ALGEBRAIC NUMBER THEORY W4043

Homework, week 7, due November 5

Let n be a positive integer. A *Dirichlet character* modulo n is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

(1) $\chi(ab) = \chi(a)\chi(b)$.
(2) $\chi(a)$ depends only on the residue class of a modulo n.
(3) $\chi(a) = 0$ if and only if a and n have a non-trivial common factor.

It follows that a Dirichlet character modulo n can also be considered a function $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$.

Let $X(n)$ denote the set of distinct Dirichlet characters modulo n. We consider $X(p)$ when p is prime and show it forms a cyclic group with identity element χ_0 defined by $\chi_0(a) = 1$ if $(a, p) = 1$, $\chi_0(a) = 0$ if $p \mid a$.

1. Show that for any $\chi \in X(p)$, $\chi(1) = 1$, and $\chi(a)$ is a $(p-1)$st root of 1 for all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.

2. For all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, show that $\chi(a^{-1}) = \bar{\chi}(a)$ where $\bar{\chi}$ is the complex conjugate function.

3. Show that $\sum_{a\in\mathbb{Z}/p\mathbb{Z}} \chi(a) = 0$ if $\chi \neq \chi_0$.

4. Show that the Legendre symbol $a \mapsto \begin{pmatrix} a \\ p \end{pmatrix}$ for $(a, p) = 1$, extended to take the value 0 at integers divisible by p, defines a Dirichlet character modulo p that is different from χ_0.

5. We show that $X(p)$ is a cyclic group of order $p - 1$ and that, for any $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$. there exists $\chi \in X(p)$ such that $\chi(a) \neq 1$.

(a) Bearing in mind that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group, show that $X(p)$ has at most $p - 1$ elements.

(b) Show that $X(p)$ has the structure of abelian group.

(c) Let g be a cyclic generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ and define a function $\lambda : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ by

$$\lambda(g^k) = e^{\frac{2\pi ik}{p-1}};\ \lambda(0) = 0.$$

Show that $\lambda \in X(p)$ and that, if n is the smallest positive integer such that $\lambda^n = \chi_0$, then $n = p - 1$. Conclude that λ is a cyclic generator of $X(p)$.

(d) If $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $a \neq 1$ then $\lambda(a) \neq 1$.

6. Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a \neq 1$. Show that $\sum_{\chi\in X(p)} \chi(a) = 0$.

7. Let d be a divisor of $p - 1$. Show that the set of $\chi \in X(p)$ such that $\chi^d = \chi_0$ is a subgroup of order d.