MODERN ALGEBRA II W4042

Homework, week 9, due November 19

1. Let K be the extension of $\mathbb{Q}$ generated by a root ζ of the polynomial $X^5 - 1$ that is *primitive*, i.e. $\zeta \neq 1$.

(a) Let $u = \zeta + \zeta^4$. Show that

$$u^2 = \zeta^3 + \zeta^2 + 2$$

and that u is a root of the polynomial $X^2 + X - 1$.

(b) Let $L = \mathbb{Q}(\sqrt{5})$. Show that $L \subset K$.

(c) Determine the group $Gal(K/\mathbb{Q})$ and identify the subgroup $Gal(K/L)$.

(d) Let $E = L(\sqrt[4]{5})$. Show that E is a Galois extension of L but not of $\mathbb{Q}$.

(e) Let $F = K(\sqrt{-1}, \sqrt[4]{5})$. Show that F is a Galois extension of $\mathbb{Q}$.

2. Let $\alpha, \beta \in \mathbb{C}$. Suppose $\alpha + \beta$ and $\alpha\beta$ are both algebraic numbers. Show that α and β are algebraic.

3. Rotman's book, p. 63, exercises 79, 80.