Homework, week 11, due December 10

1. Let \( P = X^5 - 4X + 2 \in \mathbb{Q}[X] \), and let \( K \supset \mathbb{Q} \) be a splitting field for \( P \) contained in \( \mathbb{C} \).
   (a). Show that \( P \) is irreducible.
   (b) Show that \( K \) contains an extension \( L \) of degree 5. Show that 5 divides the order of \( \text{Gal}(K/\mathbb{Q}) \).
   (c) Show using differential calculus that \( P \) has exactly 3 real roots. (Hint: compute \( P(0) \) and \( P(1) \).) Use this to show that complex conjugation defines a non-trivial element of \( \text{Gal}(K/\mathbb{Q}) \).
   (d) Denote by \( S_5 \) the group of permutations of 5 letters. We admit the following lemma:

   **Lemma.** Let \( H \subset S_5 \) be a subgroup containing a cycle of length 5 and a cycle of length 2. Then \( H = S_5 \).

   Use this to show that \( \text{Gal}(K/\mathbb{Q}) = S_5 \), and thus \( P \) cannot be solved by radicals.

2. Suppose \( p = 2^k + 1 \) is an odd prime. Show that \( k \) is a power of 2.

3. Rotman’s book, pp. 94-95, exercises 95, 96, 97, 98.

The following questions are optional; they are included to provide background to the theory of solvable Galois extensions

4. Let \( G \) and \( H \) be finite groups, and let \( \phi : G \to H \) be a surjective homomorphism.
   (a) Suppose \( G \) is solvable; prove that \( H \) is solvable.
   (b) Suppose \( H \) and \( \ker \phi \) are solvable; prove that \( G \) is solvable.

5. Let \( \Gamma \) and \( \Delta \) be finite groups, and suppose \( \Gamma \) acts by automorphisms on \( \Delta \): there exists a homomorphism \( r : \Gamma \to \text{Aut}(\Delta) \), where \( \text{Aut}(\Delta) \) is the set of automorphisms of \( \Delta \). Define \( G = \Delta \rtimes \Gamma \) to be the semidirect product. Let \( H \subset G \) be a normal subgroup and let \( L/K \) be a Galois extension with group \( H \). Let \( E \subset L \) be the fixed field of \( \Gamma \cap H \) and let \( F \subset L \) be the fixed field of \( \Delta \cap H \). Show that \( F/K \) is a Galois extension and determine the kernel of the composite map

\[
\Gamma \cap H \to H \to \text{Gal}(F/K)
\]
where the second arrow is the restriction of an automorphism of $L$ to $F$. 