Let $n$ be a positive integer. A Dirichlet character modulo $n$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\chi(ab) = \chi(a)\chi(b)$.
2. $\chi(a)$ depends only on the residue class of $a$ modulo $n$.
3. $\chi(a) = 0$ if and only if $a$ and $n$ have a non-trivial common factor.

It follows that a Dirichlet character modulo $n$ can also be considered a function $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$.

Let $X(n)$ denote the set of distinct Dirichlet characters modulo $n$. We consider $X(p)$ when $p$ is prime and show it forms a cyclic group with identity element $\chi_0$ defined by $\chi_0(a) = 1$ if $(a, p) = 1$, $\chi_0(a) = 0$ if $p | a$.

1. Show that for any $\chi \in X(p)$, $\chi(1) = 1$, and $\chi(a)$ is a $(p-1)$st root of 1 for all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.
2. For all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, show that $\chi(a^{-1}) = \overline{\chi}(a)$ where $\overline{\chi}$ is the complex conjugate function.
3. Show that $\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \chi(a) = 0$ if $\chi \neq \chi_0$.
4. Show that the Legendre symbol $a \mapsto \left(\frac{a}{p}\right)$ for $(a, p) = 1$, extended to take the value 0 at integers divisible by $p$, defines a Dirichlet character modulo $p$ that is different from $\chi_0$.
5. We show that $X(p)$ is a cyclic group of order $p-1$ and that, for any $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$, there exists $\chi \in X(p)$ such that $\chi(a) \neq 1$.
   (a) Bearing in mind that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group, show that $X(p)$ has at most $p-1$ elements.
   (b) Show that $X(p)$ has the structure of an abelian group.
   (c) Let $g$ be a cyclic generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ and define a function $\lambda : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ by $\lambda(g^k) = e^{\frac{2\pi ik}{p-1}}$; $\lambda(0) = 0$.
   Show that $\lambda \in X(p)$ and that, if $n$ is the smallest positive integer such that $\lambda^n = \chi_0$, then $n = p-1$. Conclude that $\lambda$ is a cyclic generator of $X(p)$.
   (d) If $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $a \neq 1$ then $\lambda(a) \neq 1$.
6. Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$. Show that $\sum_{\chi \in X(p)} \chi(a) = 0$.
7. Let $d$ be a divisor of $p-1$. Show that the set of $\chi \in X(p)$ such that $\chi^d = \chi_0$ is a subgroup of order $d$. 