ALGEBRAIC NUMBER THEORY W4043

Homework, week 9, due November 15

The following exercises study the basic properties of the p-adic norm. Let $x = \frac{a}{b} \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$. If $x \neq 0$, write $x = p^r \cdot \frac{a'}{b'}$ with p relatively prime to both a' and b' and $r \in \mathbb{Z}$ (not necessarily positive), and define

$$|x|_p = p^{-r}.$$

Thus $|p|_p = p^{-1}$. We also define $|x|_p = 0$.

1. Show that $|\bullet|_p$ has the properties of a metric:
   1. For all $x \in \mathbb{Q}$, $|x|_p \geq 0$, with $|x|_p = 0$ if and only if $x = 0$.
   2. For all $x, y \in \mathbb{Q}$, $|xy|_p = |x|_p|y|_p$.
   3. For all $x, y \in \mathbb{Q}$, $|x + y|_p \leq \sup(|x|_p, |y|_p)$.

Item 3. is the *non-archimedean* (or ultrametric) property; it is stronger than the usual triangle inequality. It allows us to define $\mathbb{Q}_p$ as the set of equivalence classes of infinite series

$$\sum_{i=0}^{\infty} a_i, a_i \in \mathbb{Q}; \lim_{i \to \infty} |a_i|_p = 0$$

where $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are defined to be equivalent if

$$\lim_{N \to \infty} |\sum_{i=0}^{N} a_i - \sum_{i=0}^{N} b_i|_p = 0.$$

The p-adic norm extends to $\mathbb{Q}_p$ by setting

$$|\sum_{i=0}^{\infty} a_i|_p = \lim_{N \to \infty} |\sum_{i=0}^{N} a_i|_p.$$

2. (a) Show that $|\sum_{i=0}^{\infty} a_i|_p$ is always either 0 or a power (positive or negative) of p.

(b) Show that $|\sum_{i=0}^{\infty} a_i|_p = |\sum_{i=0}^{\infty} b_i|_p$ if $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are equivalent.

3. (a) $a \in \mathbb{Q}$, $a \neq 0$. Show that $|a|_p = 1$ for all but finitely many prime numbers p.

(b) (*The product formula*) Let $a \in \mathbb{Q}$, $a \neq 0$. Let $|a|$ be the usual absolute value (equal to a if $a > 0$ and to $-a$ if $a < 0$. Show that

$$|a| \cdot \prod_p |a|_p = 1,$$

where the product is taken over all prime numbers. (By (a), this is actually a finite product.)

4. Define the *adèle group* $\mathbf{A}$ to be the subgroup of the direct product $\mathbb{R} \otimes \prod_p \mathbb{Q}_p$, where the product is taken over all prime numbers, of elements $(a_{\mathbb{R}}, (a_p))$ such that $|a_p|_p \leq 1$ for all but finitely many p. (The number of p such that $|a_p|_p > 1$ depends on the element but it must always be finite.

Show that there is an injective homomorphism $i : \mathbb{Q} \to \mathbf{A}$. Show that the set of $x \in \mathbb{Q}$ such that $|x|_p \leq 1$ for all p is equal to $\mathbb{Z}$.