REPRESENTATION THEORY W4044

Homework, week 7, due March 30 (NOTE CHANGE OF DATE)

1. James and Liebeck book, Chapter 15, Exercises 1, 2, 3; Chapter 16, Exercises 4, 5.

2. (a) Let U be a vector space, H_1, H_2 two groups, and $\tau_1 : H_1 \to Aut(U)$, $\tau_2 : H_2 \to Aut(U)$ two representations such that

$$\tau_1(h_1) \circ \tau_2(h_2) = \tau_2(h_2) \circ \tau_1(h_1) \forall h_1 \in H_1, h_2 \in H_2.$$

Show that the map

$$\tau : H_1 \times H_2 \to Aut(U); \tau(h_1, h_2) = \tau_1(h_1) \circ \tau_2(h_2), h_1 \in H_1, h_2 \in H_2$$

defines a representation of $H_1 \times H_2$.

(b) In the situation of (a), let

$$U^{H_1} = \{u \in U \mid \tau_1(h_1)(u) = u \forall h_1 \in H_1\}.$$

Show that $U^{H_1} \subset H$ is a subrepresentation of τ_2; in other words, if $u \in U^{H_1}$, then $\tau_2(h_2)(u) \in U^{H_1}$ for all $h_2 \in H^2$.

Now Let H be a subgroup of the finite group G. In what follows, we let ℓ and r denote the right and left regular representations of G on $\mathbb{C}[G]$. Let (σ, W) be a representation of H and view $(\ell, \mathbb{C}[G])$ as a representation of H via restriction. Consider the tensor product $(res_H^G \ell \otimes \sigma, \mathbb{C}[G] \otimes W)$ as a representation of H.

(c) Let $triv_W$ denote the trivial representation of G on W: for all $g \in G$ and $w \in W$, $triv_W(g)(w) = w$. Show that $r \otimes triv_W$ is a representation of G on $\mathbb{C}[G] \otimes W$ with the property that, for all $h \in H, g \in G$,

$$(r \otimes triv_W)(g) \circ (res_H^G \ell \otimes \sigma)(h) = (res_H^G \ell \otimes \sigma)(h) \circ (r \otimes triv_W)(g).$$

(d) Show that there is a representation $\tau : H \times G \to \mathbb{C}[G] \otimes W$ such that

$$\tau(h, g) = (res_H^G \ell \otimes \sigma)(h) \circ (r \otimes triv_W)(g),\ h \in H, g \in G.$$

Show that $Ind_H^G \sigma$ can be identified with the representation of G on $(\mathbb{C}[G] \otimes W)^H$.