1. Homework, week 3, due February 11


(Exercises 3, 5, and 6 concern representations of the dihedral group. Let $n \geq 2$ be an integer. The dihedral group $D_{2n}$ is a group of order $2n$ with two generators $a, b$ such that $a^n = e, b^2 = e$ (here $e$ denotes the identity element) and the relation $bab^{-1} = a^{-1}$. This can be seen as the group consisting of rotations of the euclidean plane that fix a regular $n$-gon $X$ inscribed in the unit circle, together with the reflections in the perpendicular bisectors of each of the $n$-sides of $X$.)

2. Let $V$ be the $n$-dimensional vector space $K^n$ over $K$, with $K = \mathbb{R}$ or $\mathbb{C}$. Let $M(n, K)$ denote the space of $n \times n$ matrices over $V$. We want to show that, if $L : \text{End}(V) = M(n, K) \to K$ is a linear map such that $L(AB) = L(BA)$ for all $A, B \in \text{End}(V)$, then $L$ is a scalar multiple of the trace.

(a) First suppose $n = 2$ and define the following matrices:

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Verify that

$$HE_{12} - E_{12}H = 2E_{12}; \ HE_{21} - E_{21}H = -2E_{21}; \ E_{12}E_{21} - E_{21}E_{12} = H.$$ 

Deduce that, if $L : \text{End}(V) = M(2, K) \to K$ is a linear map such that $L(AB) = L(BA)$ for all $A, B \in \text{End}(V)$, then $L(H) = L(E_{12}) = L(E_{21}) = 0$. Thus $\ker L$ is of dimension 3. Show that this implies that $L$ is a scalar multiple of the trace.

(b) Now imitate the argument for $n = 2$ for general $n$. 
