

ALGEBRAIC NUMBER THEORY W4043

HOMEWORK, WEEK 9, DUE NOVEMBER 14

The following exercises study the basic properties of the p -adic norm. Let $x = \frac{a}{b} \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$. If $x \neq 0$, write $x = p^r \cdot \frac{a'}{b'}$ with p relatively prime to both a' and b' and $r \in \mathbb{Z}$ (not necessarily positive), and define

$$|x|_p = p^{-r}.$$

Thus $|p|_p = p^{-1}$. We also define $|x|_p = 0$.

1. Show that $|\bullet|_p$ has the properties of a metric:
 1. For all $x \in \mathbb{Q}$, $|x|_p \geq 0$, with $|x|_p = 0$ if and only if $x = 0$.
 2. For all $x, y \in \mathbb{Q}$, $|xy|_p = |x|_p|y|_p$.
 3. For all $x, y \in \mathbb{Q}$, $|x + y|_p \leq \sup(|x|_p, |y|_p)$.

Item 3. is the *non-archimedean* (or ultrametric) property; it is stronger than the usual triangle inequality. It allows us to define \mathbb{Q}_p as the set of equivalence classes of infinite series

$$\sum_{i=0}^{\infty} a_i, a_i \in \mathbb{Z}; \lim_{i \rightarrow \infty} |a_i|_p = 0$$

where $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are defined to be equivalent if

$$\lim_{N \rightarrow \infty} \left| \sum_{i=0}^N a_i - \sum_{i=0}^N b_i \right|_p = 0.$$

The p -adic norm extends to \mathbb{Q}_p by setting

$$\left| \sum_{i=0}^{\infty} a_i \right|_p = \lim_{N \rightarrow \infty} \left| \sum_{i=0}^N a_i \right|_p.$$

2. (a) Show that $|\sum_{i=0}^{\infty} a_i|_p$ is always either 0 or a power (positive or negative) of p .

(b) Show that $|\sum_{i=0}^{\infty} a_i|_p = |\sum_{i=0}^{\infty} b_i|_p$ if $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are equivalent.

3. (a) $a \in \mathbb{Q}$, $a \neq 0$. Show that $|a|_p = 1$ for all but finitely many prime numbers p .

(b) (*The product formula*) Let $a \in \mathbb{Q}$, $a \neq 0$. Let $|a|$ be the usual absolute value (equal to a if $a > 0$ and to $-a$ if $a < 0$). Show that

$$|a| \cdot \prod_p |a|_p = 1,$$

where the product is taken over all prime numbers. (By (a), this is actually a finite product.)

4. Define the p -adic logarithm

$$\log_p(1+x) = \sum_{i=1}^{\infty} (-1)^{i-1} x^i / i.$$

(a) If $x \in \mathbb{Q}_p$, show that $\log_p(1+x)$ converges (all convergence in the p -adic numbers is absolute) if and only if $|x|_p < 1$.

(b) Show that if $p > 2$ and $|x|_p < 1$, then $|\log_p(1+x)|_p = |x|_p$.

(c) Show that the subset U_1 of elements of \mathbb{Q}_p of the form $1+x$, with $|x|_p < 1$, is a multiplicative subgroup of \mathbb{Q}_p^\times . Show that if $u, v \in U_1$, then $\log_p(uv) = \log_p(u) + \log_p(v)$. Thus \log_p is a homomorphism from U_1 to the additive group of \mathbb{Q}_p . Show that it is injective if $p > 2$.