REPRESENTATION THEORY W4044

1. Homework, week 4, due February 18

1. James and Liebeck book, chapter 9, exercises 2, 3, 4.

2. Let G be a finite group and let V be a representation of G over $\mathbb{C}$. For any non-zero vector $v \in V$, let $Gv \subset V$ denote the orbit of v in V, and let $[Gv] \subset V$ denote the subspace spanned by Gv.

(a) Show that $[Gv]$ is a subrepresentation of V.

(b) Show that $|Gv| \geq \dim[Gv]$. Conclude that if V is irreducible, then $|Gv| \geq \dim V$.

(c) Suppose $H \subset G$ is an abelian subgroup, not necessarily normal, and V is an irreducible representation of G. Show that $\dim V \leq (G : H)$.

(d) Let D_n be the dihedral group of order $2n$. Show that every irreducible representation of D_n is of degree 1 or 2.

(e) Let $C_n \subset D_n$ denote the cyclic subgroup of order n. Let (ρ, V) be an irreducible representation of D_n. Suppose V contains a vector v that is a simultaneous eigenvector for all elements of C_n: for all $c \in C_n$, there is a scalar $\alpha(c)$ such that $\rho(c)v = \alpha(c)v$. Show that $\alpha : C_n \to \mathbb{C}^\times$ is a homomorphism. Show that V contains a vector w such that, for all $c \in C_n$, $\rho(c)w = \alpha(c)^{-1}w$.

3. Let (ρ, V) be a finite-dimensional representation of G. Consider the representation of G on $End(V) = Hom(V, V)$. (a) What is the dimension of $End(V)^G$ if $V = \oplus_{i=1}^r V_i$ with V_i irreducible subrepresentations, no two equivalent?

(b) What is the dimension of $End(V)^G$ if $V = \oplus_{i=1}^r V_i$ with V_i irreducible subrepresentations, but now with no restriction on possible equivalences?