MODERN ALGEBRA II W4042

HOMEWORK, WEEK 11, DUE DECEMBER 10

1. Let $P = X^5 - 4X + 2 \in \mathbb{Q}[X]$, and let $K \supset \mathbb{Q}$ be a splitting field for P contained in $\mathbb{C}$.

(a). Show that P is irreducible.

(b) Show that K contains an extension L of degree 5. Show that 5 divides the order of $Gal(K/\mathbb{Q})$.

(c) Show using differential calculus that P has exactly 3 real roots. (Hint: compute $P(0)$ and $P(1)$.) Use this to show that complex conjugation defines a non-trivial element of $Gal(K/\mathbb{Q})$.

(d) Denote by S_5 the group of permutations of 5 letters. We admit the following lemma:

Lemma. *Let $H \subset S_5$ be a subgroup containing a cycle of length 5 and a cycle of length 2. Then $H = S_5$.*

Use this to show that $Gal(K/\mathbb{Q}) = S_5$, and thus P cannot be solved by radicals.

2. Suppose $p = 2^k + 1$ is an odd prime. Show that k is a power of 2.

3. Rotman's book, pp. 94-95, exercises 95, 96, 97, 98.

THE FOLLOWING QUESTIONS ARE OPTIONAL; THEY ARE INCLUDED TO PROVIDE BACKGROUND TO THE THEORY OF SOLVABLE GALOIS EXTENSIONS

4. Let G and H be finite groups, and let $\phi : G \to H$ be a surjective homomorphism.

(a) Suppose G is solvable; prove that H is solvable.

(b) Suppose H and $\ker \phi$ are solvable; prove that G is solvable.

5. Let Γ and Δ be finite groups, and suppose Γ acts by automorphisms on Δ: there exists a homomorphism $r : \Gamma \to Aut(\Delta)$, where $Aut(\Delta)$ is the set of automorphisms of Δ. Define $G = \Delta \rtimes \Gamma$ to be the semidirect product. Let $H \subset G$ be a normal subgroup and let L/K be a Galois extension with group H. Let $E \subset L$ be the fixed field of $\Gamma \cap H$ and let $F \subset L$ be the fixed field of $\Delta \cap H$. Show that F/K is a Galois extension and determine the kernel of the composite map

$$\Gamma \cap H \to H \to Gal(F/K)$$

where the second arrow is the restriction of an automorphism of L to F.