1. Homework, week 4, due October 3

This assignment establishes some of the basic properties of quadratic forms attached to ideals in imaginary quadratic fields. A quadratic space of rank $n$ over $\mathbb{Z}$ is a pair $(M, q)$, where $M$ is a free rank $n$ $\mathbb{Z}$-module (free abelian group on $n$ generators) and $q : M \to \mathbb{Z}$ is a quadratic form, i.e. a function satisfying

1. $q(am) = a^2q(m)$, $a \in \mathbb{Z}$, $m \in M$;
2. The function $B_q : M \times M \to \mathbb{Z}$, defined by $B_q(m, m') = q(m + m') - q(m) - q(m')$ is a bilinear form, i.e.
3. $B_q(m, m') = B_q(m', m)$;
4. $B_q(am + bm', m'') = aB_q(m, m'') + bB_q(m', m'').$

We only consider the case $n = 2$ and identify $M$ with $\mathbb{Z}^2$. If $\{e_1, e_2\}$ is the standard $\mathbb{Z}$-basis of $\mathbb{Z}^2$, $B_q$ is determined by the $2 \times 2$ symmetric matrix $(b_{ij})$ where $B_q(e_i, e_j) = b_{ij}$ (and you can check that this in turn determines $q(m) = \frac{1}{2}B_q(m, m)$). We identify $q$ with a polynomial in two variables $(X, Y)$ by setting

$$q(X, Y) = q(Xe_1 + Ye_2).$$

A (binary) quadratic form $q(X, Y) = aX^2 + bXY + cY^2$

Say $(M, q)$ and $(M', q')$ are isomorphic if there is an isomorphism $f : M \to M'$ of abelian groups such that $q' \circ f = q$. Define the discriminant of the quadratic form $q$ by $\Delta(q) = -\det(b_{ij})$ and check for yourselves (without writing it down) that two isomorphic quadratic spaces have the same discriminant.

1. Consider $q_1(X, Y) = X^2 + 15Y^2$, $q_2(X, Y) = 3X^2 + 5Y^2$. Show that $q_1$ and $q_2$ have the same discriminant but don’t define isomorphic quadratic spaces. 2. Let $d$ be a positive squarefree integer. Let $K = \mathbb{Q}(\sqrt{-d})$, with integer ring $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ if $d \equiv 3 \pmod{4}$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{-d}]$ if $d \equiv 1, 2 \pmod{4}$. We write $\Delta_d = -d$ if $d \equiv 3 \pmod{4}$ and $\Delta_d = -4d$ if $d \equiv 1, 2 \pmod{4}$ (this is the discriminant of the field $K$).

(a) Show that the quadratic form $q = q_{\mathcal{O}_K}$ on the rank 2 $\mathbb{Z}$-module $\mathcal{O}_K$, defined by $q(x) = N_{K/\mathbb{Q}}(x)$, has discriminant $\Delta_d$. Moreover, $q$ is positive definite: $q(x) > 0$ for all $x \neq 0$.

(b) Show that the bilinear form $B_q$ associated to $q$ is given by

$$B_q(x, y) = Tr_{K/\mathbb{Q}}(x\sigma(y)) = x\sigma(y) + \sigma(x)y$$

where $\sigma \in Gal(K/\mathbb{Q})$ is the non-trivial element.
(c) In general, let $I \subset \mathcal{O}_K$ be an ideal, $N(I) = [\mathcal{O}_K : I] = |\mathcal{O}_K/I|$. Define $q_I : I \to \mathbb{Q}$ by $q_I(x) = N_{\mathcal{K}/\mathbb{Q}}(x)/N(I)$. Show that $q_I$ takes values in $\mathbb{Z}$ and the pair $(I, q_I)$ is a quadratic space over $\mathbb{Z}$.

(d) Show that $(I, q_I)$ is of discriminant $\Delta_d$.

3. A (binary) quadratic form $q(X, Y) = aX^2 + bXY + cY^2$ is called \textit{primitive} if $a$, $b$, and $c$ have no common divisors. Show that $q_I$ is \textit{primitive}.

4. Suppose $I$ and $J$ are two ideals of $\mathcal{O}_K$. Show that $(I, q_I)$ and $(J, q_J)$ are isomorphic if $I$ and $J$ are equivalent in the ideal class group $\text{Cl}(K)$.