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Factorization of Period Integrals

Hervé Jacquet[1]

Columbia University, New York, New York 10027
E-mail: hj@math.columbia.edu

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TO THE MEMORY OF YASUKO JACQUET

We show that for certain quadratic extensions E/F of number fields the period integral of a cusp form of $GL(3, E)$ over the unitary group H_0 in three variables is a product of local linear forms.

Contents.

1. GLOBAL RESULTS

Let E/F be a quadratic extension of number fields. We will denote by σ the non-trivial element of the Galois group of E/F and will often write $\sigma(z) = \bar{z}$. We will denote by $\mathbb{U}_1$ the unitary group in 1 variable. We assume that every Archimedean place of F *splits* in E. We let H_0 be the unitary group for the 3×3 identity matrix. Recall that a cuspidal automorphic representation Π of $GL(3, E_{\mathbb{A}})$ is said to be *distinguished* by H_0 if the linear form:

$$\mathscr{P}(\phi) := \int_{H_0(F) \backslash H_0(F_{\mathbb{A}})} \phi(h)\, dh,$$

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is non identically zero on the space $\mathscr{V}(\Pi)$ of smooth vectors of Π. Our main result is that if this linear form is non-zero, then it can be written as a tensor product of local linear forms. This is a rather startling result, since there is no local property of uniqueness which guarantees in advance the existence of such a decomposition.

In more detail, if Π is distinguished, then the representation Π^σ defined by $\Pi^\sigma(g)=\Pi(g^\sigma)$ is equivalent to Π. Moreover, the following condition is satisfied for every place v_0 of F inert in E; let v be the corresponding place of E and $\mathscr{H}_{v_0}=\mathscr{H}(\Pi_v, H_{0,v})$ be the space of linear forms on the space $\mathscr{V}(\Pi_v)$ of smooth vectors of Π_v which are invariant under H_{0,v_0}. Then $\mathscr{H}_{v_0}\neq 0$. If v_0 is a place of F which split into v_1 and v_2 in E let similarly $\mathscr{H}_{v_0}=\mathscr{H}(\Pi_{v_1}\otimes\Pi_{v_2}, H_{0,v_0})$ be the space of linear forms on the space $\mathscr{V}(\Pi_{v_1}\otimes\Pi_{v_2})$ of smooth vectors for the tensor product $\Pi_{v_1}\otimes\Pi_{v_2}$ which are invariant under H_{0,v_0}. Now H_{0,v_0} is isomorphic to the group of pairs (g_{v_1}, g_{v_2}) such that $g_{v_1}={}^t g_{v_2}^{-1}$, $g_{v_1}\in GL(3, F_{v_0})$. Furthermore $\Pi_{v_1}=\Pi_{v_2}$. Thus the dimension of $\mathscr{H}_{v_0}$ is actually one.

Let S_0 be a finite set of places of F containing all the places at infinity, the even places, and the places which ramify in E. Let S_i be the set of places in S_0 which are inert in E and let S_s be the set of split places. Let S be the set of places of E above a place of S_0. Let Π be a distinguished representation. Suppose that Π is unramified outside S. We let $\mathscr{V}^S(\Pi)$ or simply $\mathscr{V}^S$ be the subspace of vectors in $\mathscr{V}(\Pi)$ which are invariant under $K^S:=\prod_{v\notin S}K_v$, $K_v=GL(3,\mathscr{O}_v)$. We consider the elements of $\mathscr{V}^S(\Pi)$ which are *pure tensors*. They can be described in terms of the Whittaker models as follows. Let ψ be a non-trivial character of $F_{\mathbb{A}}/F$. Set $\psi_E(z)=\psi(z+\bar{z})$. Denote by N the group of upper triangular matrices with unit diagonal, and by θ the character of $N(F_{\mathbb{A}})$ defined by:

$$\theta(n)=\psi(n_{1,2}+n_{2,3}).$$

Similarly, define a character $n\mapsto\theta(n\bar{n})$ on $N(E_{\mathbb{A}})$ by:

$$\theta(n\bar{n})=\psi_E(n_{1,2}+n_{2,3}).$$

For $\phi\in\mathscr{V}(\Pi)$, set

$$\mathscr{W}(\phi)=\int_{N(E)\backslash N(E_{\mathbb{A}})}\phi(n)\,\theta^{-1}(n\bar{n})\,dn,\qquad W(g)=\mathscr{W}(\Pi(g)\,\phi).$$

Then

$$\phi(g)=\sum_{\gamma\in N(2,E)\backslash GL(2,E)} W\left[\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}g\right],$$

where $N(2)$ denotes the group of upper triangular matrices with unit diagonal in $GL(2)$. If ϕ is a pure tensor in $\mathscr{V}^S$, then the corresponding function W has the form

$$W(g) = \prod W_v(g_v),$$

where W_v is in the Whittaker model $\mathscr{W}(\Pi_v)$ of Π_v for every place v, and, for $v \notin S$, $W_v = W^{K_v}$ is the unique element of $\mathscr{W}(\Pi_v)$ which is invariant under K_v and equal to 1 on K_v. Whenever convenient, we identify the space $\mathscr{V}(\Pi_v)$ with $\mathscr{W}(\Pi_v)$. Then our precise result is the following theorem:

THEOREM 1. *There exist a constant $c \neq 0$ and, for each $v_0 \in S_0$, a non-zero element $\mathscr{P}_{v_0} \in \mathscr{H}_{v_0}$ such that, for any pure tensor ϕ in $\mathscr{V}^S$,*

$$\mathscr{P}(\phi) = c \prod_{v_0 \in S_i} \mathscr{P}_{v_0}(W_v) \prod_{v_0 \in S_s} \mathscr{P}_{v_0}(W_{v_1} \otimes W_{v_2}).$$

We remark that the proof of the theorem will provide us with a specific choice of the local linear forms and also a specific value for the constant c in terms of L-functions.

Since the spaces $\mathscr{H}_{v_0}$ with $v_0 \in S_i$ are not one-dimensional in general, the existence of such a decomposition is *not formal.* Moreover, the theorem implies the non-trivial result that the linear form $\mathscr{P}$ is non-zero on $\mathscr{V}^S$, for any S satisfying the above conditions. A priori, one can only say that if it is non-zero, then it is non-zero on a space $\mathscr{V}^S$, with a large enough S.

The paper is arranged as follows. We review the relative trace formula of [JY] in Section 3. We prove the main theorem in Section 4. In Section 5 and 7 we prove local results. Let E/F be a local quadratic extension of non-Archimedean fields. If Π is a supercuspidal representation of $GL(3, E)$ we prove in Section 5 that the dimension of $\mathscr{H}(\Pi, H_0(F))$ is at most one. In Section 6, we review the local theory for $GL(2)$. In Section 7 we show, that, at least for certain irreducible representations Π of $GL(3, E)$, the dimension of $\mathscr{H}(\Pi, H_0(F))$ is equal to the number of irreducible representations of $GL(3, F)$ (with a given central character) which base change to Π. Conjecturally, this should always be the case. Finally, in Section 8, we state general conjectures.

We note that a recent work of E. Lapid and J. Rogawski treat a related question ([LR]). Roughly speaking, they investigate the notion of distinguished representation and period integrals for non-cuspidal automorphic representations. Finally, we refer to [yF2] for the discussion of representations distinguished by $GL(n, F)$.

We now explain our notations. In general G denotes the group $GL(3)$ regarded as an algebraic group. The context indicates whether we regard G as an algebraic group over F or over E. For instance, if v_0 is a place of F then G_{v_0} denotes the group $GL(3, F_{v_0})$. Likewise if v is a place of E then G_v denotes the group $GL(3, E_v)$. Moreover K_{v_0} and K_v denote the corresponding standard maximal compact subgroups. Throughout the paper S_0 and S are finite sets of places of F and E respectively satisfying the previous conditions. Then G_{S_0} is the product of the groups G_{v_0} with $v_0 \in S_0$ and G^{S_0} the (restricted) product of the groups G_{v_0} with $v_0 \notin S_0$. The notations F_{S_0}, $F^\times_{S_0}$, K_{S_0}, K^{S_0}, G_S, G^S, K_S, K^S have a similar meaning.

2. L^2-NORM OF A PURE TENSOR

We keep to the notations of the introduction. We recall how to compute the L^2-norm of a pure tensor in a cuspidal representation π of $GL(3, F_{\mathbb{A}})$. We assume that π is unramified outside S_0. An invariant scalar product on each space $\mathscr{W}_{v_0}(\Pi_{v_0})$ is given by:

$$(W_1, W_2) = \int_{N(2, F_{v0})\backslash GL(2, F_{v0})} W_1 \bar{W}_2 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right] dg.$$

This is a result of Bernstein ([B]) in the non-Archimedean case and of Baruch in the Archimedean case ([Ba]). More precisely, Baruch shows that the restriction of the unitary representation Π_{v_0} to the group P_{v_0} of triangular matrices with last row $(0, 0, 1)$ is irreducible (as a unitary representation). On the other hand in [JS] it is shown that the irreducible representation τ of P_{v_0} induced by the character θ_{v_0} of N_{v_0} occurs in this restriction. The conclusion follows. In fact this assertion is also a consequence of the global theory as we are going to see (once one knows that the above Hermitian form is defined by a *convergent integral*).

This being so, there is a constant $c(F, S_0)$, which depends on F, S_0, and the choice of the Haar measures, but not on π, such that:

$$\|\phi\|^2 = c(F, S_0)\, L^{S_0}(1, \pi, \mathrm{Ad}) \prod_{v_0 \in S_0} \|W_{v_0}\|^2. \tag{1}$$

Here $L^{S_0}(s, \pi, \mathrm{Ad})$ denotes the partial adjoint L-function attached to π.

Indeed, let $\Phi = \prod \Phi_{v_0}$ be a Schwartz–Bruhat function in three variables where Φ_{v_0} is the characteristic function of $\mathcal{O}^3_{v_0}$ for $v_0 \notin S_0$. Consider the Epstein–Eisenstein series:

$$E(g, \Phi, s) = \int_{F^\times\backslash F^\times_{\mathbb{A}}} \sum_{\xi \in F^3 - \{0\}} \Phi(t\xi g)\, |t|^{3s}\, d^\times t\, |\det g|^s.$$

Then, for $\Re s \gg 0$, if ϕ_1, ϕ_2 are pure tensors,

$$\int \phi_1 \bar{\phi}(g)\, E(g, \Phi, s)\, dg$$

$$= L^{S_0}(s, 1_F)\, L^{S_0}(s, \pi, \mathrm{Ad}) \prod_{v_0 \in S_0} \int W_{1, v_0} \overline{W_{2, v_0}}(g)\, \Phi_{v_0}[(0, 0, 1)\, g]\, |\det g|^s\, dg.$$

Taking the residue at $s = 1$ we obtain

$$\int \phi_1 \bar{\phi}(g)\, dg \prod_{v_0 \in S_0} \int_{F_{v0}^3} \Phi_{v_0}(x)\, dx$$
$$= \mathrm{Res}_{s=1} L^{S_0}(s, 1_F)\, L^{S_0}(1, \pi, \mathrm{Ad})$$
$$\times \prod_{v_0 \in S_0} \int W_{1, v_0} \overline{W_{2, v_0}} \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk\, dg.$$

In the last integral k is in K_{S_0}, $g \in GL(2, F_{S_0})$, $t \in F_{S_0}^\times$. Now for $v_0 \in S_0$ the integrals

$$\int_{F_{v0}^3} \Phi_{v_0}(x)\, dx, \qquad \int \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk$$

are equal (up to a scalar factor). Moreover, the left hand side of the previous formula is an invariant Hermitian form on the space $\mathscr{V}(\Pi)$. If follows that for $v_0 \in S_0$ there is an Hermitian form β_{v_0} on $\mathscr{W}(\Pi_{v_0})$ such that

$$\int W_1 \bar{W}_2 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk\, dg$$

$$= \beta_{v_0}(W_1, W_2) \int \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk.$$

It follows in turn that for every smooth function f_{v_0} on K_{v_0} invariant on the left under $P_{v_0} Z_{v_0} \cap K_{v_0}$, Z denoting the center,

$$\int W_1 \bar{W}_2 \left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] f_{v_0}(k)\, dk\, dg = \beta_{v_0}(W_1, W_2) \int f_{v_0}(k)\, dk.$$

The same relation is then true for every smooth function on K_{v_0}. In turn this implies that the Hermitian form

$$(W_1, W_2)$$

is invariant under K_{v_0} and thus under G_{v_0} a claimed. Then

$$\int W_1 \bar{W}_2\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k\right] \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk\, dg$$

$$= (W_1, W_2) \int \Phi_{v_0}[tk]\, |\det t|^3\, d^\times t\, dk,$$

and relation (1) follows.

Thus we can construct an orthonormal basis of $\mathscr{V}^{S_0}$ as follows: for each $v_0 \in S_0$, we choose an orthonormal basis $(W_{\alpha_{v_0}})$, $\alpha_{v_0} \in A_{v_0}$, of $\mathscr{W}(\Pi_{v_0}, \psi_{v_0})$. We then set $A = \prod_{v_0 \in S_0} A_{v_0}$. For each $\alpha \in A$ we define

$$W_\alpha(g) = \prod_{v_0 \in S_0} W_{\alpha_{v_0}}(g_{v_0}) \prod_{v_0 \notin S_0} W^{K_{v0}}(g_{v_0})$$

and then we set

$$\phi_\alpha(g) = \frac{1}{\sqrt{c(F, S_0)\, L^{S_0}(1, \pi, \mathrm{Ad})}} \sum_{\gamma \in GL(2, F)} W_\alpha\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Then (ϕ_α) is indeed an orthonormal basis of $\mathscr{V}^{S_0}$.

We introduce the matrix

$$w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and a Bessel distribution on $GL(3, F_{S_0})$. We let $\mathscr{W}'$ be the linear form on $\mathscr{V}(\pi)$ defined by:

$$\mathscr{W}'(\phi') = \int_{N(F)\backslash N(F_{\mathbb{A}})} \phi'(n)\, \theta^{-1}(n)\, dn.$$

We define the *global Bessel distribution* attached to π as follows. If f' is a smooth function of compact support on $GL(3, F_{S_0})$ we set

$$\mathscr{B}_\pi(f') := \sum_\alpha \mathscr{W}'(\pi(f')\, \phi_\alpha)\, \overline{\mathscr{W}'(\pi(w)\, \phi_\alpha)}. \tag{2}$$

Remark. To make the above sum finite, we have to assume that $f = \prod_{v_0 \in S_0} f_{v_0}$ where f_{v_0} is K_{v_0}-finite if v_0 is Archimedean. However with a little more effort, one can show that there is distribution whose value on a function of this type is given by the above expression.

On the other hand, for every place $v_0 \in S_0$ we denote by $\mathscr{W}_{v_0}$ the local Whittaker linear form (evaluation at e) and introduce the *local Bessel distribution* $\mathscr{B}_{\pi_{v0}}$ or simply $\mathscr{B}_{v_0}$:

$$\mathscr{B}_{v_0}(f'_{v_0}) := \sum_{\alpha_{v0}} \mathscr{W}_{v_0}(\pi_{v_0}(f')\, W_{\alpha_{v0}})\, \overline{\mathscr{W}_{v_0}(\pi_{v_0}(w)\, W_{\alpha_{v0}})}. \tag{3}$$

Then if $f' = \prod_{v_0 \in S_0} f'_{v_0}$ we have

$$\mathscr{B}_\pi(f') = c(\pi) \prod_{v_0 \in S_0} \mathscr{B}_{v_0}(f'_{v_0}),$$

where we have set

$$c(\pi) := \frac{1}{c(F, S_0)\, L^{S_0}(1, \pi, \mathrm{Ad})}. \tag{4}$$

Similar results are true for a cuspidal automorphic representation Π of $GL(3, E_{\mathbb{A}})$ and a pure tensor $\phi \in \mathscr{V}^S$:

$$\|\phi\|^2 = c(E, S)\, L^{S_0}(1, \pi, \mathrm{Ad}) \prod_{v \in S} \|W_v\|^2 \tag{5}$$

with

$$c(\Pi) := \frac{1}{c(E, S)\, L^{S_0}(1, \Pi, \mathrm{Ad})}. \tag{6}$$

3. MATCHING OF ORBITAL INTEGRALS

The main theorem will a consequence of the relative trace formula of [JY]. We recall the geometric side of the trace formula in question. We fix an idele class character ω of F.

Let F^+ be the set of elements of $F^\times$ which are norm of an element of $E^\times$ and, for each inert place v_0 of F, let $F^+_{v_0}$ be the set of $x \in F^\times_{v_0}$ which are a norm of an element of $E^\times_v$. We set $F^\times_{S_0} = \prod_{v_0 \in S_i} F^+_{v_0} \prod_{v_0 \in S_s} F^\times_{v_0}$. Similarly, let $F^+_{\mathbb{A}}$ be the set of ideles x such that $x_{v_0} \in F^+_{v_0}$ for every inert place v_0. Then $F^+ = F^\times \cap F^+_{\mathbb{A}}$.

Let $\mathfrak{S}$ be the variety of Hermitian matrices in $GL(3, E)$ and $\mathfrak{S}^+(F)$ the set of elements of $\mathfrak{S}(F)$ whose determinant is in F^+. For each inert place v_0, let $\mathfrak{S}^+_{v_0}$ be the set of $s \in \mathfrak{S}_{v_0}$ whose determinant is in $F^+_{v_0}$. Note that $GL(3, \mathcal{O}_v) \cap \mathfrak{S}_{v_0}$ is contained in $\mathfrak{S}^+_{v_0}$ if v_0 is odd, inert, and unramified. Let $\mathfrak{S}^+_{S_0}$ be the set of $s \in \mathfrak{S}_{S_0}$ such that $\det s \in F^+_{S_0}$. Let also $\mathfrak{S}^+_{\mathbb{A}}$ be the set of elements of $\mathfrak{S}(F_{\mathbb{A}})$ whose determinant is in $F^+_{\mathbb{A}}$. We define a distribution

$J(\bullet)$ on $\mathfrak{S}^+_{\mathbb{A}}$ as follows. If Φ is a smooth function of compact support on $\mathfrak{S}^+_{\mathbb{A}}$ we set:

$$J(\Phi) := \int_{N(E_{\mathbb{A}})/N(E)} \int_{F^+_{\mathbb{A}}/F^+} \left(\sum_{\xi \in \mathfrak{S}^+(F)} \Phi(n\xi {}^t\bar{n} z) \right) \omega(z)\, dz \theta(n\bar{n})\, dn. \tag{7}$$

On the other hand, let $G^+(F)$ be the group of $g \in GL(3, F)$ such that $\det g \in F^+$. Define similarly, for any inert place v_0, the group $G^+_{v_0}$ of $g_{v_0} \in GL(3, F_{v_0})$ with $\det g_{v_0} \in F^+_{v_0}$. Finally, let $G^+(F_{\mathbb{A}})$ be the set of $g \in GL(3, F_{\mathbb{A}})$ such that $\det g$ is in $F^+_{\mathbb{A}}$. We define a distribution $J'(\bullet)$ on $G^+(F_{\mathbb{A}})$:

$$J'(f') := \int_{(N(F_{\mathbb{A}})/N(F))^2} \int_{F^+_{\mathbb{A}}/F^+} \left(\sum_{\xi \in \mathfrak{S}^+(F)} f'(n_1 \xi z {}^t n_2) \right) \times \omega(z)\, dz \theta(n_1 n_2)\, dn_1\, dn_2. \tag{8}$$

We have a notion of *matching orbital integrals*; if Φ and f' have matching orbital integrals, then:

$$J(\Phi) = J'(f'). \tag{9}$$

In a precise way, we assume that Φ and f' are products of local functions which themselves have *matching orbital integrals*. This means the following. If v_0 is an inert place, we say that Φ_{v_0} and f'_{v_0} have matching orbital integrals, if, for any diagonal matrix a whose determinant is a norm:

$$\int_{N(E_v)} \Phi_{v_0}(na {}^t\bar{n})\, \theta_{v_0}(n\bar{n})\, dn = \gamma(a, \psi_{v_0}) \int_{N(F_{v_0}) \times N(F_{v_0})} f'_{v_0}(n_1 a {}^t n_2)\, \theta_{v_0}(n_1 n_2)\, dn_1\, dn_2. \tag{10}$$

Here the *transfer factor* $\gamma(a, \psi_{v_0})$ is defined by:

$$\gamma(\mathrm{diag}(a_1, a_2, a_3), \psi) = \omega_{E/F}(a_2).$$

One can show that, this relation implies, in turn, that there are similar relations between the other relevant orbital integrals [J3]. In an earlier paper [JY4], we have shown that given Φ_{v_0} there is a function f'_{v_0} satisfying the above conditions and conversely. For instance, if v_0 is odd, E_v/F_{v_0} is unramified and the character ψ_{v_0} has for conductor the ring of integers, then, if Φ_{v_0} is the characteristic function of $K_v \cap \mathfrak{S}_{v_0}$ we may take for f'_{v_0} the characteristic function of K_{v_0}.

If f_v is a smooth function of compact support on $GL(3, E_v)$ and Φ_{v_0} is the function on $\mathfrak{S}_{v_0}^+$ defined by

$$\Phi_{v_0}(g_v^t \bar{g}_v) = \int f_v(g_v h_{v_0})\, dh_{v_0},$$

we say that f_v and f'_{v_0} have matching orbital integrals provided Φ_{v_0} and f'_{v_0} do. If f_v is a Hecke function and f'_{v_0} is its image by the base change homomorphism, then f'_{v_0} is supported on $G_{v_0}^+$ and f_v and f'_{v_0} have matching orbital integrals.

If v_0 splits into v_1, v_2 then we may identify $\mathfrak{S}_{v_0}$ to the set of pairs $(s, {}^ts)$ with $s \in GL(3, F_{v_0})$ and H_{v_0} to the set of pairs $(h, {}^th^{-1})$ with $h \in GL(3, F_{v_0})$. Thus we may identify $GL(3, F_{v_0})$, $\mathfrak{S}_{v_0}$ and H_{v_0}. Then we take the condition of matching orbital integrals to be:

$$f'_{v_0}(g) = \Phi_{v_0}(g). \tag{11}$$

We say that f_{v_0} and $f_{v_1} \times f_{v_2}$ have matching orbital integrals if:

$$f'_{v_0}(g) = \Phi_{v_0}(g) = \int f_{v_0}(gh)\, f_{v_2}({}^th^{-1})\, dh.$$

For Hecke functions this means that f_{v_0} is the convolution of f_{v_1} and f_{v_2}.

Identity (9) follows readily from the condition of matching.

To the functions f and f' we attach kernels in the usual way:

$$K_f(x, y) := \int_{E_{\mathbb{A}}^\times / E^\times} \sum_{\xi \in GL(3, E)} f(x^{-1}\xi z y)\, \Omega(z)\, dz,$$

$$K_{f'}(x, y) := \int_{F_{\mathbb{A}}^+ / F^+} \sum_{\xi \in G^+(F)} f'(x^{-1}\xi z y)\, \Omega(z)\, dz.$$

Then

$$J(\Phi) = \int_{N(E)\backslash N(E_{\mathbb{A}}) \times H(F)\backslash H(F_{\mathbb{A}})} K_f(n, h)\, \theta^{-1}(n\bar{n})\, dn\, dh,$$

$$J'(f') = \int_{N(F)\backslash N(F_{\mathbb{A}}) \times N(F)\backslash N(F_{\mathbb{A}})} K_{f'}(n_1, {}^tn_2)\, \theta^{-1}(n_1)\, \theta(n_2)\, dn_2.$$

From identity (9) follows the equality of the integral of the two kernels. In turn, this implies the equality of the integrals of the corresponding spectral kernels and, finally, the equality of the integrals of the kernels attached to a cuspidal representation π and its base change.

Now let Ω be the base change of ω, that is, $\Omega(z)=\omega(z\sigma(z))$. Let π a cuspidal representation of $GL(3, F_{\mathbb{A}})$ with central character ω and let Π its base change. Let K_f^{Π} and $K_{f'}^{\pi}$ be the kernels attached to the representations Π and π respectively:

$$K_f^{\Pi}(x, y)=\sum \Pi(f)\,\phi_i(x)\,\overline{\phi_i(y)},$$

$$K_{f'}^{\pi}(x, y)=\sum \pi(f')\,\phi_i'(x)\,\overline{\phi_i'(y)},$$

where the sums are over orthonormal bases of $\mathscr{V}(\Pi)$ and $\mathscr{V}(\pi)$ respectively.

Then, if f and f' have matching orbital integrals,

$$\int\int K_f^{\Pi}(nh)\,\theta^{-1}(n\bar{n})\,dn\,dh=\int\int K_{f'}^{\Pi}(n_1, {}^t n_2)\,\theta^{-1}(n_1)\,\theta(n_2)\,dn_1\,dn_2. \qquad (12)$$

Recall that on the space of $\mathscr{V}(\Pi)$ we have introduced the following linear forms:

$$\mathscr{W}(\phi)=\int \phi(n)\,\theta^{-1}(n\bar{n})\,dn, \qquad \mathscr{P}(\phi)=\int \phi(h)\,dh.$$

Thus the integral of K_f^{Π} can be written as

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}(\phi_i)}.$$

On the other hand:

$$\begin{aligned}\int_{N(F_{\mathbb{A}})/N(F)} \overline{\phi'}({}^t n')\,\theta(n')\,dn' &= \overline{\int_{N(F_{\mathbb{A}})/N(F)} \phi'({}^t n')\,\theta^{-1}(n')\,dn'}\\ &= \overline{\int_{N(F_{\mathbb{A}})/N(F)} \phi'(w{}^t n' ww)\,\theta^{-1}(n')\,dn'}\\ &= \overline{\int_{N(F)\backslash N(F_{\mathbb{A}})} \phi'(n'w)\,\theta^{-1}(n')\,dn'}\\ &= \overline{\mathscr{W}'(\pi(w')\,\phi')}.\end{aligned}$$

Thus the integral of $K_{f'}^{\pi}$ can also be written:

$$\sum_{\phi_i'} \mathscr{W}'(\pi(f')\,\phi_i')\,\overline{\mathscr{W}'(\pi(w)\,\phi_i')}.$$

We arrive at the identity:

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}(\phi_i)} = \sum_{\phi'_i} \mathscr{W}'(\pi(f')\,\phi'_i)\,\overline{\mathscr{W}'(\pi(w)\,\phi^i_i)}, \tag{13}$$

whenever f and f' have matching orbital integrals.

Remark. If v_0 is an Archimedean place of F, by hypothesis, the place v_0 splits into v_1 and v_2. We assume that the functions f_{v_1} and f_{v_2} are in fact $K_{v_1} \simeq K_{v_2} \simeq K_{v_0}$ finite so as to have only finite sums in the above identity. In fact, both sides may be viewed as distributions and then the identity is true without restriction at the infinite places.

4. PROOF OF THE MAIN THEOREM

We now prove the theorem stated in the first section. Thus we let Π be a distinguished cuspidal representation of $GL(3, E_{\mathbb{A}})$ with central character Ω. It is thus the base change of a *unique* cuspidal representation π of $GL(3, F_{\mathbb{A}})$ with central character ω. We let S_0 and S be as before. If f is a smooth function of compact support on G_S and K_S-finite, we set:

$$\mathscr{R}_\Pi(f) = \sum_{\phi_i} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}(\phi_i)},$$

the sum over an orthonormal basis (ϕ_i) of $\mathscr{V}^S$. The sum does not depend on the choice of the orthonormal basis. We think of this linear form as being the *relative Bessel distribution* attached to Π.

Recall

$$G^+_{S_0} := \prod_{v_0 \in S_s} G_{v_0} \prod_{v_0 \in S_i} G^+_{v_0}.$$

We have defined the global Bessel distribution attached to π. We can compute its value on a function f' smooth and of compact support, on the group $G^+_{S_0}$:

$$\mathscr{B}_\pi(f') = \sum_{\phi'_i} \mathscr{W}'(\pi(f')\,\phi'_i)\,\overline{\mathscr{W}'(\pi(w)\,\phi'_i)},$$

where the sum is over an orthonormal basis of $\mathscr{V}^{S_0}(\pi)$. Actually, as before, to make the sum finite we have to assume that $f' = \prod_{v_0 \in S_0} f'_{v_0}$ where f_{v_0} is K_{v_0}-finite for v_0 infinite. We think of $\mathscr{B}_\pi$ as the *Bessel distribution* attached to π.

If f and f' have matching orbital integrals then it follows from the previous section that:

$$\mathscr{R}_{\Pi}(f) = \mathscr{B}_{\pi}(f'). \tag{14}$$

For $v_0 \in S_i$ we define a distribution $\mathscr{B}_v$ on $GL(3, E_v)$ as follows: given a smooth function of compact support f_v, we choose a function f'_{v_0} with matching orbital integrals and we set:

$$\mathscr{R}_v(f_v) = \mathscr{B}_{\pi_{v0}}(f'_{v_0}).$$

We must check that the right hand side is independent of the choice of f'_{v_0}. But if f''_{v_0} is another choice then f'_{v_0} and f''_{v_0} have the same orbital integrals, or, what amounts to the same, all the orbital integrals of the difference vanish. However, the orbital integrals are weakly dense in the space of distributions on $GL(3, F_{\mathbb{A}})$ which transform on the left and on the right under the character θ of $N(F_{v_0})$ ([GK], principle of localization). Thus the distribution $\mathscr{B}_{\pi_{v0}}$ takes the same value on both functions, and the distribution $\mathscr{R}_v$ is well defined.

At a place $v_0 \in S_s$ we set

$$\mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2}) = \mathscr{B}_{v_0}(f'_{v_0}),$$

where f'_{v_0} have matching orbital integrals with $f_{v_1} \otimes f_{v_2}$, that is, $f'_{v_0} = f_{v_1} * {}^t f_{v_2}$.

Recall the decomposition

$$\mathscr{B}_{\pi}(f') = c(\pi) \prod_{v \in S_0} \mathscr{B}_{v_0}(f'_{v_0}).$$

It follows that we can write:

$$\mathscr{R}_{\Pi}(f) = c(\pi) \prod_{v_0 \in S_i} \mathscr{R}_{v_0}(f_v) \prod_{v_0 \in S_s} \mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2}). \tag{15}$$

The theorem will follow from (15) and a careful analysis of the distributions $\mathscr{R}_{v_0}$ (See the next two lemmas.).

LEMMA 1. *For every $v_0 \in S_i$ there is a unique element $\mathscr{P}_{v_0}$ of $\mathscr{H}_{v_0}$ such that*

$$\mathscr{R}_{v_0}(f_v) = \sum_{u_i} \mathscr{W}_v(\Pi_v(f_v)\, u_i)\, \overline{\mathscr{P}_{v_0}(u_i)},$$

where the sum is over an orthonormal basis of $\mathscr{V}(\Pi_v)$.

Proof of the lemma. We first prove the uniqueness. Suppose $\mathscr{P}_0$ is any linear form such that

$$\sum_{u_i} \mathscr{W}_v(\Pi_v(f_v)\, u_i)\, \overline{\mathscr{P}_0(u_i)} = 0,$$

for any function f_v. Our task is to show that, for any vector u_0, $\mathscr{P}_0(u_0) = 0$. We may as well assume that u_0 is a unit vector and even a member of the orthonormal basis (u_i). We then choose a vector u' such that $\mathscr{W}(u') \neq 0$ and a function f_v such that $\Pi_v(f_v)\, u_0 = u'$ and $\Pi_v(f_v)\, u_i = 0$ for $i \neq 0$. We then obtain our conclusion by applying the hypothesis to f_v.

To prove the existence, we fix a place $w_0 \in S_i$ and let w be the corresponding place of E. Let Π^w be the restricted tensor product of the unitary representations Π_v with $v \neq w$. We fix a unitary intertwining operator $A\colon \Pi^w \otimes \Pi_w \to \Pi$. In a precise way, the space of smooth vectors of Π^w can be identified with the space $\mathscr{V}^w$ spanned by the functions of the form $W^w(g^w) = \prod_{v \neq w} W_v(g_v)$ on the group G^w, the restricted product of the groups G_v with $v \neq w$, where W_v is in $\mathscr{W}(\Pi_v)$ and $W_v = W^{K_v}$ for $v \notin S$. Then A has the form:

$$A(W^w \otimes W_w) = d\phi(g),$$

$$\phi(g) = \sum_{\gamma \in N(2, E)\backslash GL(2, E)} W\left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right],$$

$$W(g) = W^w(g^w)\, W_w(g_w).$$

The constant d is chosen to make the map a unitary operator. Let (u_α) be an orthonormal basis of $\mathscr{W}(\Pi_w)$ and (m^β) be an orthonormal basis of $\mathscr{V}^w$. Then $A(m^\beta \otimes u_\alpha)$ is an orthonormal basis of Π. If we set $\mathscr{W}^w(W^w(g)) = dW^w(e)$ then

$$\mathscr{W}(A(m \otimes u)) = \mathscr{W}^w(m)\, \mathscr{W}_w(u).$$

For every vector m in $\mathscr{V}(\Pi^w)$, the linear form

$$u \mapsto \mathscr{P}(A(m \otimes u))$$

is invariant under $H_0(F_{w_0})$ thus belongs to $\mathscr{H}_{w_0}$. Thus there is a linear map $A_H\colon \mathscr{V}^w \to \mathscr{H}_{w_0}$ such that

$$\mathscr{P}(A(m \otimes u)) = A_H(m)(u).$$

The global distribution $\mathscr{R}_\Pi$ can be written

$$\mathscr{R}_\Pi(f) = \sum_\beta \sum_\alpha \mathscr{W}^w(\Pi^w(f^w)\, m^\beta)\, \mathscr{W}_w(\Pi_w(f_w)\, u_\alpha)\, \overline{A_H(m^\beta)(u_\alpha)}.$$

Let $\mathscr{P}_\kappa$ be a basis for the space $\mathscr{H}_{w_0}$. Note that, at this point, we do not know that the space is finite dimensional in general. We can write

$$A_H(m) = \sum_\kappa A_\kappa(m)\, \mathscr{P}_\kappa,$$

where the A_κ are suitable linear forms on $\mathscr{V}^w$. Note that for a given m, $A_\kappa(m) = 0$ for all but finitely many κ's. We get:

$$\mathscr{R}_\Pi(f) = \sum_\kappa \left(\sum_\beta \mathscr{W}^w(\Pi^w(f^w)\, m^\beta)\, A_\kappa(m^\beta) \right) \mathscr{R}_\kappa(f_v)$$

where we have set:

$$\mathscr{R}_\kappa(f_v) = \sum_\alpha \mathscr{W}_v(\Pi_v(f_v)\, u_\alpha)\, \overline{\mathscr{P}_\kappa(u_\alpha)}.$$

Now for each $v_0 \in S_i$, $v_0 \neq w_0$ we can choose a function f_v such that $\mathscr{R}_{v_0}(f_v) \neq 0$ and, for $v_0 \in S_s$, functions f_{v_1}, f_{v_2} such that $\mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2}) \neq 0$. Thus the distribution $\mathscr{R}_{w_0}$ is a linear combination of the distributions $\mathscr{R}_\kappa$:

$$\mathscr{R}_{w_0} = \sum_\kappa c_\kappa \mathscr{R}_\kappa.$$

Now we set

$$\mathscr{P}_{w_0} = \sum_\kappa c_\kappa \mathscr{R}_\kappa,$$

and then $\mathscr{R}_{w_0}$ has the required form. ∎

We need an analog of the previous lemma for a place $v_0 \in S_s$. It is in fact formal. We define an element $\mathscr{P}_{v_0} \in \mathscr{H}_{v_0}$ as follows: we may identify the representations π_{v_1}, π_{v_2}, π_{v_0}; their common space is the space $\mathscr{W}(\pi_{v_0})$. Define an antilinear map A from that space to itself by:

$$AW(g) = \overline{W(w\,{}^tg^{-1})}$$

Then $A\pi_{v_0}(g) = \pi_{v_0}({}^tg^{-1})\, A$ and $A^2 = 1$. Now consider the unitary representation attached to π_{v_0} and let $\mathscr{H}$ be its Hilbert space. Of course, we use the same notation for the unitary representation attached to π_{v_0}. We regard the representation $\pi_{v_0}({}^tg^{-1})$ has a representation on the conjugate Hilbert space. This new representation and the representation π_{v_0} have the same character thus are equivalent. It follows that there is a unitary antilinear operator U such that $U\pi_{v_0}(g) = \pi_{v_0}({}^tg^{-1})\, U$. Then U^2 is a unitary operator which commutes to π_{v_0}. It is therefore a scalar μ with

$|\mu|=1$. Dividing U by the square root of μ we may as well assume $U^2=1$. We have then (on the space of smooth vectors) $A=\lambda U$ with $\lambda\in\mathbb{C}^\times$. Since $A^2=1$ we get $\lambda=\pm 1$ and so A preserves the norm. This being so, for u_1, $u_2\in\mathscr{W}(\pi_{v_0})$, we set

$$\mathscr{P}_{v_0}(u_1\otimes u_2)=(u_1, Au_2).$$

Then

$$\mathscr{P}_{v_0}(\pi_{v_0}(g)\,u_1\otimes u_2)=\mathscr{P}_{v_0}(u_1\otimes\pi_{v_0}({}^t g)\,u_2).$$

We then define a distribution

$$\mathscr{R}'_{v_0}(f_{v_1}\otimes f_{v_2}):=\sum_{\alpha,\,\beta}\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_1})\,u_\alpha)\,\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_2})\,m^\beta)\,\overline{\mathscr{P}_{v_0}(u_\alpha\otimes m^\beta)}.$$

The sum is over orthonormal bases (u_α) and (m^β) of π_{v_0}.

LEMMA 2. *In fact*:

$$\mathscr{R}'_{v_0}=\mathscr{R}_{v_0}$$

Proof of the Lemma. Indeed, let us take $m^\beta=A(u_\beta)$. Then

$$\mathscr{P}_{v_0}(u_\alpha\otimes m^\beta)=\langle u_\alpha, A^2u_\beta\rangle=\langle u_\alpha, u_\beta\rangle=\delta_{\alpha,\,\beta}.$$

Then:

$$\begin{aligned}\mathscr{R}'_{v_0}(f_{v_1}\otimes f_{v_2})&=\sum_\alpha\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_1})\,u_\alpha)\,\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_2})\,Au_\alpha)\\&=\sum_\alpha\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_1})\,u_\alpha)\,\mathscr{W}_{v_0}(A\pi_{v_0}({}^t f^*_{v_2})\,u_\alpha)\\&=\sum_\alpha\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_1})\,\pi_{v_0}({}^t f_{v_2})\,u_\alpha)\,\mathscr{W}_{v_0}(Au_\alpha).\end{aligned}$$

But $\mathscr{W}_{w_0}(Au)=\overline{\mathscr{W}}_{w_0}(\pi_{v_0}(w)\,u)$. Thus we get at last:

$$\mathscr{R}'_{v_0}(f_{v_1}\otimes f_{v_2})=\sum_\alpha\mathscr{W}_{v_0}(\pi_{v_0}(f_{v_1}*{}^t f_{v_2})\,u_\alpha)\,\overline{\mathscr{W}_{v_0}}(\pi_{v_0}(w)\,u_\alpha),$$

which is indeed $\mathscr{R}_{v_0}(f_{v_1}\otimes f_{v_2})$. ∎

Now we prove the theorem. Let $\tilde{\mathscr{P}}$ be the linear form on $\mathscr{V}^S(\Pi)$ defined by

$$\tilde{\mathscr{P}}(\phi)=\prod_{v_0\in S_i}\mathscr{P}_{v_0}(W_v)\prod_{v_0\in S_s}\mathscr{P}_{v_0}(W_{v_1}\otimes W_{v_2})$$

when ϕ is a pure tensor. Taking into account the computation of the norm of a pure tensor, we get:

$$\sum_{\phi} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\tilde{\mathscr{P}}(\phi_i)} = c(\Pi) \prod_{v_0 \in S_i} \mathscr{R}_{v_0}(f_v) \prod_{v_0 \in S_s} \mathscr{R}_{v_0}(f_{v_1} \otimes f_{v_2})$$
$$= \frac{c(\Pi)}{c(\pi)} \mathscr{R}_{\Pi}(f)$$
$$= \frac{c(\Pi)}{c(\pi)} \sum_{\phi} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}(\phi_i)}.$$

Since Π is irreducible (See the proof of uniqueness in Lemma (1)) we get

$$\mathscr{P} = \frac{c(\pi)}{c(\Pi)} \tilde{\mathscr{P}} \tag{16}$$

and the theorem follows.

5. LOCAL RESULTS: SUPERCUSPIDAL CASE

Now we consider a local quadratic extension E/F of non-Archimedean local fields. We only consider irreducible unitary generic representations of $GL(3, E)$. We say that such a representation Π is *distinguished* if the space $\mathscr{H}(\Pi, H)$ of H-invariant linear forms is non-zero. Then the central character Ω of Π is itself distinguished, that is, trivial on $\mathbb{U}_1$. We then fix a character ω of $F^{\times}$ such that $\Omega(z) = \omega(z\bar{z})$.

THEOREM 2. *Suppose Π is supercuspidal. Then Π is distinguished if and only $\Pi^{\sigma} = \Pi$. The dimension of $\mathscr{H}(\Pi, H)$ is then one. Let Ω and ω as above. Then Π is the base change of a unique cuspidal representation π of $GL(3, F)$ with central character ω and there exists a unique element $\mathscr{P}_{\pi}$ of $\mathscr{H}(\Pi, H)$ such that*

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}_{\pi}(\phi_i)} = \mathscr{B}_{\pi}(f'), \tag{17}$$

each time f and f' have matching orbital integrals.

We first prove a result of density.

LEMMA 3. *If Π is supercuspidal and distinguished, let $\mathscr{H}^c_{\Pi}$ be the space spanned by the linear forms $\mathscr{P}$ defined by:*

$$\mathscr{P}(u) = \int_{H(F)} (\Pi(h)\,u, \tilde{u})\,dh,$$

where $\tilde{u}$ *is a* (*smooth*) *vector. Then for any* $\mathscr{P}_1 \in \mathscr{H}(\Pi, H)$, *the space* $\operatorname{Ker}(\mathscr{P}_1)$ *contains the intersection* $\bigcap_{\mathscr{P} \in \mathscr{H}^c_\Pi} \operatorname{Ker}(\mathscr{P})$.

Proof. We use Bernstein theory to write the space $\mathscr{C}$ of smooth functions on $GL(3, E_v)$ transforming under the character Ω and compactly supported modulo the center, as a direct sum of bi-invariant subspaces

$$\mathscr{C} = \mathscr{C}(\Pi) \oplus \mathscr{C}^{\Pi},$$

where $\mathscr{C}(\Pi)$ denotes the space spanned by the matrix coefficients of Π_v, that is, the functions of the form

$$g \mapsto (\Pi(g)\, u, \tilde{u}).$$

If we choose a linear basis (u_i) of the space $\mathscr{V}(\Pi)$ we can decompose further $\mathscr{C}(\Pi)$ as direct sum of the invariant spaces $\mathscr{C}_i$,

$$\mathscr{C}(\Pi) = \bigoplus \mathscr{C}_i,$$

where $\mathscr{C}_i$ is the space spanned by the functions of the form: $g \mapsto (\Pi(g)\, u, u_i)$. Choosing an index i_0, we may identify the space $\mathscr{C}_{i_0}$ to the space $\mathscr{V}$ and view $\mathscr{P}_1$ as a linear form on $\mathscr{C}_{i_0}$. We may extend $\mathscr{P}_1$ to $\mathscr{C}$ by demanding that it be zero on $\mathscr{C}_i$ with $i \neq i_0$, and then zero on $\mathscr{C}^{\Pi}$. Thus we now view the given linear form $\mathscr{P}_1$ as a distribution invariant on the right under $H(F)$. Its value on the function $f(g) = \langle \Pi(g)\, u, u_{i_0} \rangle$ is equal to $\mathscr{P}_1(u)$. Since u_{i_0} is invariant under a compact open subgroup K' this distribution is invariant on the left under K'. It follows that there exists a distribution μ on $G(E)/H(F)$ such that, for any $f \in \mathscr{C}$,

$$\mathscr{P}_1(f) = \int \left(\int f(xh)\, dh \right) d\mu(x).$$

Moreover, this distribution is invariant on the left under K'. Thus, if (x_j) is a set of representatives for the double cosets $Z(E)\, K' \backslash G(E)/H(F)$ we have, for suitable constants λ_j,

$$\mathscr{P}_1(f) = \sum_j \lambda_j \int \int f(k' x_j h)\, dh\, dk'.$$

For a given function f, there are only finitely many non-zero terms on the right. Coming back to the original linear form $\mathscr{P}_1$, we see that

$$\mathscr{P}_1(u) = \sum_j \lambda_j \int (\Pi_v(x_j h)\, u, u_{i_0})\, dh,$$

the sum on the right having only finitely many non-zero terms. If we set $u_j = \Pi(x_j^{-1})(u_{i_0})$ we get finally:

$$\mathscr{P}_1(u) = \sum_j \lambda_j \int (\Pi(h)\, u, u_j)\, dh.$$

The lemma follows. ∎

To finish the proof we write our local extension in the form E_v/F_{v_0} where E/F is a quadratic extension of number fields and v_0 a place of F inert in E, v the corresponding place of E. We assume that all the infinite places of F split in E. We write the given supercuspidal representation as Π_v.

Suppose first that $\Pi_v^\sigma = \Pi_v$. Then Π_v is the base change of a supercuspidal representation π_{v_0}. In turn we may write π_{v_0} as the local component of a cuspidal automorphic representation π. Let Π be the base change of π. Then Π_v is the local component of Π at the place v. Since Π is globally distinguished, it follows that Π_v is (locally) distinguished.

Now we suppose that Π_v is distinguished. Then its central character Ω_v is distinguished and we write, as before, $\Omega_v(z) = \omega_{v_0}(z\bar{z})$. We may further choose an idele class character ω of F whose component at v_0 is ω_{v_0}. We then set $\Omega(z) = \omega(z\bar{z})$. We first show that Π_v is the local component at v of a distinguished cuspidal automorphic representation Π with central character Ω. Since $\Pi^\sigma = \Pi$ then, it will follow that, as claimed, $\Pi_v^\sigma = \Pi_v$. From the previous density lemma, it follows there is a smooth vector u_1 in the space of Π_v such that the linear form $\mathscr{P}_1$ defined by

$$\mathscr{P}_1(u) = \int_{H_{v0}} (\Pi_v(h)\, u, u_1)\, dh$$

is non-zero. For any smooth vector u in the space of Π_v we set:

$$f_v^u(g_v) = (\Pi_v(g_v)\, u, u_1).$$

On the other hand, we let f^v be a smooth function on the group G^v, transforming under the character Ω^v of Z^v, and compactly supported, modulo the center. We set $f(g) = f^v(g^v)\, f_v^u(g_v)$. We define a function ϕ^{u, f^v} on $GL(3, E_{\mathbb{A}})$ as follows:

$$\phi^{u, f^v}(g) = \sum_{\gamma \in Z(E)\backslash G(E)} f(\gamma g).$$

The resulting function is invariant under $G(E)$ on the left, compactly supported modulo $Z(E_{\mathbb{A}})\, G(E)$ and cuspidal. Let (Π_α) be the family of cuspidal representations with central character Ω, and for each α, let $\mathscr{V}_\alpha$ be the space of smooth vectors of Π_α. Let ϕ_α^{u, f^v} be the orthogonal projection of ϕ^{u, f^v} on $\mathscr{V}_\alpha$. Then

$$\phi^{u, f^v}(g) = \sum_\alpha \phi_\alpha^{u, f^v}(g).$$

The series converges in the space of rapidly decreasing functions on the quotient $G(E)\, Z(E_{\mathbb{A}})\backslash G(E_{\mathbb{A}})$. Thus we may write:

$$\int \phi^{u, f^v}(h)\, dh = \sum_\alpha \int \phi_\alpha^{u, f^v}(h)\, dh.$$

Moreover, Schur orthogonality relations imply the existence of a constant $d > 0$ such that

$$\int_{Z_v\backslash G_v} (\Pi_v(g_0 g_v)\, u, u_0)(u_1, \Pi_v(g_v)\, u)\, dg_v = d(u, u)(\Pi_v(g_0)\, u_1, u_0).$$

This implies that

$$\int_{Z_v\backslash G_v} \phi^{u, f^v}(g g_v)(u_1, \Pi_v(g_v)\, u)\, dg_v = d(u, u)\, \phi^{u_1, f^v}(g).$$

Thus, for each α

$$\int_{Z_v\backslash G_v} \phi_\alpha^{u, f^v}(g g_v)(u_1, \Pi_v(g_v)\, u)\, dg_v = d(u, u)\, \phi_\alpha^{u_1, f^v}(g).$$

It follows that if the projection ϕ_α^{u, f^v} is not zero (for some choice of u and f^v) then the representation Π_α has the form $\Pi_v \otimes \Pi_\alpha^v$. We claim further that at least one of the representations Π_α is distinguished. Indeed, suppose not. Thus, for all α,

$$\int \phi_\alpha^{u, f^v}(h)\, dh = 0.$$

It follows that for any function f^v

$$\int \phi^{u, f^v}(h)\, dh = 0.$$

Explicitly:

$$\sum_{\gamma \in G(E)/H(F)\, Z(E)} \int_{H_{v_0}} (\Pi_v(\gamma h_{v_0})\, u, u_0)\, dh_{v_0} \int_{H^{v_0}} f^v(\gamma h^{v_0})\, dh^{v_0} = 0.$$

Now let Φ_{v_0} be the function on $\mathfrak{S}_{v_0}^+$ defined by

$$\Phi_{v_0}(g^t\bar{g}) = \int_{H_{v0}} (\Pi_v(gh_{v_0})\, u, u_0)\, dh_{v_0}.$$

Similarly, let Φ^{v_0} be the function on $\mathfrak{S}^{v_0,+}$ defined by:

$$\Phi^{v_0}(g^t\bar{g}) = \int_{H^{v_0}} f^v(gh^{v_0})\, dh^{v_0}.$$

Finally let Φ be the product of Φ_{v_0} and Φ^{v_0}. Thus $\Phi(sz) = \Phi(s)\,\omega(z)$ for $z \in F_{\mathbb{A}}^+$. Moreover the support of Φ is contained in a set of the form $MF_{\mathbb{A}}^+$ where M is a compact set. The above relation reads:

$$\sum_{\xi \in S^+(F)/F^+} \Phi(\xi) = 0.$$

We can choose Φ^{v_0} in such a way that the above relation reduces to $\Phi_{v_0}(1) = 0$ or

$$\int_{H_{v0}} (\Pi_v(h_{v_0})\, u, u_0)\, dh_{v_0} = 0,$$

that is, $\mathcal{P}_1(u) = 0$, which is a contradiction. Thus one of the representations Π_α is distinguished. As we have remarked before, this implies that $\Pi_\alpha^\sigma = \Pi_\alpha$ and thus as claimed, $\Pi_v^\sigma = \Pi_v$.

Finally, it remains to prove that if Π_v is distinguished then the dimension of $\mathscr{H}(\Pi_v, H_v)$ is one. We have just seen that Π_v is the local component of a distinguished cuspidal representation Π, which is itself the base change of a cuspidal representation π. In particular, Π_v is the local base change of the local component π_{v_0}; in fact π_{v_0} is the unique irreducible representation with central character ω_{v_0} whose base change is Π_v. By lemma there is a unique element $\mathcal{P}_0$ of $\mathscr{H}_{\Pi_v}$ such that (17) is satisfied. By the density result, it will suffice to prove that if u is a vector in the kernel of $\mathcal{P}_0$ then $\mathcal{P}_1(u) = 0$ for every element $\mathcal{P}_1$ of $\mathscr{H}^c_{\Pi_v}$. We may assume that $\mathcal{P}_1$ has the form:

$$\mathcal{P}_1(u) = \int_{H_{v0}} (\Pi_v(h)\, u, u_1)\, dh.$$

Now if Π is *any* cuspidal automorphic representation of $GL(3, E_{\mathbb{A}})$ with local factor Π_v and ϕ is any smooth vector of Π which is a pure tensor of the form $u \otimes u^v$ then

$$\int \phi(h)\, dh = 0.$$

Indeed, this is clear if Π is not distinguished and follows from the factorization of the global period and the uniqueness of π_{v_0} otherwise. We now apply the previous construction. Each function ϕ_α^{u, f^v} corresponds to a pure tensor vector of the form $u \otimes u^v$. Then

$$\int \phi_\alpha^{u, f^v}(h)\, dh = 0.$$

As we have seen this implies that in fact $\mathscr{P}_1(u) = 0$, as claimed. Thus the theorem is completely established.

6. SUPERCUSPIDAL REPRESENTATIONS FOR $GL(2, E)$

We briefly review the case of the group $GL(2, E)$ (Cf. [HLR], [jH], [jHyF], [yF2], [P]). Let H_1 be a split-unitary group in two variables. Denote by $\tilde{H}_1$ the corresponding similitude group and by λ the similitude ratio. Thus $H_1 Z(E)$ has index two in $\tilde{H}_1$. Let h_1 be an element of $\tilde{H}_1 - H_1 Z(E)$. We say that an irreducible admissible (unitary) representation Π of $GL(2, E)$ is distinguished by H_1 if the space $\mathscr{H}(\Pi, H_1)$ of linear forms invariant under H_1 is non-zero.

Proposition 1. *Suppose that Π is supercuspidal. Then Π is distinguished by H_1 if and only if $\Pi^\sigma = \Pi$. Then* $\dim(\mathscr{H}(\Pi, H_1)) = 1$. *Moreover, let π be a supercuspidal representation whose base change is Π and let ω be the central character of π. Then in fact, for $\mathscr{P} \in \mathscr{H}(\Pi, H_1)$ and $h \in \tilde{H}_1$:*

$$\mathscr{P}(\Pi(h)\, u) = \omega\omega_{E/F}((\lambda(h))\, \mathscr{P}(u).$$

Proof of the Proposition. If Π is distinguished by H_1 then its central character Ω is distinguished by $\mathbb{U}_1$ and so has the form $\Omega(z) = \omega(z\bar{z})$ for a suitable ω. For $\mathscr{P} \in \mathscr{H}(\Pi, H_1)$ set

$$\mathscr{P}_\omega(u) = \tfrac{1}{2}(\mathscr{P}(\Pi(h_1)\, u) + \omega(\lambda(h_1))\, \mathscr{P}(u)), \tag{18}$$

$$\mathscr{P}_{\omega_{E/F}\omega}(u) = \tfrac{1}{2}(\mathscr{P}(\Pi(h_1)\, u) + \omega_{E/F}\omega(\lambda(h_1))\, \mathscr{P}(u)). \tag{19}$$

Then, for any $h \in \tilde{H}_1$ and any vector u:

$$\mathscr{P}_\omega(\Pi(h)\, u) = \omega(\lambda(h))\, \mathscr{P}(u), \tag{20}$$

$$\mathscr{P}_{\omega_{E/F}\omega}(\Pi(h)\, u) = \omega_{E/F}\omega(\lambda(h))\, \mathscr{P}(u). \tag{21}$$

We denote by $\mathscr{H}(\Pi, H_1, \omega)$ (resp. $\mathscr{H}(\Pi, H_1, \omega_{E/F}\omega)$) the space of linear forms satisfying (20) (resp. (21)).

LEMMA 4. *The dimension of the space $\mathcal{H}(\Pi, H_1, \omega)$ is at most one.*

Indeed, $\tilde{H}_1$ is conjugate to $GL(2, F)\, Z(E)$ by an element of $GL(2, E)$ and the conjugation takes the similitude ratio to $gz \mapsto \det gz\bar{z}$. Thus it suffices to prove that the space $\mathcal{H}(\Pi, GL(2, F), \omega)$ of linear forms $\mathcal{Q}$ such that

$$\mathcal{Q}(\Pi(g)\, u) = \omega(\det g)\, \mathcal{Q}(u)$$

for $g \in GL(2, F)$ has dimension at most one. We may extend ω to a character ω_1 of $E^\times$ and replace Π by $\Pi \otimes \omega_1^{-1}$. We are then reduced to proving that the space $\mathcal{H}(\Pi \otimes \omega_1^{-1}, GL(2, F))$ of linear forms invariant under $GL(2, F)$ by the representation $\Pi \otimes \omega_1^{-1}$ has dimension at most one; this is known (see [jH] and [yF2]). Moreover if that space is non-zero then (loc. cit.)

$$(\Pi \otimes \omega_1^{-1})^\sigma = \widetilde{\Pi \otimes \omega_1^{-1}} = \Pi \otimes \Omega^{-1} \omega_1^2.$$

This relation is in fact equivalent to $\Pi^\sigma = \Pi$. Thus if Π is distinguished by H_1 then it is invariant by σ.

Now we recall some global results. Let again E/F be a global quadratic extension of number fields. Let Π be a cuspidal representation. Let ω be an idele class character of F. The two following conditions are equivalent ([HLR]): (i) the representation Π is the base change of a cuspidal representation π with central character $\omega\omega_{E/F}$; (ii) the restriction of the central character Ω of Π to $F_{\mathbb{A}}^\times$ is ω^2 and there is ϕ in the space of Π such that

$$\int_{Z(F_{\mathbb{A}})\, G(F) \backslash G(F_{\mathbb{A}})} \phi(g)\, \omega^{-1}(\det g)\, dg \neq 0.$$

Note that the central character Ω verifies then $\Omega(z) = \omega(z\bar{z})$. Thus, the second condition amounts to: (iii) the central character Ω has the form $\Omega = \omega \circ \mathrm{Norm}$ and there is ϕ in the space of Π such that

$$\int_{Z(E_{\mathbb{A}})\, \tilde{H}(F) \backslash H(F_{\mathbb{A}})} \phi(g)\, \omega^{-1}(\lambda(h))\, dh \neq 0.$$

Now we go back to the local problem and again write our local extension in the form E_v/F_{v_0} and write Π_v instead of Π. If Π_v is given and $\mathcal{P}_{\omega_v} \neq 0$, then we can argue as before and find a cuspidal representation Π of which Π_v is the local component at the place v and Π satisfies (ii). Then Π is the base change of a representation π with central character $\omega\omega_{E/F}$. Thus Π_v is the base change of a representation π_{v_0} with central character

$\omega_{v_0}\omega_{E_v/F_{v0}}$. The only other representation of which Π_v is the base change is the representation $\pi_{v_0}\otimes\omega_{E_v/F_{v_0}}$ and it has the same central character. Likewise, if $\mathscr{P}_{\omega_{v0}\,\omega_{E_v}/F_{v0}}\neq 0$ then Π_v is the base change of a representation π_{v_0} with central character ω_{v_0}.

We conclude that $\mathscr{P}_{\omega_v}$ and $\mathscr{P}_{\omega_{v0}\,\omega_{E_v}/F_{v0}}$ cannot be both non-zero. This already prove that if $\mathscr{H}(\Pi_v, H_{1,v_0})\neq 0$ then it has dimension one. Moreover Π_v is then a base change of a representation π_{v_0}, $\Pi_v=\Pi_v^\sigma$ and the central character has the required properties.

If $\Pi_v^\sigma=\Pi_v$ then, as before, Π_v is the base change of a supercuspidal representation π_{v_0}. We can find a cuspidal representation of which π_{v_0} is a component. We base change π to Π and apply the previously recalled result to conclude that Π is distinguished by H_1 and Π_v by H_{1,v_0}.

Remark. We could argue as in the previous section using the trace formula described in [JY] (Cf. [yF2]).

Now let H_2 be a unitary group which is not split. Let also $\tilde{H}_2$ be the corresponding similitude group and λ the similitude ratio. We define again $\mathscr{H}(\Pi, H_2)$ as the space of linear forms on $\mathscr{V}(\Pi)$ which are invariant under H_2. Then:

PROPOSITION 2. *Suppose that Π is supercuspidal. Then Π is distinguished by H_2 if and only if $\Pi^\sigma=\Pi$. Then* $\dim(\mathscr{H}(\Pi, H_2))=1$. *Moreover, let π be a supercuspidal representation whose base change is Π and let ω be the central character of π. Then in fact, for $\mathscr{P}\in\mathscr{H}(\Pi, H_2)$ and $h\in\tilde{H}_2$:*

$$\mathscr{P}(\Pi(h)\,u)=\omega\omega_{E/F}((\lambda(h))\,\mathscr{P}(u).$$

Let $G'(F)\subset GL(2, E)$ be the multiplicative group of a quaternion algebra. It is known that the dimension of $\mathscr{H}(\Pi, G'(F))$ is at most one. Moreover, $\mathscr{H}(\Pi, G'(F))\neq 0$ if and only if $\mathscr{H}(\Pi, GL(2, F))\neq 0$ (see [jH] and [jHyF]). Arguing as before we can reduce this proposition to the previous one.

7. REPRESENTATIONS INDUCED FROM A CUSPIDAL REPRESENTATION

For the other unitary generic representations of $GL(3, E)$ we propose the following conjecture:

Conjecture 1. Suppose Π is a unitary irreducible generic representation of $GL(3, E)$. Then Π is distinguished by H if and only $\Pi^\sigma=\Pi$. Let Ω and ω as above. For each irreducible admissible representation π of $GL(3, F)$

with central character ω whose base change is Π, there exists a unique element $\mathscr{P}_\pi$ of $\mathscr{H}(\Pi, H)$ such that

$$\sum_{\phi_i} \mathscr{W}(\Pi(f)\,\phi_i)\,\overline{\mathscr{P}_\pi(\phi_i)} = \mathscr{B}_\pi(f'), \tag{22}$$

each time f and f' have matching orbital integrals. Moreover, the linear forms $\mathscr{P}_\pi$ form a basis of $\mathscr{H}(\Pi, H)$

We have established this conjecture when Π is supercuspidal. We prove it in another case. Suppose that Π is induced by a supercuspidal representation. In a precise way, let $P = MU$ be the Levi decomposition of the parabolic subgroup P of type $(2, 1)$ (upper generalized triangular matrices). Let Π_1 is a supercuspidal representation of $GL(2, E)$ and Π_2 a character of $E^\times$. Thus we may regard $\Pi_1 \times \Pi_2$ as a representation of $M(E) \simeq GL(2, E) \times E^\times$. We assume that Π is the corresponding normalized induced representation:

$$\Pi = \mathrm{Ind}(\Pi_1, \Pi_2).$$

Thus Π operates by right shifts on the space of smooth maps ϕ: $GL(3, E) \to \mathscr{V}(\Pi_1)$ such that

$$\phi(ph) = \delta_P(p)^{1/2}\, \Pi_1 \times \Pi_2(p)\, \phi(g)$$

for every $p \in P(E)$; here δ_P is the module of $P(E)$. We will content ourselves with proving the conjecture in this case. We begin with a lemma:

LEMMA 5. *The dimension of $\mathscr{H}(\Pi, H)$ is at most two. Moreover, if Π is distinguished by H then Π_1 is distinguished by a split group in two variables, Π_2 is distinguished by $\mathbb{U}_1$ and $\Pi^\sigma = \Pi$.*

Proof of Lemma. Let P_0 be the group of upper triangular matrices. We first study the orbits of $P_0(E)$ on $\mathfrak{S}(F)$; a system of representatives is given by the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix},$$

where the elements b and b_i take their values in $F^\times/F^+$. Next, a set of representatives for the orbits of $P(E)$ on $\mathfrak{S}(F)$ is given by the matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_3 \end{pmatrix},$$

with b, b_1, b_3 as before. A system of representatives for the orbits of $P(E)$ on $\mathfrak{S}^+(F)$ is then given by the following matrices:

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} -b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -b^{-1} \end{pmatrix},$$

with $b \in F^\times - F^+$. For each element σ_i of the above type, let ξ_i be such that $\xi_i {}^t\overline{\xi_i} = \sigma_i$. Let δ_{ξ_i} be the module of the group $P_{\xi_i} := P(E) \cap \xi_i H(F)\, \xi_i^{-1}$ and $V(\xi_i)$ be the space of linear forms μ on the space $\mathscr{V}(\Pi_1)$ such that

$$\delta_P^{1/2}(p)\, \mu(\Pi_1 \times \Pi_2(p)\, v) = \delta_{\xi_i}(p)\, v,$$

for every vector u and every $p \in P_{\xi_i}$. Then

$$\dim(\mathscr{H}(\Pi, H)) \leqslant \sum_i \dim(V(\xi_i)).$$

Now P_{ξ_0} contains the subgroup

$$\left\{ \begin{pmatrix} 1 & x & \dfrac{x\bar{x}}{2} \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

whose intersection with (or, more correctly, projection on) $M(E)$ contains the unipotent radical U_1 of the parabolic subgroup of type $(1, 1)$ of M. Thus for any $\mu \in V(\xi_0)$, any $u \in U_1$ and any vector v:

$$\mu(\Pi_1(u)\, v) = \mu(v)$$

Because Π_1 is supercuspidal, this implies $\mu = 0$. Thus $V(\xi_0) = 0$.

Next, consider the case of the element σ_1. Then P_{ξ_1} is the set of matrices

$$\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

with $h_1 \in H_1(F)$ and $h_2 \in \mathbb{U}_1(F)$, where H_1 is the unitary group in $GL(2, E)$ for the Hermitian matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $V(\xi_1)=0$ unless Π_2 is distinguished and then $V(\xi_1)$ may be identified with the space of linear forms μ on $\mathscr{V}(\Pi_1)$ such that

$$\mu(\Pi_1(h_1)\, v)=\mu(v)$$

for any $h_1 \in H_1$ and any vector v. Thus $\dim(V(\xi_1)) \leqslant 1$ by the previous section. Moreover if $V(\xi_1) \neq 0$ then Π_1 is distinguished by H_1 and Π_2 by $\mathbb{U}_1$. In particular, it follows then that $\Pi_1^\sigma=\Pi_1$ and $\Pi_2^\sigma=\Pi_2$, and thus $\Pi^\sigma=\Pi$ as well.

Now consider the case of the element σ_2. We let H_2 be the unitary group for the matrix

$$\begin{pmatrix} -b & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $V(\xi_2)=0$ unless Π_2 is distinguished by $\mathbb{U}_1$ and then $V(\xi_2)$ is isomorphic to the space of linear forms μ on $\mathscr{V}(\Pi_1)$ which are invariant under H_2. Thus $\dim(V(\xi_2)) \leqslant 1$ again. Furthermore, if $V(\xi_2) \neq 0$ then again Π_1 and Π_2 are invariant under σ and so is Π.

Thus we do get $\dim(\mathscr{H}(\Pi, H)) \leqslant 2$. Moreover if Π is distinguished by H then $\Pi^\sigma=\Pi$.

Now suppose that $\Pi^\sigma=\Pi$. Then $\Pi_1^\sigma=\Pi_1$ and $\Pi_2^\sigma=\Pi_2$. The representation Π_1 is the base change of two supercuspidal representations, say π_1^1 and $\pi_1^2 := \pi_{1,v}^1 \otimes \omega_{E/F}$. Let ω_1 be their common central character and set $\omega_2=\omega\omega_1^{-1}$. Thus Π is the base change of exactly two irreducible representations with central character ω, the central character of Π being $\Omega=\omega \circ \mathrm{Norm}$. The two representations in question are the induced representations

$$\pi^i=\mathrm{Ind}(\pi_1^i, \pi_2), \qquad i=1, 2.$$

It remains to show that there are two elements $\mathscr{P}_i$, $i=1, 2$, *of* $\mathscr{H}(\Pi, H)$ with the following property. Define as before distributions $\mathscr{R}_i$, $i=1, 2$, by

$$\mathscr{R}_i(f)=\mathscr{B}_{\pi^i}(f'),$$

for (f, f') with matching orbital integrals. Then

$$\mathscr{R}_i(f)=\sum \mathscr{W}_v(\Pi_v(f)\, u_\alpha)\, \overline{\mathscr{P}_i(u_\alpha)}.$$

Note that f' is an arbitrary smooth function of compact support on G^+ and the restrictions of the representations π^i to G^+ inequivalent. If follows that the distributions $\mathscr{B}_{\pi^i}$ are linearly independent. The same is true of the distributions $\mathscr{R}_i$. It follows that the linear forms $\mathscr{P}_i$ are linearly independent (see Lemma (1)).

To apply the global theory, we again write our local extension in the form E_v/F_{v_0}, write Π_v for Π and so on. Our assertion follows from the global theory if there exist two cuspidal representations π^i, $i=1, 2$, with components $\pi^i_{v_0}$ at v_0. Indeed, the corresponding base change representations Π_i are then cuspidal and distinguished and we can argue as before. Of course, this will not be the case in general.

To remedy the situation, for every $z=q_{v_0}^{-s}$ with $s\in\mathbb{C}$, we set:

$$\pi^{i,z}_{v_0}=\mathrm{Ind}(\pi^i_{1,v_0}\otimes\alpha^s_{v_0}, \alpha^{-2s}_{v_0}\pi_{2,v_0}),$$

where α_{v_0} denotes the module of F_{v_0}. We recall a standard lemma:

LEMMA 6. *Fix an index i. Let X be the set of complex numbers z of module* 1 *such that there is a cuspidal automorphic representation* π *of* $GL(3, F_{\mathbb{A}})$ *whose component at* v_0 *is* $\pi^{i,z}_{v_0}$. *The set X is infinite.*

Proof of the Lemma. For the convenience of the reader we provide a proof. For the proof of the lemma we fix the index i and drop it from the notations and consider the representation

$$\pi^z_{v_0}:=\mathrm{Ind}(\pi_{1,v_0}\otimes\alpha^s_{v_0}, \alpha^{-2s}_{v_0}\pi_{2,v_0}).$$

Let $\mathscr{C}(G_{v_0}, \omega^{-1}_{v_0})$ be the space of smooth functions transforming under the character $\omega^{-1}_{v_0}$ and compactly supported modulo the center. We recall that as a consequence of Bernstein's theory, there is a bi-invariant subspace $\mathscr{C}_0$ of the space $\mathscr{C}(G_{v_0}, \omega^{-1}_{v_0})$ with the following properties. There is a direct sum decomposition

$$\mathscr{C}(G_{v_0}, \omega^{-1}_{v_0})=\mathscr{C}_0\oplus\mathscr{C}^0$$

where $\mathscr{C}^0$ is also bi-invariant. Let $f\in\mathscr{C}_0$. For any irreducible admissible representation π_1 of $GL(3, F_{v_0})$ with central character ω_{v_0} we have $\pi_1(f)=0$ unless $\pi=\pi^z_{v_0}$ for some z. If f is given and f_0 is its projection on $\mathscr{C}_0$ in the above decomposition then $\pi^z_{v_0}(f)=\pi^z_{v_0}(f_0)$ for any z. Given $m+1$ complex numbers $z_0, z_1, z_2, ..., z_m$ one can find $f\in\mathscr{C}(G_{v_0}, \omega^{-1}_{v_0})$ such that $\pi^{z_0}_{v_0}(f)\neq 0$ but $\pi^{z_i}_{v_0}(f)=0$, $1\leqslant i\leqslant m$. This follows from the fact the representations are irreducible and inequivalent. Thus there is an element of $\mathscr{C}_0$ with the same property, namely f_0. In particular, suppose Y is a set of complex numbers of module 1 with the following property of density: for $f\in\mathscr{C}_0$ the relations $\pi^y_{v_0}(f)=0$ for all $y\in Y$ imply $f=0$. Then the set Y must be infinite. Thus it suffices to show X has precisely this property of density.

To show this is the case, consider an element $f_0\in\mathscr{C}_0$. Define a function

$$f_{v_0}(g)=\int_{G_{v0}}\overline{f_0(gx)}\,f_0(x)\,dx$$

on G_{v_0}. Choose another place v_1 and a cuspidal element f_{v_1} of $\mathscr{C}(G_{v_1},\omega_{v_1})$ with $f_{v_1}(e)\neq 0$. Choose also an element f^{v_1,v_2} of $\mathscr{C}(G^{v_0,v_1},\omega^{v_0,v_1})$ and set

$$f=f_{v_0}f_{v_1}f^{v_0,v_1},\ f'=\overline{f_0}\,f_{v_1}f^{v_0,v_1}.$$

Consider the sums

$$\phi(g)=\sum_{\gamma\in Z(F)\backslash G(F)}f(\gamma g),\ \phi'(g)=\sum_{\gamma\in Z(F)\backslash G(F)}f'(\gamma g).$$

Both functions are cuspidal. Let again π_α denote the set of cuspidal representations with central character ω and let ϕ_α,ϕ'_α denote the corresponding orthogonal projections. Then

$$\phi_\alpha=\pi_{\alpha,v_0}(f_0)\,\phi'_\alpha.$$

Thus if $\phi_\alpha\neq 0$ then $\pi_{\alpha,v_0}=\pi^z_{v_0}$ for some $z\in X$. Now suppose that f_0 is such that $\pi^z_{v_0}(f_0)=0$ for all $z\in X$. Then $\phi_\alpha=0$ for all α and so $\phi=0$. In particular

$$\sum f(\gamma)=0,$$

for all choices of f^{v_0,v_1}. This implies $f_{v_0}(e)=0$ and so $f_0=0$. Thus X is infinite, as claimed. ▌

For any z of module 1, let Π^z_v be the base change of the representation $\pi^{i,z}_{v_0}$. Let $\mathscr{R}^{i,z}$ be the distribution corresponding to $\mathscr{B}_{\pi^{i,z}_{v0}}$. For $z\in X$, there is thus a linear form in $\mathscr{H}(\Pi^z_v,H)$ with the required property. More precisely, let $\mathscr{W}^z$ be any non-zero Whittaker linear form on $\mathscr{V}(\Pi^z_v)$, then there is a unique $\mathscr{P}^i_z\in\mathscr{H}(\Pi^z_v,H)$ such that

$$\mathscr{R}^{i,z}(f)=\sum_\alpha\mathscr{W}^z(\Pi^z_v(f)\,W_\alpha)\,\overline{\mathscr{P}^i_z(W_\alpha)}.$$

Our task is to show that for every z (of module 1) there is such a linear form.

At this point, we may as well revert to a local notation, writing our extension as E/F and writing simply Π^z rather than Π^z_v and so on. We may regard the representations $\pi^{i,z}$ as a fiber bundle of representations. In a precise way, we set $K' = GL(3, \mathcal{O}_F)$ and let $\mathscr{V}'$ be the space of smooth functions $\phi: K' \to \mathscr{V}(\pi_1)$ such that $\phi(mk) = \pi_1^i \times \pi_2(m)$ for $m \in M(F) \cap K'$. Then we may regard all the representations $\pi^{i,z}$ as operating on $\mathscr{V}'$. For every $u \in \mathscr{V}'$ the map $z \mapsto \pi^{i,z}(f')\, u$ takes its values in a fixed finite dimensional vector space and is a polynomial function of z. Similarly, there is a holomorphic family of Whittaker linear forms $\mathscr{W}'_z$ on $\mathscr{V}'$. More precisely, for every $u \in \mathscr{V}'$ the map $z \mapsto \mathscr{W}'_z(u)$ is a polynomial in z. Now suppose that f' is bi-invariant under the compact open subgroup K'_1. Then the Bessel distribution corresponding to $\mathscr{W}'_z$ is given by:

$$\mathscr{B}_{\pi_i^z}(f') = \sum_\alpha \mathscr{W}'_z(\pi_i^z(f')\, u_\alpha)\, \overline{\mathscr{W}'_z(u_\alpha)}$$

where u_α is a fixed orthonormal basis of the space of K'_1-invariant vectors in $\mathscr{V}$. It follows that

$$z \mapsto \mathscr{B}_{\pi_i^z}(f')$$

is a polynomial in z. The same is therefore true of the map $z \mapsto \mathscr{R}^{i,z}(f)$.

We introduce the notation of generalized vector and write

$$\mathscr{W}^z(W) = (W, \mathscr{W}^z),\ \mathscr{P}^i_z(W) = (W, \mathscr{P}^i_z).$$

In terms of generalized vectors, we see that for $z \in X$:

$$(\Pi^z_v(f)\, \mathscr{P}^i_z, \mathscr{W}^z) = \mathscr{R}^{i,z}(f).$$

Fix a compact open subgroup K_1 and let dk_1 denote the Haar measure of mass one on K_1. Set

$$f^{K_1}(g) = \int f(gk_1)\, dk_1.$$

Let also $P^z_{K_1}$ denote the corresponding projection operator:

$$P^z_{K_1} = \int \Pi^z(k_1)\, dk_1.$$

Then for *any* z, we can write:

$$\mathscr{R}^{i,z}(f^{K_1}) = \int W_z(g)\, f(g)\, dg$$

where W_z is a function such that

$$W_z(ngk_1) = \theta(n\bar{n})\, W_z(g),$$

for $n \in N(E)$ and $k_1 \in K_1$. For every g the map $z \mapsto W_z(g)$ is a polynomial in z.

We claim that W_z belongs to the Whittaker model of Π^z. This is true for $z \in X$. Indeed, for $z \in X$, we have:

$$W_z(g) = (\Pi^z(g)\, P^z_{K_1} \mathscr{P}^i_z, \mathscr{W}^z).$$

Again the representations Π^z form a fiber bundle of representations all operating on the same space $\mathscr{V}$ smooth of functions on $K = GL(3, \mathcal{O}_E)$ with values in $\mathscr{V}(\Pi_1)$. There is also an analytic family of Whittaker linear forms $\mathscr{W}^z$ on $\mathscr{V}$. Let e_μ, $1 \leqslant \mu \leqslant M$, be a basis of the space of vectors in $\mathscr{V}$ invariant under K_1. The functions W^z_μ defined by:

$$W^z_\mu(g) = \mathscr{W}^z(\Pi^z(g)\, e_\mu)$$

form a basis of the space of K_1 invariant elements in the Whittaker model of Π^z_v. Thus for $z \in X$ we have a unique decomposition:

$$W_z(g) = \sum_{1 \leqslant \mu \leqslant M} \lambda_\mu(z)\, W^z_\mu(g).$$

Let z_0 be a point not in X. Now choose M elements (g^ν) such that

$$D(z) := \det(W^z_\mu(g^\nu))$$

is non-zero at z_0. Then $D(z) \neq 0$ on a subset X_0 of X which is also infinite. For $z \in X_0$ the scalars $(\lambda_\mu(z))$ are solutions of the Cramer system

$$W_z(g^\nu) = \sum_{1 \leqslant \mu \leqslant M} \lambda_\mu(z)\, W^z_\mu(g^\nu).$$

Thus there exist polynomials (P_μ) such that

$$\lambda_\mu(z) = \frac{P_\mu(z)}{D(z)}$$

for $z \in X_0$. Then

$$W_z(g) = \sum_\mu \frac{P_\mu(z)}{D(z)}\, W^z_\mu(g)$$

for $z \in X_0$. Thus the same relation is true at z_0. Our assertion follows.

This being so the above result for $z=0$ amounts to saying that for every K_1 there is a unique vector u_{K_1} invariant under K_1 such that, for any function f,

$$\mathscr{R}^i(f^{K_1}) = (\Pi(f)\, u_{K_1}, \mathscr{W}).$$

Now if $K_2 \supseteq K_1$ then $(f^{K_2})^{K_1} = f^{K_2}$. Thus

$$\mathscr{R}^i(f^{K_2}) = (\Pi(f^{K_2})\, u_{K_1}, \mathscr{W}) = (\Pi(f)\, P_{K_2} u_{K_1}, \mathscr{W}).$$

It follows that $P_{K_2} u_{K_1} = u_{K_2}$. It follows there is a generalized vector $\mathscr{P}_i$ such that

$$\mathscr{R}^i(f) = (\Pi(f)\, \mathscr{P}_i, \mathscr{W}).$$

By definition the distribution $\mathscr{R}^i$ is invariant under H on the right. Thus the generalized vector $\mathscr{P}_i$ is also invariant and we are done.

8. CONCLUDING REMARKS

The same technics can be used to prove the conjecture for other representations. As a matter of fact, this is done in [LR]. However, it is difficult to prove the conjecture for all representations thus we prefer to limit ourself to the above cases.

One expects the above results to generalize in a straightforward way to the groups $GL(n)$ with n odd. For n even, the situation is more complicated. Even if we assume that the infinite places of F split in E, we have to deal with more than one unitary group. Thus it is reasonable to conjecture that a cuspidal representation which is a base change is distinguished by some unitary group; it is then a separate issue to show that it is in fact distinguished with respect to the quasi-split group H, that is the unitary group for the Hermitian matrix w with entries $w_{i,j} = \delta_{i+j,n+1}$. Assuming that it is the case, it is best to introduce the similitude group $\tilde{H}$ and global period integrals of the form:

$$\int_{Z(E_{\mathbb{A}}) \backslash \tilde{H}(E_{\mathbb{A}})} \phi(h)\, \omega^{-1}(\lambda(h))\, dh.$$

Such an integral should be non-identically zero if and only if Π is the base change of a cuspidal representation π with central character $\omega \omega_{E/F}^{n/2}$. It should factor as product of local invariant linear forms.

Now we discuss the local situation for n even. So let E/F be a local non-Archimedean quadratic extension. For every generic irreducible representation Π, we should introduce, for a character ω of $F^\times$, the space $\mathscr{H}(\Pi, \tilde{H}, \omega)$ of linear forms $\mathscr{P}$ such that

$$\mathscr{P}(\Pi(\tilde{h})\, u) = \omega(\lambda(\tilde{h}))\, \mathscr{P}(u).$$

Then it should be that $\mathscr{H}(\Pi, \tilde{H}, \omega) \neq 0$ if and only if Π is the base change of at least one representation π with central character $\omega\omega_{E/F}^{n/2}$. This conjecture is motivated by the following property of the local transfer factor $\gamma(a, \psi_{v_0})$, at an inert place v_0: the transfer factor is a function defined on the group $A(F_{v_0})$ of diagonal matrices with entries in F_{v_0}. If n is even, for any scalar matrix $z \in F_{v_0}^\times$,

$$\gamma(az, \psi) = \gamma(a, \psi)\, \omega_{E/F}(z)^{n/2}.$$

Going back to a local situation, for global purposes, we deal with functions Φ supported on the set of Hermitian matrices in the orbit of w, that is, whose determinant is in $\det wF^+$. The matching functions f' are supported on the set wG^+. It is more convenient to consider the symmetric space $\mathfrak{S}_w$ of matrices s such that $s = s^*$ where we have set $g^* := w\,{}^t g^\sigma w$. The group $GL(n, E)$ operates on $\mathfrak{S}_w$ by $s \mapsto gsg^*$. We consider the orbit of w, that is, the set $\mathfrak{S}_w^+$ of matrices with $\det s \in F^+$. Then the condition of matching reads

$$\Phi(gg^*) = \int f(gh)\, dh,$$

$$\int \Phi(nawn^*)\, \theta(n\bar{n})\, dn = \gamma(a, \psi) \int f'(n_1 awn_2)\, \theta(n_1)\, \theta(n_2)\, dn_1\, dn_2.$$

The relative Bessel distribution is defined in the same way as before but the Bessel distribution is now defined by:

$$\mathscr{B}_\pi(f') := \sum_\alpha \mathscr{W}(\pi(f')\, W_\alpha)\, \overline{\mathscr{W}(W_\alpha)}.$$

The distributions $\mathscr{B}_\pi$ and $\mathscr{B}_{\pi \otimes \omega_{E/F}}$ have the same restriction to G^+. We have the following lemma:

Lemma 7. *The restriction of $\mathscr{B}_\pi$ to G^+ is non-zero. Let π_j, $1 \leqslant i \leqslant m$ be a family of irreducible generic representations of $GL(n, F)$ such that for any pair (i, j), $i \neq j$, the representations π_i and π_j (resp. π_i and $\pi_j \otimes \omega_{E/F}$) are* *inequivalent. Then the restrictions of the Bessel distributions $\mathscr{B}_{\pi_i}$ to G^+ are linearly independent.*

Proof of the Lemma. Indeed, the first assertion is clear if the restriction of π to G^+ is irreducible (see Lemma (1)). Suppose it is not. Then π is induced by an irreducible representation π_0 of G^+ and $\pi = \pi \otimes \omega_{E/F}$. Suppose that $W \in \mathscr{W}(\pi)$ is such that $\mathscr{W}(W) \neq 0$. Then the function W_1 defined by:

$$W_1(g) = \tfrac{1}{2}(W(g) + W(g)\, \omega_{E/F}(\det g))$$

is a non-zero element of $\mathscr{W}$ supported on G^+. It follows that

$$\mathscr{W}(\pi) = \mathscr{W}^+ \oplus \pi(r)\, \mathscr{W}^+,$$

where $\mathscr{W}^+$ is the space of elements of $\mathscr{W}(\pi)$ supported on G^+ and $\det r \notin F^+$. It follows that the representation π^+ on $\mathscr{W}^+$ by right shifts is equivalent to π_0 or π_0^r and is irreducible. Since $\mathscr{W}$ vanishes on $\pi(r)\, \mathscr{W}^+$, for f' supported on G^+, we may take the sum defining the Bessel distribution over a basis of $\mathscr{W}^+$. The first assertion follows.

To prove the second assertion, for each i, denote by $\pi_{i,0}$ the restriction of π_i to G^+, if this restriction is irreducible, or the irreducible representation of G^+ which induces π_i if not. Then the representations $\pi_{i,0}$ are irreducible and pairwise inequivalent. By the first part of the proof, the restriction of the Bessel distribution $\mathscr{B}_{\pi_i}$ to G^+ may be viewed as a (generalized) matrix coefficient of $\pi_{i,0}$. The second assertion follows. ∎

The lemma implies that in the (still conjectural) relative trace formula, for a global extension E/F of number fields, where the infinite places of F split in E, a cuspidal automorphic representation of $GL(n, F_{\mathbb{A}})$ cannot give a zero contribution, even if we consider only functions f' supported on $G^+_{\mathbb{A}}$. This will prove that any base change representation of $GL(n, E\mathbb{A})$ is distinguished by a quasi-split unitary group. Note that this argument is insufficient if some real place of F is inert in E.

The lemma also suggest the following conjecture: suppose that $\mathscr{H}(\Pi, \tilde{H}, \omega)$ is not zero. Let $\pi_1, \pi_2, \pi_3, ..., \pi_m$ be representations of $GL(n, F)$ with central character $\omega\omega_{E/F}^{n/2}$ which base change to Π. We assume that for any other representation π which base change to Π there is exactly one index i such that either $\pi = \pi_i$ or $\pi = \pi_i \otimes \omega_{E/F}$. Then, for each index i, there is a unique element $\mathscr{P}_i$ of $\mathscr{H}(\Pi, \tilde{H}, \omega)$ such that

$$\mathscr{R}_\Pi(f) = \mathscr{B}_{\pi_i}(f')$$

if f and f' have matching orbital integrals. Moreover, the vectors $\mathscr{P}_i$, $1 \leqslant i \leqslant m$, form a basis of $\mathscr{H}(\Pi, \tilde{H}, \omega)$.

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