ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 3, due September 26

1. Let K/k be a quadratic extension of fields of characteristic different from 2, so that we can write $K = k(\sqrt{d})$ for some $d \in k$. Let $s : K \to K$ be the non-trivial element of $Gal(K/k)$,

$$s(a + b\sqrt{d}) = a - b\sqrt{d},$$

and let $Tr : K \to k$ be the k-linear trace map, $Tr(\alpha) = a + s(a)$. Let $V = K$, viewed as a 2-dimensional vector space over k. Define a bilinear form $B : V \times V \to k$ by

$$B(\alpha, \beta) = Tr(\alpha\beta).$$

A non-zero vector $v \in V$ is *isotropic* if $B(v, v) = 0$.

(a) Suppose $k = \mathbb{R}$ and $K = \mathbb{C}$. Show that V contains non-zero isotropic vectors. (b) Suppose $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-2})$. Does V contain non-zero isotropic vectors?

2. Hindry's book, Exercise 6.15, p. 119. (Use the Vandermonde determinant.)

3. In the notation of question 2, we let p be a prime number, r a positive integer, and let $F(X) = (X^{p^r} - 1)/(X^{p^{r-1}} - 1)$, a polynomial of degree $\phi(p^r)$ where ϕ denotes the Euler function. Let $K = \mathbb{Q}(\zeta)$ be the splitting field of F where ζ is a root of F, and therefore a primitive p^rth root of unity.

(a) Suppose $r = 1$. Show that the discriminant of the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{p-2}\}$ is equal to $\pm p^{p-2}$.

(b) Determine the sign in (a).

(c) Now for any r, show that the discriminant of the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{\phi(p^r)-1}\}$ is equal to $\pm p^{p^{r-1}(pr-r-1)}$. (You will find it convenient to use the result of (a).)