ALGEBRAIC NUMBER THEORY W4043

HOMEWORK, WEEK 9, DUE NOVEMBER 19

The following exercises study the basic properties of the p-adic norm. Let $x = \frac{a}{b} \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$. If $x \neq 0$, write $x = p^r \cdot \frac{a'}{b'}$ with p relatively prime to both a' and b' and $r \in \mathbb{Z}$ (not necessarily positive), and define

$$|x|_p = p^{-r}.$$

Thus $|p|_p = p^{-1}$. We also define $|x|_p = 0$.

1. Show that $|\bullet|_p$ has the properties of a metric:
   1. For all $x \in \mathbb{Q}$, $|x|_p \geq 0$, with $|x|_p = 0$ if and only if $x = 0$.
   2. For all $x, y \in \mathbb{Q}$, $|xy|_p = |x|_p |y|_p$.
   3. For all $x, y \in \mathbb{Q}$, $|x + y|_p \leq \sup(|x|_p, |y|_p)$.

Item 3. is the *non-archimedean* (or ultrametric) property; it is stronger than the usual triangle inequality. It allows us to define $\mathbb{Q}_p$ as the set of equivalence classes of infinite series

$$\sum_{i=0}^{\infty} a_i, a_i \in \mathbb{Z}; \lim_{i \to \infty} |a_i|_p = 0$$

where $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are defined to be equivalent if

$$\lim_{N \to \infty} |\sum_{i=0}^{N} a_i - \sum_{i=0}^{N} b_i|_p = 0.$$

The p-adic norm extends to $\mathbb{Q}_p$ by setting

$$|\sum_{i=0}^{\infty} a_i|_p = \lim_{N \to \infty} |\sum_{i=0}^{N} a_i|_p.$$

2. (a) Show that $|\sum_{i=0}^{\infty} a_i|_p$ is always either 0 or a power (positive or negative) of p.

(b) Show that $|\sum_{i=0}^{\infty} a_i|_p = |\sum_{i=0}^{\infty} b_i|_p$ if $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are equivalent.

3. (a) $a \in \mathbb{Q}$, $a \neq 0$. Show that $|a|_p = 1$ for all but finitely many prime numbers p.

(b) (*The product formula*) Let $a \in \mathbb{Q}$, $a \neq 0$. Let $|a|$ be the usual absolute value (equal to a if $a > 0$ and to $-a$ if $a < 0$. Show that

$$|a| \cdot \prod_p |a|_p = 1,$$

where the product is taken over all prime numbers. (By (a), this is actually a finite product.)

4. Define the p-adic logarithm

$$log_p(1+x) = \sum_{i=1}^{\infty}(-1)^{i-1}x^i/i.$$

(a) If $x \in \mathbb{Q}_p$, show that $log_p(1+x)$ converges (all convergence in the p-adic numbers is absolute) if and only if $|x|_p < 1$.

(b) Show that if $p > 2$ and $|x|_p < 1$, then $|log_p(1+x)|_p = |x|_p$.

(c) Show that the subset U_1 of elements of $\mathbb{Q}_p$ of the form $1+x$, with $|x|_p < 1$, is a multiplicative subgroup of ${\mathbb{Q}_p}^{\times}$. Show that if $u, v \in U_1$, then $log_p(uv) = log_p(u) + log_p(v)$. Thus log_p is a homomorphism from U_1 to the additive group of $\mathbb{Q}_p$. Show that it is injective if $p > 2$.