigma \widehat{\otimes} I_\tau$ and Schwartz functions Φ_i such that*

$$\sum_i \Psi(s, W_{v_i}, \Phi_i) = F(s)\,.$$

PROOF. Recall that the integrals $\Psi(s,W,\Phi)$ converge for $\Re s > 0$. We first claim that given s with $\Re s > 0$, we can choose v $K \times K$-finite in $I_\sigma \otimes I_\tau$ and Φ such that

$$\Psi(s, W_v, \Phi) \neq 0\,.$$

Indeed suppose that, for all such v, $\Psi(s, W_v, \Phi) = 0$ for all Φ.

Indeed, this integral can be written as

$$\int \left(\int W_v \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right] |a|^{s-1} d^\times a \int f\,[b 1_2 k]\,\omega(b) d^\times b \right) dk$$

where ω is the product of the central characters and

$$f(g) = \Phi[(0,1)g]\,.$$

Any function f invariant under the subgroup

$$\left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$$

and compactly supported modulo this subgroup can be written as $f(g) = \Phi[(0,1)g]$ for a suitable Φ. Thus we find

$$\int W_v \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] |a|^{s-1} d^\times a = 0$$

for all $K \times K$-finite v. By continuity, this is then true for all vectors v in the tensor product $I_\sigma \widehat{\otimes} I_\tau$; in particular, this is true when v is a pure tensor. Thus we find

$$\int W_{v_1} \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] W_{v_2} \left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] |a|^{s-1} d^\times a = 0\,,$$

for all $v_1 \in I_\sigma$ and $v_2 \in I_\tau$. But this is a contradiction, because given functions f_1, f_2 in $\mathcal{S}(\mathbb{R}_+^\times)$, we can find v_1, v_2 such that, for $a > 0$,

$$W_{v_1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = f_1(a)\,,\; W_{v_2} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = f_2(a)\,,$$

and

$$W_{v_1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = W_{v_2} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = 0$$

for $a < 0$.

Thus the entire functions

$$\frac{\Psi(s, W_{v_1}, W_{v_2}, \Phi)}{L(s, \sigma \otimes \tau)}$$

with v_1, v_2 K-finite and Φ an arbitrary Schwartz function have no common zero for $\Re s > 0$. By continuity of the integral as a function of Φ, it follows that the above entire functions for v_1, v_2 K-finite and Φ a standard function have no common zero for $\Re s > 0$. By the functional equation, they have no common zero for $\Re s < 1$ as well, that is, they have no common zero.

Now we claim that there are K-finite vectors v_i, v_i' and standard Schwartz functions Φ_i such that

$$\sum_i \Psi(s, W_{v_i}, W_{v_i'}, \Phi_i) = L(s, \sigma \otimes \tau)\,.$$

This is checked by direct computation in [**12**], but we give a more conceptual proof. The representations π_σ and π_τ are contained in induced representations I_{μ_1,μ_2} and I_{ν_1,ν_2} respectively with $\mu_i = \mu_1^0 \alpha^{s_i}$, $\nu_i = \nu_1^0 \alpha^{t_i}$, $s_1 < s_2$, $t_1 < t_2$ (see the Appendix).

For K-finite vectors v_1, v_2 in I_{μ_1,μ_2}, I_{ν_1,ν_2}, respectively, and Φ standard, we have proved that

$$\Psi(s, W_{v_1}, W_{v_2}, \Phi) = P(s) \prod L(s, \mu_i \nu_j)\,,$$

where P is a polynomial. The vector space spanned by the polynomials P for v_1, v_2 K-finite in I_σ, I_τ respectively and Φ standard is an ideal. Let P_0 be a generator and set

$$L_0(s) = P_0(s) \prod L(s, \mu_i \nu_j)\,.$$

By direct computation (see the Appendix), we have

$$L(s, \sigma \otimes \tau) = Q_0(s) \prod L(s, \mu_i \nu_j)\,,$$

where Q_0 is another polynomial. Thus

$$L_0(s) = \frac{P_0(s)}{Q_0(s)} L(s, \sigma \otimes \tau)\,.$$

But $L_0(s)$ is a holomorphic multiple of $L(s, \sigma \otimes \tau)$. Thus

$$L_0(s) = R_0(s) L(s, \sigma \otimes \tau)$$

where R_0 is another polynomial. Hence every integral $\Psi(s, W_{v_1}, W_{v_2}, \Phi)$ with v_1, v_2 K-finite and Φ standard is a polynomial multiple of $R_0(s)L(s, \sigma \otimes \tau)$. Thus any zero of R_0 is a common zero of the ratios

$$\frac{\Psi(s, W_{v_1}, W_{v_2}, \Phi)}{L(s, \sigma \otimes \tau)}\,.$$

Hence R_0 is a constant which proves our assertion.

Now let $F \in \mathcal{L}(\sigma \otimes \tau)$. By Lemma 12.2, there are $F_i \in \mathcal{L}(0)$, $i = 1, 2$, such that

$$F(s) = F_1(s) L(s, \sigma \otimes \tau) + F_2(s)\,.$$

Let $f \in \mathcal{S}(\mathbb{R}_+^\times)$ such that

$$\int_0^\infty f(t) \omega^{-1}(t) t^{-2s} d^\times t = F_1(s)\,.$$

Recall that we have found K-finite vector v_i, v_i' and standard Schwartz functions Φ_i such that

$$\sum_i \Psi(s, W_{v_i}, W_{v_i'}, \Phi_i) = L(s, \sigma \otimes \tau)\,.$$

We set

$$\Phi_i^0(x, y) = \int \Phi_i[t(x, y)] f(t^{2m}) d^\times t\,.$$

These functions are still Schwartz functions as follows from the following lemma.

Lemma 12.5. *Let V be a finite dimensional F-vector space. Let $\Phi \in \mathcal{S}(V)$ and $f \in \mathcal{S}(\mathbb{R}_+^\times)$. The function*

$$\Phi^0(v) := \int_0^\infty \Phi(tv) f(t) d^\times t$$

is in $\mathcal{S}(V)$.

Proof. The integral converges and represents a continuous function of v. For every N,

$$|\Phi(v)| \leq \frac{C_N}{(1+||v||^2)^N}\,.$$

For $t \geq 1$,

$$|\Phi(tv)| \leq \frac{C_N}{(1+||v||^2)^N}\,.$$

Thus

$$\int_1^\infty |\Phi(tv)||f(t)|d^\times t \leq \int_1^\infty |f(t)|d^\times t \frac{C_N}{(1+||v||^2)^N}\,.$$

For $t \leq 1$,

$$|\Phi(tv)| \leq t^{-N}\frac{C_N}{(1+||v||^2)^N}\,.$$

Thus

$$\int_0^1 |\Phi(tv)||f(t)|d^\times t \leq \int_0^1 |f(t)|t^{-N} d^\times t \frac{C_N}{(1+||v||^2)^N}\,.$$

Hence

$$|\Phi^0(v)| \leq \frac{C'_N}{(1+||v||^2)^N}\,.$$

If D is a constant vector field on V, then $D\Phi^0$ exists and is given by

$$D\Phi^0(v) = \int f(t)tD\Phi(tv)d^\times t\,.$$

Thus $D\Phi^0$ is of the same form as Φ, with f replaced by $f(t)t$ and Φ by $D\Phi$. Inductively, it follows that Φ^0 is a Schwartz function. □

Now we compute

$$\Psi(s, W_{v_i}, W_{v'_i}, \Phi_i^0) = \int W_{v_i}(g)W_{v'_i}(g)\left(\int \Phi_i[(0,1)gt]f(t)d^\times t\right)|\det g|^s dg\,.$$

Exchanging the order of integration and changing g into $gt^{-1}1_2$, we find

$$\int f(t)\omega(t^{-1})t^{-2s}d^\times t\ \Psi(s, W_{u_i}, W_{u'_i}, \Phi_i) = F_1(s)\Psi(s, W_{v_i}, W_{v'_i}, \Phi_i)\,.$$

We conclude that

$$\sum_i \Psi(s, W_{v_i}, W_{v'_i}, \Phi_i^0) = F_1(s)L(s, \sigma\otimes\tau)\,.$$

Now

$$F_2(s) = \int h(a)|a|^s d^\times a$$

with $h \in \mathcal{S}(F^\times)$. We may apply the Dixmier-Malliavin Lemma to the translation representation of $\mathbb{R}^\times$ on $\mathcal{S}(\mathbb{R}^\times)$ to conclude that

$$h(x) = \sum_\alpha \int_0^\infty h_\alpha(xt)f_\alpha(t)d^\times t\,,$$

with $h_\alpha \in \mathcal{S}(F^\times)$ and $f_\alpha \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. After a change of notations we see than we can write

$$F_2(s) = \sum_\alpha \int h_\alpha(a)|a|^s d^\times a\ \int f_\alpha(b)|b|^{2s}\omega(b)d^\times b$$

with $h_\alpha \in \mathcal{S}(\mathbb{R}^\times)$ and $f_\alpha \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. Now $h_\alpha(a) = k_\alpha(a)\Omega\Omega'(a)$ with $k_\alpha \in \mathcal{S}(F^\times)$. Now we have the following lemma.

LEMMA 12.6. *Any element h of $\mathcal{S}(\mathbb{R})$ can be written as a sum*

$$h(x) = \sum h_\xi(a)k_\xi(a)$$

with h_ξ, k_ξ in $\mathcal{S}(\mathbb{R})$. Any element of $\mathcal{S}(\mathbb{R}^\times)$ can be written as a sum

$$\sum h_\xi(a)k_\xi(a)$$

with h_ξ, k_ξ in $\mathcal{S}(\mathbb{R})$.

PROOF. For the first assertion, replacing the function by its Fourier transform, it suffices to show that h is a finite sum of convolutions $\sum h_\xi * k_\xi$ with h_ξ, k_ξ in $\mathcal{S}(\mathbb{R})$. Applying the Dixmier-Malliavin Lemma to the translation representation of $\mathbb{R}$ on $\mathcal{S}(\mathbb{R})$, we obtain our assertion (with $k_\xi \in C_c^\infty$). For the second part of the lemma, we remark that any h in $\mathcal{S}(\mathbb{R}^\times)$ can be written as

$$h(x) = h_1(x)h_2(x^{-1})$$

with $h_i \in \mathcal{S}(\mathbb{R})$. We then apply the first part of the lemma. □

Coming back to the proof of the proposition, we see that we have written

$$F_2(s) = \sum_\alpha \int_{\mathbb{R}^\times} h_\alpha(a)k_\alpha(a)|a|^s d^\times a \int_{\mathbb{R}^\times} f_\alpha(b)\omega(b)|b|^{2s}d^\times b\,,$$

with $h_\alpha, k_\alpha \in \mathcal{S}(\mathbb{R}^\times)$ and $f_\alpha \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. There exists vectors v_α and v'_α such that

$$W_{v_\alpha}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = h_\alpha(a)|a|^{1/2}\,,\ W_{v'_\alpha}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = k_\alpha(a)|a|^{1/2}\,.$$

Then

$$\sum_\alpha \int W_{v_\alpha}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} W_{v'_\alpha}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}|a|^{s-1}d^\times a \int f_\alpha(b)\omega(b)|b|^{2s}d^\times b = F_2(s)\,.$$

Now let us apply the Dixmier-Malliavin Lemma to the subgroup $\overline{N}$ and the representation $(\pi_\sigma \otimes \pi_\tau, I_\sigma \widehat{\otimes} I_\tau)$ restricted to $\overline{N}$. We conclude that for each α, there are vectors $\widetilde{v}_\beta$ in the tensor product and $\phi_\beta \in C_c^\infty(\mathbb{R})$ such that

$$W_{v_\alpha}(g)W_{v'_\alpha}(g) = \sum_\beta \int W_{\widetilde{v}_\beta}\left[g\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, g\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]\phi_\beta(x)dx\,.$$

Changing notations, we see that we have obtained the formula

$$\sum_\beta \int W_{\widetilde{v}_\beta}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right]|a|^{s-1}d^\times a$$

$$\times\ \phi_\beta(x)dx \int f_\beta(b)\omega(b)|b|^{2s}d^\times b = F_2(s)\,,$$

where $\phi_\beta \in C_c^\infty(\mathbb{R})$ and $f_\beta \in C_c^\infty(\mathbb{R}^\times)$. For each β, set

$$\Phi_\beta(x,y) = \phi_\beta\left(\frac{x}{y}\right)f_\beta(y)\,.$$

This is an element of $\mathcal{S}(\mathbb{R}^2)$ such that

$$\Phi_\beta(xb,b) = \phi_\beta(x)f_\beta(b)\,.$$

The above formula can then be written in the form

$$\sum_\beta \Psi(s, W_{\widetilde{v}_\beta}, \Phi_\beta) = F_2(s)\,.$$

This concludes the proof of the proposition. □

12.5. Reduction step for $GL(2)$. We will now reduce Theorem 2.6 to the case where σ and τ are irreducible. This requires further preliminary work. In this subsection, we explain the reduction step in the case of $GL(2)$. For clarity, we repeat what we want to prove

PROPOSITION 12.8. *Let μ_1, μ_2 be two normalized characters of $F^\times$, u_1, u_2 two complex numbers with $\Re u_1 \le \Re u_2$. Given $F \in \mathcal{L}(\mu_1\alpha^{u_1} \oplus \mu_2\alpha^{u_2})$, there is $v \in I_{\mu_1,\mu_2,u_1,u_2}$ such that*

$$\Psi(s, W_v) = F(s)\,.$$

PROOF. The space of the representation I_{μ_1,μ_2,u_1,u_2} is the space of C^∞ functions f on $GL(2,F)$ such that

$$f\left[\begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} g\right] = \mu_1(a_1)|a_1|^{u_1-1/2}\mu_2(a_2)|a_2|^{u_2+1/2} f(g)\,.$$

Let $\Phi \in \mathcal{S}(F)$. Define a function f by the following rule. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a \neq 0,$$

then we can write G uniquely in the form

$$g = \begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix};$$

we then define

$$f(g) = \mu_1(a_1)|a_1|^{u_1-1/2}\mu_2(a_2)|a_2|^{u_2+1/2}\Phi(y)\,.$$

If $a = 0$, then we define $f(g) = 0$. We claim the function f is C^∞. Let Ω_1 be the set of g such that $a \neq 0$ and Ω_2 the set of g such that $b \neq 0$. It will suffice to show that the restriction of f to each open set if C^∞. This is clear for Ω_1. If g is in Ω_2, then g can be written uniquely in the form

$$g = \begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} w_2 \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

It will suffice to show that

$$f\left[w_2\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right]$$

is a C^∞ function of z. Now, if $z \neq 0$, then

$$w_2\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 1 & -z^{-1} \end{pmatrix}\begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus, if z is not zero,

$$f\left[w_2\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right] = \Phi(z^{-1})|z|^{u_1-u_2-1}\mu_1(z)\mu_2(-z^{-1})\,.$$

On the other hand, for $z = 0$ we find $f(w_2) = 0$. It easily follows that f is a C^∞ function of z, even at the point $z = 0$. We will write f as $f_{\Phi,u}$.

If $\Re u_1 < \Re u_2$, then we define a ψ linear form λ_u on I_{μ_1,μ_2,u_1,u_2} by the convergent integral

$$\lambda_u(f) = \int f\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\overline{\psi}(x)dx\,.$$

In particular,

$$\lambda_u(f_{\Phi,u}) = \widehat{\Phi}(1)\,.$$

By analytic continuation, this formula remains true even for $\Re u_1 = \Re u_2$. Set

$$W_{\Phi,u}(g) = \lambda_u(\pi_{\mu_1,\mu_2,u_1,u_2}(g)f_{\Phi,u})\,.$$

A simple computation shows that

$$W_{\Phi,u}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}|a|^{-1/2} = \widehat{\Phi}(a)\mu_1(a)|a|^{u_1}\,.$$

Thus

$$\Psi(s, W_{\Phi,u}) = \int \widehat{\Phi}(a)\mu_1(a)|a|^{s+u_1}d^\times a\,.$$

As we have seen before, for any $F \in \mathcal{L}(\mu_1\alpha^{u_1})$ we can choose Φ such that the right hand side is equal to F. Hence we have proved that for every $F \in \mathcal{L}(\mu_1\alpha^{u_1})$ there is a vector $v \in I_{\mu_1,\mu_2,u_1,u_2}$ such that $\Psi(s, W_v) = F(s)$.

Consider now the representation $(\pi_{\mu_2^{-1},\mu_1^{-1},-u_2,-u_1}, I_{\mu_2^{-1},\mu_1^{-1},-u_2,-u_1})$. Likewise, for every $\Phi \in \mathcal{S}(F)$ there is a vector v' such that

$$\Psi(s, W_{v'}) = \int \widehat{\Phi}(a)\mu_2^{-1}(a)|a|^{s-u_2}d^\times a\,.$$

Now $\widetilde{W_{v'}} = W_v$ for a suitable v and

$$\begin{aligned}\Psi(s, W_v) =&\mu_1\mu_2(-1)\gamma(s+u_1,\mu_1,\psi)^{-1}\gamma(s+u_2,\mu_2,\psi)^{-1} \\ &\times \int \widehat{\Phi}(a)\mu_2^{-1}(a)|a|^{1-s-u_2}d^\times a\,.\end{aligned}$$

Using the functional equation of the Tate integral, we can write this as

$$\Psi(s, W_v) = \mu_1\mu_2(-1)\gamma(s+u_1,\mu_1,\psi)^{-1}\int \Phi(a)\mu_2(a)|a|^s d^\times a\,.$$

Thus for every $F \in \mathcal{L}(\mu_2\alpha^{u_2})$, there is $v \in I_{\mu_1,\mu_2,u_1,u_2}$ such that

$$\Psi(s, W_v) = F(s)\frac{L(s+u_1,\mu_1)}{L(1-s-u_1,\mu_1^{-1})}\,.$$

To finish the proof we appeal to the following lemma, which is a special case of Proposition 12.3.

LEMMA 12.7.

$$\mathcal{L}(\mu_1\alpha^{u_1}\oplus\mu_2\alpha^{u_2}) = \mathcal{L}(\mu_1\alpha^{u_1}) + \frac{L(s+u_1,\mu_1)}{L(1-s-u_1,\mu_1^{-1})}\mathcal{L}(\mu_2\alpha^{u_2})\,.$$

This concludes the proof of the proposition. □

13. Bruhat Theory

In this section, we prove that certain naturally defined functions belong to the induced representations at hand. This result is due to Casselman. According to Casselman, the methods developed in [**4**] can be used to prove the result that we need. For the sake of completeness, we have included an elementary proof.

13.1. Preliminaries.

LEMMA 13.1. *Suppose that* $Y_n \in G_n(F)$ *and*

$$\lim_{n\to+\infty} Y_n = Y_0$$

where $\det Y_0 = 0$. *Then*

$$\lim_{n\to+\infty} ||Y_n^{-1}||_e = +\infty\,.$$

PROOF. Indeed, at the cost of replacing Y_0 by $g_1 Y_0 g_2$ with g_1, g_2 invertible, we may assume that the first column of Y_0 is 0. We proceed by contradiction. Let e_i, $1 \le i \le n$, be the canonical basis of F^n. If the assertion is not true, then, at the cost of replacing Y_n by a subsequence, we may assume that $||Y_n^{-1}||_e \le K$ for all n. Then

$$||Y_n^{-1} Y_n e_1||_e \le K||Y_n e_1||_e \to K||Y_0 e_1|| = 0\,.$$

However, $||Y_n^{-1} Y_n e_1||_e = ||e_1||_e = 1$, so we get a contradiction. □

LEMMA 13.2. *Let* $Y_0 \in M(n\times n, F)$ *with* $\det Y_0 = 0$. *Let* $Y_1 \in M(n \times n, F)$. *Either* $\det(Y_0+tY_1) = 0$ *for all* t *or there is, for* $t \in \mathbb{R}$ *small enough, a* C^∞ *function* $B(t)$ *with values in* $GL(n, F)$ *and an integer* $r > 0$ *such that, for* $t \neq 0$ *and small enough,*

$$(Y_0 + tY_1)^{-1} = \frac{B(t)}{t^r}\,.$$

PROOF. Indeed, assume $\det(Y_0 + tY_1)$ is not identically zero. Then

$$\det(Y_0 + tY_1) = t^r Q(t)\,,\ r > 0\,,\ Q \in \mathbb{C}[t], Q(0) \neq 0\,.$$

Let $A(t)$ be the adjugate of $Y_0 + tY_1$. Thus

$$(Y_0 + tY_1)A(t) = t^r Q(t) 1_n\,.$$

For $t \neq 0$ small enough, $Q(t) \neq 0$ and

$$(Y_0 + tY_1)^{-1} = \frac{B(t)}{t^r}\,,\ B(t) = \frac{A(t)}{Q(t)}\,.$$

□

The lemma implies that if Φ is a Schwartz function on $M(n\times n, F)$, the function defined by

$$\Psi(Y) = \begin{cases} \Phi(Y^{-1}) & \text{if } \det Y \neq 0 \\ 0 & \text{if } \det Y = 0 \end{cases}$$

is C^∞. We consider a more general situation. Let $\mathcal{V}$ be a Frechet space and V a finite dimensional complex vector space with Euclidean norm $||\cdot||$. Let Φ be a C^∞ function

$$\Phi : M(n \times n, F) \times V \to \mathcal{V}\,.$$

We assume that for any differential operator D with constant coefficients, and any continuous semi-norm μ on $\mathcal{V}$, there is M and, for each N, a constant C such that

$$\mu(D\Phi(Y,Z)) \le C\frac{(1 + ||Z||^2)^M}{(1 + ||Y||_e^2)^N}$$

We let P be a polynomial function on $M(n \times n, F) \times F$. Finally, we let τ be a smooth, moderate growth, representation of $G_n(F)$ on $\mathcal{V}$. Let $X \in \mathfrak{U}(G_n(F))$.

PROPOSITION 13.1. *The function*

$$\Psi : M(n \times n, F) \times V \to \mathcal{V}$$

defined by

$$\Psi(Y, Z) = \begin{cases} \tau(Y)(d\tau(X)\Phi)(Y^{-1}, Z)P(Y, \det Y^{-1}) & \text{if } \det Y \neq 0 \\ 0 & \text{if } \det Y = 0 \end{cases}$$

is C^∞.

PROOF. The function Ψ is continuous on $G_n \times V$. Let us prove that it is continuous on $M(n \times n, F) \times V$. Let Y_0 with $\det Y_0 = 0$ and $Z_0 \in V$. Let $Y_n \to Y_0$ and $Z_n \to Z_0$. We have to show that

$$\Psi(Y_n, Z_n) \to 0 .$$

If $\det Y_n = 0$, we have $\Psi(Y_n, Z_n) = 0$. Thus we may as well assume that $\det Y_n \neq 0$ for all n. Then $||Y_n^{-1}||_e \to \infty$ and $||Z_n|| \preceq 1$. If μ is a continuous semi-norm on $\mathcal{V}$, there is r and another continuous semi-norm ν such that, for all v,

$$\mu(\tau(Y)v) \leq ||Y||_H^r \nu(v) .$$

Hence

$$\mu(\Psi(Y_n, Z_n)) \leq |P(Y_n, \det Y_n^{-1})| \; ||Y_n||_H^r \nu\left((d\tau(X)\Phi)(Y_n^{-1}, Z_n)\right) .$$

Next

$$|P(Y_n, \det Y_n^{-1})| \leq ||Y||_H^{M_1}$$

for a suitable M_1. Since $v \mapsto \nu(d\tau(X)v)$ is a continuous semi-norm,

$$\nu\left((d\tau(X)\Phi)(Y_n^{-1}, Z_n)\right) \leq C(1 + ||Z_n||^2)^{M_0}(1 + ||Y_n^{-1}||_e^2)^{-N} \leq C'(1 + ||Y_n^{-1}||_e^2)^{-N} .$$

Altogether

$$\mu(\Psi(Y_n, Z_n)) \leq C_N ||Y_n||_H^{M_2}(1 + ||Y_n^{-1}||_e^2)^{-N} ,$$

for some M_2 and all N. Now $||Y_n||_H = ||Y_n||_e^2 + ||Y_n^{-1}||_e^2 \leq C'(1 + ||Y_n^{-1}||_e^2)$. Thus, if N is large enough, then $||Y_n||_H^{M_2}(1 + ||Y_n^{-1}||_e^2)^{-N} \to 0$. We conclude that $\mu(\Psi(Y_n, Z_n)) \to 0$. Hence Ψ is indeed continuous.

Now we prove that at a point where $\det Y = 0$ the partial derivatives of Ψ of order 1 exist and are 0. We start with the partial derivatives with respect to Y. Thus we have to show that

$$\lim_{t \to 0} \frac{\Psi(Y_t, Z)}{t} = 0$$

where $Y_t = Y + tY_1$, $Y_1 \in M(n \times n, F)$. This is clear if $\det Y_t = 0$ for all t because then $\Psi(Y_t, Z) = 0$ for all t. Otherwise, $||Y_t|| \sim Ct^{-r}$ with $r > 0$. As before for any continuous semi-norm μ,

$$\mu(\Psi(Y_t, Z)) \leq C(1 + ||Y_t||_e^2)^{-N} \leq C't^r .$$

Thus $\lim_{t \to 0} \frac{\Psi(Y_t, Z)}{t} = 0$. As for the partial derivatives with respect to Z, for $Z_1 \in V$,

$$\lim_{t \to 0} \frac{\Psi(Y, Z + tZ_1)}{t} = 0$$

trivially since $\Psi(Y, Z + tZ_1) = 0$ for all t.

Next we show that the partial derivatives of Ψ at a point where $\det Y \neq 0$ exists. We compute

$$\frac{d\Psi(Y + tY_1, Z)}{dt}\Big|_{t=0}$$

using the product rule. For any vector v

$$\frac{d\tau(Y+tY_1)v}{dt}|_{t=0} = \frac{d\tau(Y(1+tY^{-1}Y_1))v}{dt}|_{t=0}\,.$$

Now let Y_α be a basis of $M(n\times n,F)$, then

$$Y^{-1}Y_1 = \sum_\alpha \xi_\alpha(Y)Y_\alpha$$

where the ξ_α are polynomials in $(Y,\det Y)$. Thus

$$\frac{d\tau(Y+tY_1)v}{dt}|_{t=0} = \sum_\alpha \xi_\alpha(Y)\tau(Y)d\tau(Y_\alpha)v\,.$$

By the chain rule,

$$\frac{d\Phi((Y+tY_1)^{-1},Z)}{dt}|_{t=0} = \sum_i \Phi_i(Y^{-1},Z)P_i(Y,\det Y^{-1})$$

where the Φ_i are partial derivatives of Φ and the P_i are polynomials. By assumption, the Φ_i satisfy the same conditions as Φ. Finally,

$$\frac{dP((Y+tY_1)^{-1},Z)}{dt}|_{t=0}$$

is a polynomial in Y and $\det Y^{-1}$.

We conclude that for $\det Y\neq 0$, $\frac{d\Psi(Y+tY_1)}{dt}|_{t=0}$ do exist. Hence $\frac{d\Psi(Y+tY_1)}{dt}|_{t=0}$ exists for all Y and is a sum of functions of the same type as Ψ. The same assertion is trivially true for

$$\frac{d\Psi(Y,Z+tZ_1)}{dt}|_{t=0}\,.$$

The proposition follows then by iteration. □

13.2. The first term in Bruhat's filtration. Let n_1,n_2 be two integers such that $n=n_1+n_2$. Let P be the parabolic subgroup of G_n of matrices of the form

$$\begin{pmatrix} g_1 & 0\\ X & g_2\end{pmatrix},\quad g_i\in GL_{n_i}\,.$$

Let $G(P)$ be the open subset of matrices of the form

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix},\ A\in G_{n_1}(F)\,,\ D\in M(n_2\times n_2,F)\,.$$

Every matrix g in $G(P)$ can be written uniquely in the form

$$g=\begin{pmatrix} g_1 & 0\\ X & g_2\end{pmatrix}\begin{pmatrix} 1_{n_1} & Z\\ 0 & 1_{n_2}\end{pmatrix}.$$

More precisely, the map

$$(g_1,g_2,X,Z)\mapsto g$$

is a diffeomorphism onto $G(P)$. Let (σ_1,I_1) and (σ_2,I_2) be two Casselman-Wallach representations of $G_{n_1}(F)$ and $G_{n_2}(F)$, respectively. Let (π,I) be the representation of G_n induced by (σ_1,σ_2). Let v be a (smooth) vector in the space $I_1\widehat{\otimes}I_2$. Let Φ be a Schwartz function on $M(n_1\times n_2,F)$. Define a function f on G with values in $I_1\widehat{\otimes}I_2$ by

$$f\left[\begin{pmatrix} g_1 & 0\\ X & g_2\end{pmatrix}\begin{pmatrix} 1_{n_1} & Z\\ 0 & 1_{n_2}\end{pmatrix}\right] = \sigma_1(g_1)\otimes\sigma_2(g_2)v|\det g_1|^{-\frac{n_2}{2}}|\,|\det g_2|^{\frac{n_1}{2}}\Phi(Z)\,.$$

and $f(g) = 0$ if g is not in $G(P)$. We call such a function a **Casselman** function.

PROPOSITION 13.2. *The function f is C^∞ and belongs to the space I of the representation induced by (σ_1, σ_2).*

PROOF. Let e_i, $1 \le i \le n$, be the canonical basis of F^n. It will be convenient to write $V = F^n$ as a direct sum

$$V = V_1 \oplus V_2$$

where V_1, V_2 are subspaces spanned by the vectors $e_i, 1 \le i \le n_1$, and $e_i, n_1 + 1 \le i \le n$, respectively. Then any element M of $\mathrm{Hom}_F(V, V)$ can be represented as a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of linear operators, with

$$A \in \mathrm{Hom}_F(V_1, V_1)\,,\ B \in \mathrm{Hom}_F(V_2, V_1)\,,$$

$$C \in \mathrm{Hom}_F(V_1, V_2)\,,\ D \in \mathrm{Hom}_F(V_2, V_2)\,.$$

Then P is the set of invertible matrices with $B = 0$ and $G(P)$ is the set of invertible matrices with A invertible. To continue we write $G = GL(V)$ as a union of open sets. Each open set is attached to a direct sum decomposition

$$V_1 = V_1^1 \oplus V_1^2\,,\ V_2 = V_2^1 \oplus V_2^2$$

where each space is spanned by vectors in the canonical basis and

$$\mathrm{dim} V_1^2 + \mathrm{dim} V_2^1 = \mathrm{dim} V_1\,.$$

Then we can write

$$A = (A_1, A_2)\,,\ A_1 \in \mathrm{Hom}(V_1^1, V_1)\,,\ A_2 \in \mathrm{Hom}(V_1^2, V_1)\,,$$

$$B = (B_1, B_2)\,,\ B_1 \in \mathrm{Hom}(V_2^1, V_1)\,,\ B_2 \in \mathrm{Hom}(V_2^2, V_1)\,.$$

Then

$$(A_2, B_1) \in \mathrm{Hom}(V_1^2 \oplus V_2^1, V_1)\,.$$

The open set Ω attached to this decomposition is the set of invertible operators for which (A_2, B_1) is invertible. Our task is thus to prove that the restriction of f to Ω is a C^∞ function.

We may relable the vectors e_i so that V_1^1 is spanned by the vectors e_i, $1 \le i \le m_1$, V_1^2 is spanned by the vectors $e_i, m_1 + 1 \le i \le n_1$, V_2^1 is the space spanned by the vectors $e_i, n_1 + 1 \le i \le n_1 + m_1$ and V_2^2 by the remaining vectors. Here m_1 verifies $0 \le m_1 \le \inf(n_1, n_2)$. It is convenient to set $m_2 = n_1 - m_1$. Then Ω is the set

$$G(P)w\,,\ w = \begin{pmatrix} 0 & 0 & 1_{m_1} & 0 \\ 0 & 1_{m_2} & 0 & 0 \\ 1_{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n_2 - m_1} \end{pmatrix}.$$

Recall that every element of $G(P)$ has a unique decomposition of the form

$$\begin{pmatrix} g_1 & 0 \\ X & g_2 \end{pmatrix} \begin{pmatrix} 1_{n_1} & Z \\ 0 & 1_{n_2} \end{pmatrix}.$$

Since f transforms on the left under a representation of P, it will suffice to show that

$$f\left[\begin{pmatrix} 1_{m_1} & 0 & x_1 & y_1 \\ 0 & 1_{m_2} & x_2 & y_2 \\ 0 & 0 & 1_{m_1} & 0 \\ 0 & 0 & 0 & 1_{n_2-m_1} \end{pmatrix} w\right] \tag{13.1}$$

is a C^∞ function of x_1, y_1, x_2, y_2. Now

$$\begin{pmatrix} 1_{m_1} & 0 & x_1 & y_1 \\ 0 & 1_{m_2} & x_2 & y_2 \\ 0 & 0 & 1_{m_1} & 0 \\ 0 & 0 & 0 & 1_{n_2-m_1} \end{pmatrix} w = \begin{pmatrix} x_1 & 0 & 1_{m_2} & y_1 \\ x_2 & 1_{m_2} & 0 & y_2 \\ 1_{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n_2-m_1} \end{pmatrix}.$$

This matrix is in $G(P)$ if and only if x_1 is invertible. Then it can be written as

$$\begin{pmatrix} g_1 & 0 \\ X & g_2 \end{pmatrix}\begin{pmatrix} 1_{n_1} & Z \\ 0 & 1_{n_2} \end{pmatrix}$$

with

$$\begin{aligned} g_1 &= \begin{pmatrix} x_1 & 0 \\ x_2 & 1_{m_2} \end{pmatrix} \\ g_2^{-1} &= \begin{pmatrix} -1_{m_1} & 0 \\ 0 & 1_{n_2-m_1} \end{pmatrix}\begin{pmatrix} x_1 & y_1 \\ 0 & 1_{n_2-m_1} \end{pmatrix} \\ Z &= \begin{pmatrix} x_1 & 0 \\ x_2 & 1_{m_2} \end{pmatrix}^{-1}\begin{pmatrix} 1_{m_1} & y_1 \\ 0 & y_2 \end{pmatrix}. \end{aligned}$$

The value of f on this element is thus

$$\begin{aligned} &\sigma_1\begin{pmatrix} x_1 & 0 \\ x_2 & 1_{m_2} \end{pmatrix} \otimes \sigma_2^{-1}\begin{pmatrix} x_1 & y_1 \\ 0 & 1_{n_2-m_1} \end{pmatrix} v_1 \\ &\times |\det x_1|^{\frac{n_1-n_2}{2}} \Phi\begin{pmatrix} x_1^{-1} & x_1^{-1}y_1 \\ -x_2x_1^{-1} & -x_2x_1^{-1}y_1 + y_2 \end{pmatrix} \end{aligned}$$

where

$$v_1 = \sigma_2\begin{pmatrix} -1_{m_1} & 0 \\ 0 & 1_{n_2-m_1} \end{pmatrix} v.$$

Set

$$\tau(x_1) = |\det x_1|^{\frac{n_1-n_2}{2}} \sigma_1\begin{pmatrix} x_1 & 0 \\ 0 & 1_{m_2} \end{pmatrix} \otimes \sigma_2^{-1}\begin{pmatrix} x_1 & 0 \\ 0 & 1_{n_2-m_1} \end{pmatrix}.$$

The previous expression is the product of

$$\tau(x_1)\sigma_1\begin{pmatrix} 1_{m_1} & 0 \\ x_2 & 1_{m_2} \end{pmatrix} \otimes \sigma_2^{-1}\begin{pmatrix} 1_{m_1} & y_1 \\ 0 & 1_{n_2-m_1} \end{pmatrix} v_1$$

and the scalar factor

$$\Phi\begin{pmatrix} x_1^{-1} & x_1^{-1}y_1 \\ -x_2x_1^{-1} & -x_2x_1^{-1}y_1 + y_2 \end{pmatrix}.$$

Set

$$\begin{aligned} \Phi_0(x_1, x_2, y_1, y_2) = \Phi&\begin{pmatrix} x_1 & x_1y_1 \\ -x_2x_1 & -x_2x_1y_1 + y_2 \end{pmatrix} \\ &\times \sigma_1\begin{pmatrix} 1_{m_1} & 0 \\ x_2 & 1_{m_2} \end{pmatrix} \otimes \sigma_2^{-1}\begin{pmatrix} 1_{m_1} & y_1 \\ 0 & 1_{n_2-m_1} \end{pmatrix} v_1. \end{aligned}$$

If D is any differential operator with constant coefficients, the function

$$D\Phi_0(x_1, x_2, y_1, y_2)$$

is a sum of functions of the form

$$\Phi' \begin{pmatrix} x_1 & x_1 y_1 \\ -x_2 x_1 & -x_2 x_1 y_1 + y_2 \end{pmatrix} P(x_1, x_2, y_1, y_2)$$
$$\times\, \sigma_1 \begin{pmatrix} 1_{m_1} & 0 \\ x_2 & 1_{m_2} \end{pmatrix} \otimes \sigma_2^{-1} \begin{pmatrix} 1_{m_1} & y_1 \\ 0 & 1_{n_2 - m_1} \end{pmatrix} v' \,,$$

where Φ' is a Schwartz function, P is a polynomial function, and $v' \in \mathcal{V}$. It follows that Φ_0 and its derivatives are rapidly decreasing with respect to x_1 and slowly increasing with respect to the other variables. Now the value of f at hand (13.1) is given by

$$\tau(x_1)\Phi_0(x_1^{-1}, x_2, y_1, y_2)$$

if $\det x_1 \neq 0$ and 0 otherwise. By Proposition 13.1, the resulting function of (x_1, x_2, y_1, y_2) is C^∞. □

14. Proof of Theorem 2.6

14.1. Consequences of Bruhat Theory. Let

$$\sigma = \sigma_1 \oplus \sigma_2 \,,\; \sigma_1 \preceq \sigma_2 \,,$$

be a representation of the Weil group. Thus π_σ is equivalent to the representation induced by by $(\pi_{\sigma_1}, \pi_{\sigma_2})$. We set

$$n_1 = d(\sigma_1), n_2 = d(\sigma_2) \,.$$

PROPOSITION 14.1. *Given $v_1 \in I_{\sigma_1}$ and a Schwartz function on Φ on F^{n_1} there is a vector $v_0 \in I_\sigma$ such that, for all $g \in GL(n_1, F)$,*

$$W_{v_0} \begin{pmatrix} g & 0 \\ 0 & 1_{n_2} \end{pmatrix} = W_{v_1}(g)\Phi[(0, 0, \ldots, 0, 1)g] \; |\det g|^{\frac{n_2}{2}} \,.$$

PROOF. Let λ_1 and λ_2 be ψ linear forms on I_{σ_1} and I_{σ_2}. Let us write

$$\sigma_1 = \bigoplus_{i=1}^{r} \sigma_1^i \otimes \alpha^{u_i} \,,\; \sigma_2 = \bigoplus_{i=1}^{p} \sigma_2^j \otimes \alpha^{v_j} \,,$$

where the σ_i^j are irreducible normalized representations and

$$\Re u_1 \leq \Re u_2 \leq \ldots \leq \Re u_r \leq \Re v_1 \leq \Re v_2 \leq \ldots \leq \Re v_p \,.$$

If $\Re u_r < \Re v_1$, there is a ψ linear form λ on I_σ such that

$$\lambda(f) = \int \lambda_1 \otimes \lambda_2 \left[f \begin{pmatrix} 1_{n_1} & Y \\ 0 & 1_{n_2} \end{pmatrix} \right] \overline{\theta_\psi} \left[\begin{pmatrix} 1_{n_1} & Y \\ 0 & 1_{n_2} \end{pmatrix} \right] dY \,.$$

If f is a Casselman function, the integral converges even if $\Re u_r = \Re v_1$. By analytic continuation, we conclude that this formula remains true when f is a Casselman function and $\Re u_r \leq \Re v_1$. If f is determined by the formula

$$f \begin{pmatrix} 1_{n_1} & Y \\ 0 & 1_{n_2} \end{pmatrix} = \Phi(Y) v_0 \,,\; v_0 = v_1 \otimes v_2,$$

then the corresponding function W_f verifies

$$W_f \begin{pmatrix} g & 0 \\ 0 & 1_{n_2} \end{pmatrix} = \lambda_1 \otimes \lambda_1 \left[\int f \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n_2} \end{pmatrix} \begin{pmatrix} 1_{n_1} & g^{-1}Y \\ 0 & 1_{n_2} \end{pmatrix} \right] \psi\left(-\mathrm{Tr}(\epsilon Y)\right) dY \right]$$

where ϵ is the matrix with n_2 rows and n_1 columns whose last row is $(0,0,\dots,0,1)$ and all other rows are zero. After a change of variables, we find

$$\begin{aligned}\lambda_1(\sigma_1(g_1)v_1)\lambda_2(v_2)|\det g|^{\frac{n_2}{2}}\int \Phi(Y)\psi\left(-\mathrm{Tr}(\epsilon g Y)\right)dY \\ = \lambda_2(v_2)W_{v_1}(g)|\det g|^{\frac{n_2}{2}}\Phi_1((0,0,\dots,0,1)g)\end{aligned}$$

where Φ_1 is the Schwartz function on F^{n_1} defined by

$$\Phi_1(u) = \widehat{\Phi}(U)\,,$$

where U is the matrix with n_2 rows and n_1 columns whose last row is u and all other rows are zero. Clearly, Φ_1 is an arbitrary Schwartz function. Our assertion follows. □

One can easily establish the following variant.

Proposition 14.2. *Let σ, σ_1, σ_2 be as above. Let also τ be another representation of the Weil group of degree m. Let v be a vector in $I_{\sigma_1}\widehat{\otimes}I_\tau$ and Φ be a Schwartz function on F^{n_1}. There is a vector v_0 in $I_\sigma\widehat{\otimes}I_\tau$ such that, for all $g\in GL(n_1,F)$, $g'\in GL(m,F)$,*

$$W_{v_0}\left[\begin{pmatrix} g & 0\\ 0 & 1_{n_2}\end{pmatrix}, g'\right] = W_v(g,g')\Phi[(0,0,\dots,0,1)g]\ |\det g|^{\frac{n_2}{2}}\,.$$

The proof is similar and based on Casselman functions for the representation $\pi_\sigma\widehat{\otimes}\pi_\tau$ of the group $G_n(F)\times G_m(F)$, this tensor product being regarded as an induced representation.

14.2. Reduction step.

Proposition 14.3. *Let $\sigma=\sigma_1\oplus\sigma_2$ be a reprsentation of the Weil group of the with $\sigma_1\preceq\sigma_2$. Let n_i be the degree of σ_i, $n=n_1+n_2$. Let τ be another representation of the Weil group of degree m. Then*

$$\mathcal{I}(\sigma,\tau)\ \supseteq\ \mathcal{I}(\sigma_1,\tau)\,, \tag{14.1}$$

$$\mathcal{I}(\sigma,\tau)\ \supseteq\ \mathcal{I}(\sigma_2,\tau)\frac{L(s,\sigma_1\otimes\tau)}{L(1-s,\widetilde{\sigma_1}\otimes\widetilde{\tau})}\,. \tag{14.2}$$

Proof. We prove the first assertion of Proposition 14.3. Suppose $n_1=m$. Given $v_1\in I_{\sigma_1}\widehat{\otimes}I_\tau$ and a Schwartz function Φ on F^{n_1}, consider the integral

$$\Psi(s,W_{v_1},\Phi) = \int W_{v_1}(g,g)|\det g|^s\Phi[(0,0,\dots,0,1)g]|\det g|^s dg\,.$$

By Proposition 14.2, there is $v_0\in I_\sigma\widehat{\otimes}I_\tau$ such that

$$W_{v_1}(g,g)\Phi[(0,0,\dots,1)g] = W_{v_0}\left[\begin{pmatrix} g & 0\\ 0 & 1_{n_2}\end{pmatrix}, g\right]|\det g|^{-\frac{n_2}{2}}\,.$$

Since $n_2=n-m$, we find

$$\Psi(s,W_{v_1},\Phi)=\Psi(s,W_{v_0})$$

which proves our assertion in this case.

Now assume $m<n_1$. Then given $v_1\in I_{\sigma_1}\widehat{\otimes}I_\tau$, consider the integral

$$\Psi(s,W_{v_1}) = \int W_{v_1}\left[\begin{pmatrix} g & 0\\ 0 & 1_{n_1-m}\end{pmatrix}, g\right]|\det g|^{s-\frac{n_1-m}{2}}dg\,.$$

Applying Proposition 14.2, (with $\Phi[(0,0,\dots,1)] = 1$), we see that there is $v_0 \in I_\sigma \widehat{\otimes} I_\tau$ such that

$$W_{v_1}\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n_1-m}\end{pmatrix}, g\right] = W_{v_0}\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1_{n_1-m} & 0 \\ 0 & 0 & 1_{n_2}\end{pmatrix}, g\right] |\det g|^{-\frac{n_2}{2}}.$$

Then

$$\Psi(s, W_{v_1}) = \Psi(s, W_{v_0})$$

and we are done in this case.

Now we assume that $n_1 < m$. Recall that $\mathcal{I}(\sigma_1 \otimes \tau)$ is the space spanned by the integrals

$$\int_{N_{n_1}\backslash G_{n_1}} W_v\left[g, \begin{pmatrix} g & 0 \\ 0 & 1_{m-n_1}\end{pmatrix}\right] |\det g|^{s-\frac{m-n_1}{2}} dg$$

with v in $I_{\sigma_1} \widehat{\otimes} I_\tau$. By Proposition 6.1, it is also the space spanned by the integrals of the form

$$\int_{N_{n_1}\backslash G_{n_1}} W_v\left[g, \begin{pmatrix} g & 0 \\ 0 & 1_{m-n_1}\end{pmatrix}\right] \Phi(\epsilon_{n_1} g) |\det g|^{s-\frac{m-n_1}{2}} dg$$

with v in $I_{\sigma_1} \widehat{\otimes} I_\tau$ and Φ a Schwartz function on F^{n_1}. Thus it suffices to show such an integral belongs to $\mathcal{I}(\sigma, \tau)$. By Proposition 14.2, it has the form

$$\int W_{v_0}\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n_2}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1_{m-n_1}\end{pmatrix}\right] |\det g|^{s-\frac{n-n_1+m-n_1}{2}} dg$$

with $v_0 \in I_\sigma \widehat{\otimes} I_\tau$.

We formulate a lemma. Applying the lemma to the case $r = n_1$, we see that the previous expression is indeed in $\mathcal{I}(\sigma, \tau)$.

Lemma 14.1. *Let σ and τ be representations of the Weil group of degree n and m respectively. Suppose $r < n, r < m$. Let*

$$v \in I_\sigma \widehat{\otimes} I_\tau .$$

The integral

$$\int_{N_r\backslash G_r} W_v\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-r}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1_{m-r}\end{pmatrix}\right] |\det g|^{s-\frac{n-r+m-r}{2}} dg$$

belongs to $\mathcal{I}(\sigma, \tau)$.

It remains to prove the lemma.

Proof. Suppose $n > m$. For $r = m$, the integral of the lemma belongs to $\mathcal{I}(\sigma, \tau)$ by definition. Thus we may assume that $r < m$ and for each v the integral

$$\int_{N_{r+1}\backslash G_{r+1}} W_v\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-r-1}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1_{m-r-1}\end{pmatrix}\right] |\det g|^{s-\frac{n-(r+1)+m-(r+1)}{2}} dg$$

belongs to $\mathcal{I}(\sigma, \tau)$. Then we prove that for each v the integral

$$\int_{N_r\backslash G_r} W_v\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-r}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1_{m-r}\end{pmatrix}\right] |\det g|^{s-\frac{n-r+m-r}{2}} dg$$

belongs to $\mathcal{I}(\sigma, \tau)$. By descending induction, this will establish the lemma.

Given $v \in I_\sigma \widehat{\otimes} I_\tau$, we can find vectors $v_i \in I_\sigma \widehat{\otimes} I_\tau$ and smooth functions of compact support ϕ_i on $F^r \times F^\times$ such that

$$v = \sum_i \int \pi_\sigma \begin{pmatrix} a(h)^{-1} & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{n-m} \end{pmatrix} \otimes \pi_\tau \begin{pmatrix} a(h)^{-1} & 0 \\ x & h \end{pmatrix} v_i \quad \phi_i(x,h) dx |h| \ d^\times h \,.$$

Here

$$a(h) = \mathrm{diag}(h, \overbrace{1, 1, , \dots, 1}^{r-1}) \ .$$

Indeed, this follows from Lemma 6.1 applied to the group of matrices of the form

$$\begin{pmatrix} a(h)^{-1} & 0 \\ x & h \end{pmatrix}, \quad h \in F^\times \,, x \in F^r \,.$$

After a change of variables, the integral of the lemma becomes

$$\sum_i \int W_{v_i} \left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{n-m} \end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \right] \phi_i(x,h) dx$$
$$\times |\det g|^{s - \frac{n-(r+1)+m-(r+1)}{2} - 1} dg \ |h|^{s - \frac{n-(r+1)+m-(r+1)}{2}} d^\times h \,.$$

By Proposition 6.2, this has the form

$$\sum_i \int W_{u_i} \left[\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & 1_{n-m} \end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \right] dx$$
$$\times |\det g|^{s - \frac{n-(r+1)+m-(r+1)}{2} - 1} dg \ |h|^{s - \frac{n-(r+1)+m-(r+1)}{2}} d^\times h \,.$$

for suitable vectors u_i. Now

$$f \mapsto \int f \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} dx \ |\det g|^{-1} dg \ d^\times h$$

gives an invariant measure on $N_{r+1} \backslash G_{r+1}$. Thus we may write the above expression as

$$\sum_i \int_{N_{r+1} \backslash G_{r+1}} W_{u_i} \left[\begin{pmatrix} g & 0 \\ 0 & 1_{n_2} \end{pmatrix}, g \right] |\det g|^{s - \frac{n-(r+1)+m-(r+1)}{2}} dg \,.$$

which by hypothesis is in $\mathcal{I}(\sigma, \tau)$. We have proved the lemma in the case $n > m$.

It remains to treat the case $m = n$. The inductive argument we have just used shows that the integral of the lemma is equal to an integral of the form

$$\int_{N_{n-1} \backslash G_{n-1}} W_{v_0} \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right] |\det g|^{s-1} dg \,.$$

We use once more Lemma 6.1 to write

$$v_0 = \sum_i \int_{F^{n-1} \times F^\times} \pi_\sigma \begin{pmatrix} a(h)^{-1} & 0 \\ x & h \end{pmatrix} \otimes \pi_\tau \begin{pmatrix} a(h)^{-1} & 0 \\ x & h \end{pmatrix} v_i \ \phi_i(x,h) |h| \ dx \ d^\times h$$

with ϕ_i smooth of compact support on $F^{n-1} \times F^\times$. Then the integral takes the form

$$\sum_i \int W_{v_i} \left[\begin{pmatrix} g & 0 \\ x & h \end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \right] |\det g|^{s-1} |h|^s \phi_i(x,h) \ dx \ d^\times h \ dg \,.$$

Let Φ_i be defined by $\Phi_i(x,y) = \phi_i(x,y)$ if $y \neq 0$ and $\Phi_i(x,y) = 0$ otherwise. Then Φ_i is a Schwartz function and the integral is equal to

$$\sum_i \Psi(s, W_{v_i}, \Phi_i)\,.$$

This concludes the proof of the lemma and the proof of the first assertion of the proposition. □

For the second part of Proposition 14.3, we remark that

$$\widetilde{\sigma} = \widetilde{\sigma_2} \oplus \widetilde{\sigma_1}$$

and thus

$$\mathcal{I}(\widetilde{\sigma}, \widetilde{\tau}) \supseteq \mathcal{I}(\widetilde{\sigma_2} \otimes \widetilde{\tau})\,.$$

By the functional equation, $\mathcal{I}(\sigma,\tau)$ contains all functions of the form

$$\gamma(s, \sigma_1 \otimes \tau, \psi)^{-1} \gamma(s, \sigma_2 \otimes \tau, \psi)^{-1} f(1-s)$$

with $f \in \mathcal{I}(\widetilde{\sigma_2} \otimes \widetilde{\tau})$. Using again the functional equation, we see that $\mathcal{I}(\sigma,\tau)$ contains

$$\gamma(s, \sigma_1 \otimes \tau, \psi)^{-1} \mathcal{I}(\sigma_2 \otimes \tau)$$

or

$$\frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})} \mathcal{I}(\sigma_2 \otimes \tau)\,.$$

This concludes the proof of Proposition 14.3. □

14.3. End of Proof of Theorem 2.6.

Proof. If σ and τ are irreducible, we have already established Theorem 2.6.

Next we prove Theorem 2.6 when one representation, τ say, is irreducible, thus of the form $\tau = \tau_0 \otimes \alpha^v$ with τ_0 unitary irreducible, v real. The proof is by induction on the number of irreducible components of σ. Thus we may write

$$\sigma = \sigma_1 \oplus \sigma_2$$

where σ_1 is irreducible and $\sigma_1 \preceq \sigma_2$. The assertion of the theorem is true for the pair (σ_1, τ). By induction, we may assume it is true for the pair (σ_2, τ). We have

$$\mathcal{I}(\sigma,\tau) \supseteq \mathcal{I}(\sigma_1,\tau) + \mathcal{I}(\sigma_2,\tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})}\,.$$

By the induction hypothesis, this is

$$\mathcal{I}(\sigma_1,\tau) + \mathcal{I}(\sigma_2,\tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})} = \mathcal{L}(\sigma_1 \otimes \tau) + \mathcal{L}(\sigma_2 \otimes \tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})}\,.$$

Now

$$\sigma \otimes \tau = \sigma_1 \otimes \tau \oplus \sigma_2 \otimes \tau$$

and $\sigma_1 \otimes \tau \preceq \sigma_2 \otimes \tau$. By Proposition 12.3,

$$\mathcal{L}(\sigma_1 \otimes \tau) + \mathcal{L}(\sigma_2 \otimes \tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})} = \mathcal{L}(\sigma \otimes \tau)\,.$$

So we are done.

Now we establish our assertion by induction on the sum of the number of irreducible components of σ and the number of irreducible components of τ. We may further assume σ and τ reducible. Thus we may write

$$\sigma = \sigma_1 \oplus \sigma_2$$

with σ_1 irreducible and $\sigma_1 \preceq \sigma_2$ and

$$\tau = \tau_1 \oplus \tau_2$$

with τ_2 irreducible and $\tau_1 \preceq \tau_2$. We may further assume our assertion established for the pairs

$$(\sigma_1, \tau), (\sigma_2, \tau), (\tau_1, \sigma) .$$

As before,

$$\mathcal{I}(\sigma, \tau) \supseteq \mathcal{I}(\sigma_1, \tau) + \mathcal{I}(\sigma_2, \tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})} .$$

Also

$$\mathcal{I}(\sigma, \tau) \supseteq \mathcal{I}(\tau_1, \sigma) .$$

By the induction hypothesis,

$$\mathcal{I}(\sigma, \tau) \supseteq \mathcal{L}(\sigma_1 \otimes \tau) + \mathcal{L}(\sigma_2 \otimes \tau) \frac{L(s, \sigma_1 \otimes \tau)}{L(1-s, \widetilde{\sigma_1} \otimes \widetilde{\tau})} + \mathcal{L}(\tau_1 \otimes \sigma) .$$

By Proposition 12.4, the right hand side is $\mathcal{L}(\sigma \otimes \tau)$ and we are done. □

15. Proof of Theorem 2.7

Proof. We prove Theorem 2.7 for $(n, n-1)$. The proof for (n, n) is similar. With the notations of the theorem, the induced representation $I_{\sigma,u}$ is a closed subspace of the space $I_{\mu,v}$ of a principal series representation. Likewise $I_{\sigma',u'}$ is is a closed subspace of the space $I_{\mu',v'}$ of a principal series representation. Of course, we may have equality. Now we claim that

$$L(s, \sigma_u \otimes \sigma'_{u'}) = P_0(s) \prod_{j,k} L(s + v_i + v'_j, \mu_i \mu' j)$$

where P_0 is a polynomial. Indeed, it suffices to prove this when the tuples σ and σ' have only one element. This case is checked directly in the Appendix.

After a permutation, we may assume

$$v_1 \leq v_2 \leq \cdots \leq v_n .$$

The permutation does not change the irreducible components of the principal series representation. Thus, a priori, the representation $I_{\sigma,u}$ is now only an irreducible component of $I_{\mu,v}$, that is, is equivalent to the representation on a subquotient of $I_{\mu,v}$. But by Lemma 2.5 $I_{\sigma,u}$ is in fact a subrepresentation of $I_{\mu,v}$. Thus we can view $I_{\sigma,u}$ as a closed invariant subspace of $I_{\mu,v}$. Likewise, we may assume

$$v'_1 \leq v'_2 \leq \cdots \leq v'_n$$

and $I_{\sigma,u}$ is a closed invariant subspace of $I_{\mu',v'}$.

We have already remarked (Proposition 8.1) that for every K_n-finite $f \in I_{\mu,v}$ and every K_{n-1}-finite $f' \in I_{\mu',v'}$ the integral $\Psi(s, W_f, W_{f'})$ is a polynomial multiple of $\prod_{j,k} L(s + v_i + v'_j, \mu_i \mu' j)$, thus a rational multiple of $L(s, \sigma_u \otimes \sigma'_{u'})$. Since it is in fact a holomorphic multiple, we conclude that

$$\Psi(s, W_f, W_{f'}) = P(s) L(s, \sigma_u \otimes \sigma'_{u'})$$

where $P(s)$ is a polynomial. The vector space generated by the polynomials P is in fact an ideal. Let P_0 be a generator and s_0 a zero of P_0. Now the map

$$(f, f') \mapsto \Psi(s, W_f, W_{f'})$$

from $I_{\sigma,u} \times I_{\sigma',u'}$ to $\mathcal{L}(\sigma_u \otimes \sigma_{u'})$ is continuous or, what amounts to the same, the map

$$v \mapsto \Psi(s, W_v)$$

from $I_{\sigma,u}\widehat{\otimes}I_{\sigma',u'}$ to $\mathcal{L}(\sigma_u \otimes \sigma_{u'})$ is continuous. If s_0 is not a pole of $L(s, \sigma_u \otimes \sigma'_{u'})$, then all functions $\Psi(s, W_f, W_{f'})$, with $f \in I_{\mu,v}$ K_n-finite and $f' \in I_{\mu',v'}$ K_{n-1}-finite vanish at s_0. It follows that all functions $\Psi(s, W_v)$, with $v \in I_{\sigma,u}\widehat{\otimes}I_{\sigma',u'}$, vanish at s_0. Similarly, if s_0 is a pole of order r of $L(s, \sigma_u \otimes \sigma'_{u'})$, then the functions $\Psi(s, W_v)$, with $v \in I_{\sigma,u}\widehat{\otimes}I_{\sigma',u'}$, have a pole of order $\le r-1$ at s_0. In any case, this contradicts the fact that $L(s, \sigma_u \otimes \sigma'_{u'}) = \Psi(s, W_v)$ for a certain v. Thus P_0 is a constant. Thus we find

$$L(s, \sigma_u \otimes \sigma'_{u'}) = \sum_j \Psi(s, W_{f_j}, W_{f'_j})$$

for suitable K_n finite elements $f_j \in I_{\sigma,u}$ and K_{n-1} finite elements $f'_j \in I_{\sigma',u'}$. □

16. Appendix: the L and ϵ factors

For the convenience of the reader, we recall the precise definitions of the L and ϵ factors attached to a representation of the Weil group W_F and we prove some relations between them.

We recall the definition of the Weil group. First $W_{\mathbb{C}} = \mathbb{C}^\times$. Denote by $\kappa \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ the complex conjugation. Then $W_{\mathbb{R}}$ is the non-trivial extension

$$\mathbb{C}^\times \to W_{\mathbb{R}} \to \{1, \kappa\}\,.$$

Thus $W_{\mathbb{R}}$ contains an element κ_0 which maps onto κ and verifies

$$\kappa_0^2 = -1\,,\ \kappa_0 z \kappa_0^{-1} = \overline{z} \text{ if } z \in \mathbb{C}^\times\,.$$

Moreover

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup \mathbb{C}^\times \kappa_0\,.$$

The homomorphism

$$W_{\mathbb{R}} \to \mathbb{R}^\times$$

defined by

$$\kappa_0 \mapsto -1\,,\ z \mapsto z\overline{z}$$

is surjective. Its kernel is the derived group of $W_{\mathbb{R}}$. Thus we can view any one dimensional representation of $W_{\mathbb{R}}$ as a one dimensional representation of $\mathbb{R}^\times$.

First, for any representation σ of the Weil group W_F,

$$L(s, \sigma \otimes \alpha_F^u) = L(s+u, \sigma)\,,\ \epsilon(s, \sigma \otimes \alpha_F^u, \psi) = \epsilon(s+u, \sigma, \psi)\,.$$

Second, if $\sigma = \sigma_1 \oplus \sigma_2$, then

$$L(s, \sigma) = L(s, \sigma_1)L(s, \sigma_2)\,,\ \epsilon(s, \sigma, \psi) = \epsilon(s, \sigma_1, \psi)\epsilon(s_2, \sigma, \psi)\,.$$

Thus it suffices to define the factors for σ irreducible. We may even assume σ normalized, that is, we may assume that the restriction of σ to $\mathbb{R}_+^\times$ is trivial.

We first recall the definition of the L and ϵ factors attached to a one dimensional representation of W_F, or, equivalently, to a character μ of $F^\times$. The book [**27**] is a convenient reference. Up to a scalar, the factor $L(s, \mu)$ is essentially defined by the condition that, for any Schwartz function Φ on F, the integral

$$Z(s, \mu, \Phi) = \int \Phi(x)|x|_F^s \mu(x) d^\times x$$

be a holomorphic multiple of $L(s,\mu)$. More precisely, when Φ is standard, this integral is of the form $P(s)L(s,\mu)$, where P is a polynomial, and any polynomial P occurs for a suitable Φ.

Suppose $F=\mathbb{R}$. For $\mu = 1_{\mathbb{R}^\times}$

$$L(s,1_{\mathbb{R}^\times}) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$

and

$$Z(s,1_{\mathbb{R}^\times},\Phi_0) = L(s,1_{\mathbb{R}^\times})\,, \text{ where } \Phi_0(x)=e^{-\pi x^2}\,.$$

Denote by η the sign character of $\mathbb{R}^\times$. Then

$$L(s,\eta) = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)$$

and

$$Z(s,\eta,\Phi_\eta) = L(s,\eta)\,, \text{ where } \Phi_\eta(x) = xe^{-\pi x^2}\,.$$

Now suppose $F=\mathbb{C}$. For $\mu = 1_{\mathbb{C}^\times}$

$$L(s,1_{\mathbb{C}^\times}) = 2(2\pi)^{-s}\Gamma(s)$$

and

$$Z(s,1_{\mathbb{C}^\times},\Phi_0) = CL(s,1_{\mathbb{C}^\times}) \text{ where } \Phi_0(z) = e^{-2\pi z\overline{z}}\,,$$

and C is a suitable constant. The definition of the L factors is so chosen that

$$L(s,1_{\mathbb{C}^\times}) = L(s,1_{\mathbb{R}^\times})L(s,\eta)\,,$$

as follows from the duplication formula.

If $\mu(z) = z^m(z\overline{z})^{-\frac{m}{2}}$ where $m\ge 1$ is an integer, then

$$L(s,\mu) = 2(2\pi)^{-s-\frac{m}{2}}\Gamma\left(s+\frac{m}{2}\right)$$

and

$$Z(s,\mu,\Phi_{\overline{m}}) = CL(s,\mu) \text{ where } \Phi_{\overline{m}}(z) = \overline{z}^m e^{-2\pi z\overline{z}}\,.$$

If $\mu(z) = \overline{z}^m(z\overline{z})^{-\frac{m}{2}}$, then

$$L(s,\mu) = 2(2\pi)^{-s-\frac{m}{2}}\Gamma\left(s+\frac{m}{2}\right)$$

as before and

$$Z(s,\mu,\Phi_m) = CL(s,\mu) \text{ where } \Phi_m(z) = z^m e^{-2\pi z\overline{z}}\,.$$

The ϵ factor is defined by the functional equation

$$\frac{Z(1-s,\mu^{-1},\widehat{\Phi})}{L(1-s,\mu^{-1})} = \epsilon(s,\mu,\psi)\frac{Z(s,\mu,\Phi)}{L(s,\mu)}\,.$$

We have already indicated the dependence on ψ.

Suppose $F=\mathbb{R}$. We take $\psi_{\mathbb{R}}(x) = e^{2i\pi x}$. Then $\mathcal{F}_{\psi_{\mathbb{R}}}(\Phi_0) = \Phi_0$ and so

$$\epsilon(s,1_{\mathbb{R}^\times},\psi_{\mathbb{R}}) = 1\,.$$

On the other hand, $\mathcal{F}_{\psi_{\mathbb{R}}}(\Phi_\eta) = -i\Phi_\eta$ and so

$$\epsilon(s,\eta,\psi_{\mathbb{R}}) = -i\,.$$

Suppose $F=\mathbb{C}$. We take $\psi_{\mathbb{C}}(z) = \psi_{\mathbb{R}}(z+\overline{z}) = e^{2i\pi(z+\overline{z})}$. Then $\mathcal{F}_{\psi_{\mathbb{C}}}(\Phi_0) = \Phi_0$ and so

$$\epsilon(s,1_{\mathbb{C}^\times},\psi_{\mathbb{C}}) = 1\,.$$

On the other hand,

$$\mathcal{F}_{\psi_{\mathbb{C}}}(\Phi_m) = (-i)^m \Phi_{\overline{m}}\,,\ \mathcal{F}_{\psi_{\mathbb{C}}}(\Phi_{\overline{m}}) = (-i)^m \Phi_m$$

and so, for $\mu(z) = z^m (z\overline{z})^{-\frac{m}{2}}$ or $\mu(z) = \overline{z}^m (z\overline{z})^{-\frac{m}{2}}$, we find

$$\epsilon(s, \mu, \psi_{\mathbb{C}}) = (-i)^m\,.$$

Now let Ω be a character of $\mathbb{C}^\times$. We can induce it to $W_{\mathbb{R}}$. We obtain a two dimensional representation σ_Ω of $W_{\mathbb{R}}$. If we replace Ω by the character Ω^κ defined by $\Omega^\kappa(z) = \Omega(\overline{z})$, the class of the representation does not change. If Ω does not factor through the norm, then σ_Ω is irreducible. Within equivalence, the irreducible representations of $W_{\mathbb{R}}$ are the representations of dimension 1 and the irreducible representations of the form σ_Ω. At this point we may as well assume Ω normalized.

If $\Omega = 1_{\mathbb{C}^\times}$, then σ_Ω is reducible. In fact,

$$\sigma_\Omega = 1_{\mathbb{R}^\times} \oplus \eta\,.$$

Thus

$$L(s, \sigma_\Omega) = L(s, 1_{\mathbb{R}^\times}) L(s, \eta) = L(s, \Omega)\,.$$

On the other hand,

$$\epsilon(s, \sigma_\Omega, \psi_{\mathbb{R}}) = \epsilon(s, 1_{\mathbb{R}^\times}, \psi_{\mathbb{R}}) \epsilon(s, \eta, \psi_{\mathbb{R}}) = -i = \lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}}) \epsilon(s, \Omega, \psi_{\mathbb{C}})$$

where

$$\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}}) := -i\,.$$

This motivates the following definitions. For an arbitrary Ω,

$$L(s, \sigma_\Omega) := L(s, \Omega)\,,\ \epsilon(s, \sigma_\Omega, \psi_{\mathbb{R}}) := \lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}}) \epsilon(s, \Omega, \psi_{\mathbb{C}})\,.$$

When $F = \mathbb{R}$ we need some relations between those factors. Suppose that Ω is a normalized character of $\mathbb{C}^\times$, say $\Omega(z) = z^m (z\overline{z})^{-\frac{m}{2}}$ where $m \geq 0$ is an integer. The representation $(\pi_{\sigma_\Omega}, I_{\sigma_\Omega})$ of $GL(2, \mathbb{R})$ is a discrete series representation (or limit of discrete series if Ω is trivial). Its construction in terms of the Weil representation is described for instance in [**14**]. In the same reference, it is shown that there exists two (non-normalized) characters μ_1, μ_2 of $\mathbb{R}^\times$ such that π_{σ_Ω} is a subrepresentation of π_{μ_1, μ_2}. Thus we can view I_{σ_Ω} as a closed invariant subspace of the space I_{μ_1, μ_2} of C^∞ functions f on $GL(2, \mathbb{R})$ such that

$$f\left[\begin{pmatrix} a_1 & 0 \\ x & a_2 \end{pmatrix} g\right] = \mu_1(a_1)|a_1|^{-1/2} \mu_2(a_2)|a_2|^{1/2} f(g)\,.$$

It is in fact the only proper closed invariant subspace if Ω is non-trivial. If Ω is trivial, it is the whole space.

Suppose m is even. We have two choices for (μ_1, μ_2):

$$\mu_1(t) = |t|_{\mathbb{R}}^{\frac{m}{2}} \eta(t)\,,\ \mu_2(t) = |t|_{\mathbb{R}}^{-\frac{m}{2}}$$

and

$$\mu_1'(t) = |t|_{\mathbb{R}}^{\frac{m}{2}}\,,\ \mu_2'(t) = |t|_{\mathbb{R}}^{-\frac{m}{2}} \eta(t)\,.$$

The map $\mu \mapsto \mu^{-1}$ exchanges these two sets of characters. Suppose m is odd. We have again two choices

$$\mu_1(t) = |t|_{\mathbb{R}}^{\frac{m}{2}}\,,\ \mu_2(t) = |t|_{\mathbb{R}}^{-\frac{m}{2}}$$

and

$$\mu_1'(t) = |t|_{\mathbb{R}}^{\frac{m}{2}} \eta(t)\,,\ \mu_2'(t) = |t|_{\mathbb{R}}^{-\frac{m}{2}} \eta(t)\,.$$

We will not consider the second choice.

Lemma 16.1. *Suppose that I_{σ_Ω} is a subrepresentation of I_{μ_1,μ_2}. Then*

$$L(s,\sigma_\Omega)=P(s)L(s,\mu_1)L(s,\mu_2)\,,$$

where P is a polynomial. Moreover,

$$\gamma(s,\sigma_\Omega,\psi_\mathbb{R})=\gamma(s,\mu_1,\psi_\mathbb{R})\gamma(s,\mu_2,\psi_\mathbb{R})\,.$$

Remark 16.2. The lemma is true for any choice of (μ_1,μ_2). However, we only prove that there is a choice for which the lemma is true because this is all what we need.

Proof. Recall the duplication formula

$$2^{1-s}\Gamma(s)=\pi^{-1/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\,,$$

and the formula

$$\Gamma(t+r)=Q_r(t)\Gamma(t)$$

where

$$Q_r(t)=(t+r-1)(t+r-2)\cdots t\,. \tag{16.1}$$

Note the functional equation

$$Q_r(t)=(-1)^rQ_r(-t-r+1)\,. \tag{16.2}$$

Suppose first m is odd. Let us write $m=2r+1$ with $r\geq 0$. Then

$$L(s,\sigma_\Omega)=L(s,\Omega)=2(2\pi)^{-(s+r+\frac12)}\Gamma\left(s+r+\frac12\right)\,.$$

By the duplication formula, this is

$$L(s,\sigma_\Omega)=\pi^{-(s+r+1)}\Gamma\left(\frac{s+r+\frac12}{2}\right)\Gamma\left(\frac{s+r+\frac12+1}{2}\right)\,.$$

On the other hand, with

$$\mu_1(t)=|t|^{\frac{m}{2}}\,,\ \mu_2(t)=|t|^{-\frac{m}{2}}\,,$$

we get

$$L(s,\mu_1)L(s,\mu_2)=\pi^{-s}\Gamma\left(\frac{s+r+\frac12}{2}\right)\Gamma\left(\frac{s-r-\frac12}{2}\right)\,.$$

We have

$$\frac{s+r+\frac12+1}{2}-\frac{s-r-\frac12}{2}=r+1\,.$$

Thus we find

$$L(s,\sigma_\Omega)=\pi^{-r-1}Q_{r+1}(t)L(s,\mu_1)L(s,\mu_2)\,,\ t=\frac{s-r-\frac12}{2}\,.$$

Similarly, $\widetilde{\sigma_\Omega}$ is the representation induced by $\Omega^{-1}=\Omega^\kappa$, thus is in fact equivalent to σ_Ω. We find then

$$L(s,\widetilde{\sigma_\Omega})=\pi^{-r-1}Q_{r+1}(t)L(s,\mu_1)L(s,\mu_2)=\pi^{-r-1}Q_{r+1}(t)L(s,\mu_2^{-1})L(s,\mu_1^{-1})\,.$$

Now replacing s by $1-s$ replaces $t=\frac{s-r-\frac12}{2}$ by $-t-r$. Thus we find

$$\gamma(s,\sigma_\Omega,\psi_\mathbb{R})=\epsilon(s,\Omega,\psi_\mathbb{C})\lambda(\mathbb{C}/\mathbb{R},\psi_\mathbb{R})\frac{Q_{r+1}(-t-r)}{Q_{r+1}(t)}\frac{L(1-s,\mu_1^{-1})L(1-s,\mu_2^{-1})}{L(s,\mu_1)L(s,\mu_2)}\,.$$

Now $\frac{Q_{r+1}(-t-r)}{Q_{r+1}(t)} = (-1)^{r+1}$, $\epsilon(s, \Omega, \psi_{\mathbb{C}}) = (-i)^{2r+1}$, $\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}}) = -i$. Thus

$$\gamma(s, \sigma_\Omega, \psi_{\mathbb{R}}) = \frac{L(1-s, \mu_1^{-1})L(1-s, \mu_2^{-1})}{L(s, \mu_1)L(s, \mu_2)} = \gamma(s, \mu_1, \psi_{\mathbb{R}})\gamma(s, \mu_2, \psi_{\mathbb{R}})\,.$$

Thus we find the required identity.

Now we assume m even and we write $m = 2r$, $r \geq 0$. Then

$$L(s, \sigma_\Omega) = 2(2\pi)^{-s-r}\Gamma(s+r)\,.$$

By the duplication formula, this is

$$L(s, \sigma_\Omega) = \pi^{-s-r-\frac{1}{2}}\Gamma\left(\frac{s+r}{2}\right)\left(\frac{s+r+1}{2}\right)\,.$$

Now, with

$$\mu_1(t) = |t|^{\frac{m}{2}}\eta(t)\,,\ \mu_2(t) = |t|^{-\frac{m}{2}}\,,$$

we get

$$L(s, \mu_1)L(s, \mu_2) = \pi^{-s-\frac{1}{2}}\Gamma\left(\frac{s+r+1}{2}\right)\Gamma\left(\frac{s-r}{2}\right)\,.$$

We find

$$L(s, \sigma_\Omega) = \pi^{-r}Q_r(t)L(s, \mu_1)L(s, \mu_2)\,,\ t = \frac{s-r}{2}\,.$$

Similarly, with

$$\mu_1'(t) = |t|^{\frac{m}{2}}\,,\ \mu_2'(t) = |t|^{-\frac{m}{2}}\eta(t)\,,$$

we get

$$L(s, \mu_1')L(s, \mu_2') = \pi^{-s-\frac{1}{2}}\Gamma\left(\frac{s+r}{2}\right)\Gamma\left(\frac{s-r+1}{2}\right)\,.$$

This time we find

$$L(s, \sigma_\Omega) = \pi^{-r}Q_r(t')L(s, \mu_1')L(s, \mu_2')\,,\ t' = \frac{s-r+1}{2}\,.$$

We remark that changing s to $1-s$ changes t' to $-t-(r-1)$ where $t = \frac{s-r}{2}$. We also remark that

$$L(s, \mu_1^{-1})L(s, \mu_2^{-1}) = L(s, \mu_1')L(s, \mu_2')\,.$$

At this point, we find

$$\gamma(s, \sigma_\Omega, \psi_{\mathbb{R}}) = \epsilon(s, \Omega, \psi_{\mathbb{C}})\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})\frac{Q_r(-t-(r-1))}{Q_r(t)}\frac{L(1-s, \mu_1^{-1})L(1-s, \mu_2^{-1})}{L(s, \mu_1)L(s, \mu_2)}\,.$$

Now $\frac{Q_r(-t-(r-1))}{Q_r(t)} = (-1)^r$, $\epsilon(s, \Omega, \psi_{\mathbb{C}}) = (-i)^{2r}$, $\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}}) = -i$. Thus

$$\gamma(s, \sigma_\Omega, \psi_{\mathbb{R}}) = -i\frac{L(1-s, \mu_1^{-1})L(1-s, \mu_2^{-1})}{L(s, \mu_1)L(s, \mu_2)}\,.$$

On the other hand,

$$\epsilon(s, \mu_1, \psi_{\mathbb{R}})\epsilon(s, \mu_2, \psi_{\mathbb{R}}) = -i\,.$$

Thus we find the required relation. □

We need a more complicated lemma of the same type.

Lemma 16.3. *Let Ω and Ξ be two normalized characters of $\mathbb{C}^\times$. Choose as before (μ_1, μ_2) (resp. (ν_1, ν_2)) such that π_{σ_Ω} is a subrepresentation of π_{μ_1,μ_2} (resp. π_{ν_1,ν_2}.). Then*

$$L(s, \sigma_\Omega \otimes \sigma_\Xi) = P(s) \prod_{i,j} L(s, \mu_i \nu_j)$$

where P is a polynomial. Moreover,

$$\gamma(s, \sigma_\Omega \otimes \sigma_\Xi, \psi_\mathbb{R}) = \prod_{i,j} \gamma(s, \mu_i \nu_j, \psi_\mathbb{R}) .$$

Remark 16.4. Again, the lemma is true for any choice of the characters. We only prove it is true for one choice.

Proof. We may assume

$$\Omega(z) = z^m (z\overline{z})^{-m/2} , \ \Xi(z) = z^n (z\overline{z})^{-n/2} , \ m \geq n \geq 0 .$$

The representation $\sigma_\Omega \otimes \sigma_\Xi$ is the direct sum of the representations induced by the characters $\Omega\Xi$ and $\Omega\Xi^\kappa$ respectively. Moreover

$$\Omega\Xi(z) = z^{m+n} (z\overline{z})^{-\frac{m+n}{2}} , \ \Omega\Xi^\kappa(z) = z^{m-n} (z\overline{z})^{-\frac{m-n}{2}} .$$

Accordingly, we find

$$L(s, \sigma_\Omega \otimes \sigma_\Xi) = 2^2 (2\pi)^{-2s-m} \Gamma\left(s + \frac{m+n}{2}\right) \Gamma\left(s + \frac{m-n}{2}\right)$$

and

$$\epsilon(s, \sigma_\Omega \otimes \sigma_\Xi, \psi_\mathbb{R}) = \lambda(\mathbb{C}/\mathbb{R}, \psi_\mathbb{R})^2 (-i)^{m+n+m-n} = (-1)^{m+1} .$$

By the duplication formula,

$$L(s, \sigma_\Omega \otimes \sigma_\Xi) = \pi^{-2s-m-1} \times \Gamma\left(\frac{s + \frac{m+n}{2}}{2}\right) \Gamma\left(\frac{s + \frac{m+n}{2} + 1}{2}\right) \Gamma\left(\frac{s + \frac{m-n}{2}}{2}\right) \Gamma\left(\frac{s + \frac{m-n}{2} + 1}{2}\right) .$$

Suppose that m and n are both odd. Then $(-1)^{m+1} = 1$. The characters $\mu_i \nu_j$ are the following characters

$$|t|^{\frac{m+n}{2}} , |t|^{\frac{m-n}{2}} , |t|^{\frac{n-m}{2}} , |t|^{-\frac{m+n}{2}} .$$

The map $\mu \mapsto \mu^{-1}$ permutes this set of characters. Now

$$\prod L(s, \mu_i \nu_j) = \pi^{-2s} \Gamma\left(\frac{s + \frac{m+n}{2}}{2}\right) \Gamma\left(\frac{s + \frac{m-n}{2}}{2}\right) \Gamma\left(\frac{s + \frac{n-m}{2}}{2}\right) \Gamma\left(\frac{s - \frac{m+n}{2}}{2}\right) .$$

Thus

$$L(s, \sigma_\Omega \otimes \sigma_\Xi) = \pi^{-m-1} P(s) \prod L(s, \mu_i \nu_j)$$

where

$$P(s) = Q_{\frac{m+1}{2}} \left(\frac{s + \frac{n-m}{2}}{2}\right) Q_{\frac{m+1}{2}} \left(\frac{s - \frac{n+m}{2}}{2}\right) .$$

This proves the first assertion. We use once more the functional equation (16.2) to conclude that

$$P(1-s) = (-1)^{m+1} P(s) = P(s) .$$

Note that here the functional equation of $Q_{\frac{m+1}{2}}$ exchanges the two factors of P. We finish the proof as before. We have

$$\begin{aligned}\gamma(s,\sigma_\Omega\otimes\sigma_\Xi,\psi_{\mathbb{R}}) &= \frac{L(1-s,\sigma_\Omega\otimes\sigma_\Xi)}{L(s,\sigma_\Omega\otimes\sigma_\Xi)}\\ &= \frac{P(1-s)}{P(s)}\frac{\prod L(1-s,\mu_i\nu_j)}{\prod L(s,\mu_i\nu_j)} = \prod\gamma(s,\mu_i\nu_j,\psi_{\mathbb{R}})\,.\end{aligned}$$

Now we assume that m and n are even. Then $(-1)^{m+1}=-1$. The characters $\mu_i\nu_j$ can be taken to be the following ones:

$$|t|^{\frac{m+n}{2}}\,,\ |t|^{\frac{m-n}{2}}\eta(t)\,,\ |t|^{\frac{n-m}{2}}\eta(t)\,,\ |t|^{-\frac{m+n}{2}}\,.$$

Note that $\mu\mapsto\mu^{-1}$ is a permutation of that set. It follows that

$$\begin{aligned}\prod L(s,\mu_i\nu_j) &= (\pi)^{-2s-1}\\ &\times\ \Gamma\left(\frac{s+\frac{m+n}{2}}{2}\right)\Gamma\left(\frac{s+1+\frac{m-n}{2}}{2}\right)\Gamma\left(\frac{s+1+\frac{n-m}{2}}{2}\right)\Gamma\left(\frac{s-\frac{m+n}{2}}{2}\right)\end{aligned}$$

Thus

$$L(s,\sigma_\Omega\otimes\sigma_\Xi)=\pi^{-m}P(s)\prod L(s,\mu_i\nu_j)\,,$$

where

$$P(s)=Q_{\frac{m}{2}}\left(\frac{s+1+\frac{n-m}{2}}{2}\right)Q_{\frac{m}{2}}\left(\frac{s-\frac{m+n}{2}}{2}\right)\,.$$

Thus the first assertion is proved. For the second assertion, we get again

$$P(s)=P(1-s)(-1)^m=P(1-s)$$

and

$$\gamma(s,\sigma_\Omega\otimes\sigma_\Xi,\psi_{\mathbb{R}})=-\frac{P(1-s)}{P(s)}\frac{\prod L(1-s,\mu_i\nu_j)}{\prod L(s,\mu_i\nu_j)}=-\frac{\prod L(1-s,\mu_i^{-1}\nu_j^{-1})}{\prod L(s,\mu_i\nu_j)}\,.$$

Now

$$\prod\epsilon(s,\mu_i\nu_j,\psi)=-1\,.$$

Thus we get the required relation.

Now suppose m even and n odd. Then we have two choices for the characters corresponding to Ω. Call them as before (μ_1,μ_2) and (μ'_1,μ'_2). For Ξ we consider only the first choice. Thus $\mu_j\nu_j$ are the characters

$$|t|^{\frac{m+n}{2}}\,,\ |t|^{\frac{m-n}{2}}\,,\ |t|^{\frac{n-m}{2}}\eta(t)\,,\ |t|^{\frac{-m-n}{2}}\eta(t)$$

and $\mu'_i\nu_j$ are the characters

$$|t|^{\frac{m+n}{2}}\eta(t)\,,\ |t|^{\frac{m-n}{2}}\eta(t)\,,\ |t|^{\frac{n-m}{2}}\,,\ |t|^{\frac{-m-n}{2}}\,.$$

The map $\mu\mapsto\mu^{-1}$ exchanges the two sets. As before, we have

$$L(s,\sigma_\Omega\otimes\sigma_\Xi)=\pi^{-m}P(s)\prod L(s,\mu_i\nu_j)=\pi^{-m}P'(s)\prod L(s,\mu'_i\nu_i)$$

where

$$\begin{aligned}P(s) &= Q_{\frac{m}{2}}\left(\frac{s+1+\frac{n-m}{2}}{2}\right)Q_{\frac{m}{2}}\left(\frac{s+1-\frac{n+m}{2}}{2}\right)\\ P'(s) &= Q_{\frac{m}{2}}\left(\frac{s+\frac{n-m}{2}}{2}\right)Q_{\frac{m}{2}}\left(\frac{s-\frac{n+m}{2}}{2}\right)\,.\end{aligned}$$

We have $P(s) = (-1)^m P'(1-s) = P'(1-s)$ and

$$\gamma(s, \sigma_\Omega \otimes \sigma_\Xi, \psi_\mathbb{R}) = -\frac{P(1-s)}{P(s)} \frac{\prod L(1-s, \mu_i' \nu_j)}{\prod L(s, \mu_i \nu_j)} = -\frac{\prod L(1-s, \mu_i^{-1} \nu_j^{-1})}{\prod L(s, \mu_i \nu_j)} .$$

Now

$$\prod \epsilon(s, \mu_i \nu_j, \psi) = -1 .$$

Thus we get again the required relation.

Finally we assume m odd and n even. This time we have two choices for the characters corresponding to Ξ, ν_j and ν_j'. For Ω we only use the first choice. Thus $\mu_j \nu_j$ are the characters

$$|t|^{\frac{m+n}{2}} \eta(t) \, , \, |t|^{\frac{m-n}{2}} \, , \, |t|^{\frac{n-m}{2}} \eta(t) \, , \, |t|^{\frac{-m-n}{2}} \, ,$$

and $\mu_i \nu_j'$ are the characters

$$|t|^{\frac{m+n}{2}} \, , \, |t|^{\frac{m-n}{2}} \eta(t) \, , \, |t|^{\frac{n-m}{2}} \, , \, |t|^{\frac{-m-n}{2}} \eta(t) \, .$$

Again, the map $\mu \mapsto \mu^{-1}$ exchanges the two sets. We argue exactly exactly as in the previous case. This time

$$\begin{aligned} P(s) &= Q_{\frac{m-1}{2}} \left(\frac{s + 1 + \frac{n-m}{2}}{2} \right) Q_{\frac{m+1}{2}} \left(\frac{s - \frac{n+m}{2}}{2} \right) \\ P'(s) &= Q_{\frac{m+1}{2}} \left(\frac{s + \frac{n-m}{2}}{2} \right) Q_{\frac{m-1}{2}} \left(\frac{s + 1 - \frac{m+n}{2}}{2} \right) \end{aligned}$$

but we have again $P(s) = (-1)^m P'(1-s) = -P'(1-s)$. □

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