REPRESENTATION THEORY W4044

Homework, week 10, due April 15

1. James and Liebeck book, Chapter 22, exercises 1, 2, 3, 5.

2. Denote by $M(n, \mathbb{Z})$ the ring of $n \times n$ matrices with coefficients in $\mathbb{Z}$. Let $\alpha \in M(n, \mathbb{Z})$. Show that every element of the subring $\mathbb{Z}[\alpha] \subset M(n, \mathbb{Z})$ is integral over $\mathbb{Z}$.

3. Let S_5 be the symmetric group of permutations of 5 letters. Let (ρ, V) be the *standard* representation of S_5 on $\mathbb{C}^5/D$, where D is the 1-dimensional subspace spanned by the vector $(1,1,1,1,1)$.

(a) Show that S_5 has 7 conjugacy classes. List them. List the positive integers that divide the order of S_5.

(b) Let $\sigma : S_5 \to \mathbb{C}^\times$ be the sign character, with kernel the alternating group A_5. Compute the character of ρ and show that $\rho \otimes \sigma$ and ρ are not equivalent.

(c) Conclude that S_5 has no irreducible representation of dimension 8 or 10. Determine the dimensions of the three irreducible representations other than σ, ρ, $\rho \otimes \sigma$, and the trivial representation. Call these irreducible representations A, B, and C.

(d) Using problem 2 of last week's assignment, show that exactly two of the representations A, B, C of part (c) remain irreducible when restricted to A_5.