1. Homework, week 4, due February 18


2. Let $G$ be a finite group and let $V$ be a representation of $G$ over $\mathbb{C}$. For any non-zero vector $v \in V$, let $Gv \subset V$ denote the orbit of $v$ in $V$, and let $[Gv] \subset V$ denote the subspace spanned by $Gv$.

   (a) Show that $[Gv]$ is a subrepresentation of $V$.

   (b) Show that $|Gv| \geq \dim[Gv]$. Conclude that if $V$ is irreducible, then $|Gv| \geq \dim V$.

   (c) Suppose $H \subset G$ is an abelian subgroup, not necessarily normal, and $V$ is an irreducible representation of $G$. Show that $\dim V \leq (G : H)$.

   (d) Let $D_n$ be the dihedral group of order $2n$. Show that every irreducible representation of $D_n$ is of degree 1 or 2.

   (e) Let $C_n \subset D_n$ denote the cyclic subgroup of order $n$. Let $(\rho, V)$ be an irreducible representation of $D_n$. Suppose $V$ contains a vector $v$ that is a simultaneous eigenvector for all elements of $C_n$: for all $c \in C_n$, there is a scalar $\alpha(c)$ such that $\rho(c)v = \alpha(c)v$. Show that $\alpha : C_n \to \mathbb{C}^\times$ is a homomorphism. Show that $V$ contains a vector $w$ such that, for all $c \in C_n$, $\rho(c)w = \alpha(c)^{-1}w$.

3. Let $(\rho, V)$ be a finite-dimensional representation of $G$. Consider the representation of $G$ on $\text{End}(V) = \text{Hom}(V, V)$. (a) What is the dimension of $\text{End}(V)^G$ if $V = \bigoplus_{i=1}^r V_i$ with $V_i$ irreducible subrepresentations, no two equivalent?

   (b) What is the dimension of $\text{End}(V)^G$ if $V = \bigoplus_{i=1}^r V_i$ with $V_i$ irreducible subrepresentations, but now with no restriction on possible equivalences?