ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 6, due October 17

1. (Some calculations)

(a) Determine the discriminants of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-5})$, and of the ring $\mathbb{Z}(\sqrt[3]{2})$.

(b) Show that no odd prime ramifies in $\mathbb{Q}(\sqrt{2})$.

(c) Is 23 a quadratic residue modulo 79? Check without using the formula $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

2. Use Minkowski's theorem to show that $\mathbb{Q}(\sqrt{5})$ is a principal ideal domain. Show that Minkowski's theorem does not imply that $\mathbb{Q}(\sqrt{-5})$ has unique factorization, and show that the element 6 has two distinct factorizations in $\mathbb{Q}(\sqrt{-5})$ as products of irreducible elements.

3. (Uses the Chinese Remainder Theorem)

(a) Let $\mathcal{O}$ be the ring of integers in a number field, and let I and J be ideals in $\mathcal{O}$. Show that $I + J = \mathcal{O}$ if and only if I and J have no common prime factor. Show that if $I = \prod_i \mathfrak{p}^{a_i}$ and $J = \prod_i \mathfrak{p}^{b_i}$ then $I + J = \prod_i \mathfrak{p}^{min(a_i,b_i)}$.

(b) Use the Chinese Remainder Theorem in $\mathcal{O}$ to prove the following fact: Let $\mathfrak{p}_1, \ldots \mathfrak{p}_n$ be non-zero prime ideals on $\mathcal{O}$ and let $a_1, \ldots, a_n$ be non-negative integers. Then there exists a non-zero $\alpha \in \mathcal{O}$ such that $ord_{\mathfrak{p}_i}(\alpha) = a_i$; in other words, in the prime factorization of the principal ideal (α), the prime $\mathfrak{p}_i$ occurs with exponent exactly a_i (we are not saying that no other prime occurs!). (Hint: use the fact that the localization $\mathcal{O}_{\mathfrak{p}_i}$ of $\mathcal{O}$ at the prime ideal $\mathfrak{p}_i$ – i.e., the localization at the multiplicative set $\mathcal{O} - \{\mathfrak{p}_i\}$ – is a principal ideal domain, and that the inclusion $\mathcal{O}/\mathfrak{p}_i^a \hookrightarrow \mathcal{O}_{\mathfrak{p}_i}/\mathfrak{p}_i^a \cdot \mathcal{O}_{\mathfrak{p}_i}$ is an isomorphism for any $a \geq 0$.)

(c) Let I be any non-zero ideal of $\mathcal{O}$. Use (b) to show there exists an ideal J and an element $\alpha \in \mathcal{O}$ such that $(\alpha) = IJ$ where I and J have no common prime factors. Then construct $\alpha' \in \mathcal{O}$ such that $(\alpha') = IJ'$ where the ideals I, J, J' are relatively prime in pairs. Deduce that I is generated by α and α'.