1. Solve the optimal control problem
\[
\max \int_0^1 (x + u) dt, \quad \dot{x} = 1 - \frac{1}{2}u^2, \quad x(0) = 0, \quad x(1) \geq 0, \quad u \in (-\infty, \infty)
\]

2. Consider the control problem
\[
\max \int_0^T (x - u) dt, \quad \dot{x} = au e^{-2x} - x, \quad x(0) = x_0, \quad x(T) \text{ free}, \quad u \in [0, 1]
\]
where \(T, a, \) and \(x_0\) are positive constants.
   (a) Write down the conditions given by the maximum principle (assuming \(p_0 = 1\)). Find an explicit expression for the adjoint function \(p(t)\), and determine the possible values of the optimal control.
   (b) Put \(T = \ln 10, a = 5\) and \(x_0 = 5\) and solve the problem in this case.
   (c) What is the solution if \(T = \ln 10, a = 1/2\) and \(x_0 = 5\)?

3. Consider the control problem (\(a\) is a given constant)
\[
\max \int_0^1 -(x - u - a)^2 dt, \quad \dot{x} = u - x, \quad x(0) = 1, \quad x(1) \text{ free}, \quad u \in [0, \infty)
\]
   (a) Put \(a = 4\). Show that \(u^*(t) = 0\) for all \(t\) is an optimal control by showing that all the conditions in Mangasarian’s sufficiency theorem are satisfied.
   (b) Put \(a = 1/4\). Find the optimal control in this case. (Hint: Try \(u^*(t) > 0\) for all \(t\)).

4. Soap bubbles.

Films of soaps take shapes that minimize their surface area. We’re trying to understand what shape a soap bubble attached to two metal circles will take.

We have two metal circles \(C_0 = \{(0, y, z), y^2 + z^2 = 1\}\) and \(C_b = \{(b, y, z), y^2 + z^2 = B^2\}\). The two circles are hence in parallel planes, centered around the same axis \(y = z = 0\), at a distance \(b\) of each other, and of respective radii 1 and \(B\).

A soap bubble based on these circles will take a shape that is rotationally invariant around the axis \(y = z = 0\). Hence the whole shape of such a bubble can be described by a single function \(r(x)\) such that \(r(0) = 1\) and \(r(b) = B\). The soap film is then the collection of circles \(C_x = \{(x, y, z), y^2 + z^2 = r(x)^2\}\) for \(0 \leq x \leq b\).

Recall that the infinitesimal length of a curve \(r(x)\) is given via Pythagoras’ theorem:
\[
dl = \sqrt{dx^2 + dr^2} = \sqrt{1 + r'(x)^2} \, dx
\]

Hence the surface area of a soap bubble is given by
\[
A(r) = 2\pi \int_0^b r(x) \sqrt{1 + r'(x)^2} \, dx
\]

The optimization problem we thus have is to minimize \(A(r)\) with boundary conditions \(r(0) = 1\) and \(r(b) = B\).
(a) Show that the functions
\[ r(x) = c \cosh \left( \frac{x}{c} + d \right) \]
solve the Euler equation (where \( c \) and \( d \) are two real parameters). Recall that \( \cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2} \), \( \sinh(\theta) = \frac{e^\theta - e^{-\theta}}{2} \), and that we have the identity \( \cosh^2(\theta) = \sinh^2(\theta) + 1 \).

(b) Using one of the boundary conditions show that
\[ c = \frac{1}{\cosh(d)} \]

(c) Plugging in the second boundary condition should give us the solution. However, it turns out that this will not work. We now investigate for which values of \( B \) a solution exists.

Show that for all real numbers \( \theta \) we have \( \cosh \theta \geq |\theta| \). (Hint: \( \cosh \theta \) is an even function so it suffices to check this for non-negative \( \theta \). Consider \( \theta \geq 0 \) and find the minimum of \( f(\theta) = \cosh(\theta) - \theta \). Keep in mind the identity \( \cosh^2(\theta) = \sinh^2(\theta) + 1 \).

(d) use part (c) to show that
\[ B \geq \frac{b \cosh(d) + d}{\cosh(d)} \]

Use (c) again to show that \( B \geq b - 1 \).

(e) Suppose \( B < b - 1 \). What is the mathematical conclusion? What does this mean physically? (Hint: Fix the radius \( B \). Hold the two circles in your hands and let the distance \( b \) between them grow. What is going to happen to the soap film?)