Intro to modern algebra II

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1. Solution to problem set 7

Problem 1.

Let $w = x + iy \in \mathbb{C}$, $y \neq 0$. Show that $\{1, w\}$ gives a basis of $\mathbb{C}$ over $\mathbb{R}$. First note that $\dim_{\mathbb{R}}(\mathbb{C}) = 2$. If $\alpha \cdot 1 + \beta \cdot w = 0$ for $\alpha, \beta \in \mathbb{R}$ then, as $y \neq 0$, $\beta = \alpha = 0$, thus 1 and w are linearly independent over $\mathbb{R}$. Thus the must give a basis for $\mathbb{C}$.

Let $A(\alpha)$ be the map $z \mapsto \alpha z$, for $z \in \mathbb{C}$. Then $A(\alpha)$ is a linear map over $\mathbb{C}$, because the field $\mathbb{C}$ satisfies the associative, commutative and distributive properties. In particular for any $w, z, c \in \mathbb{C}$ $A(\alpha)(w + cz) = \alpha(w + cz) = \alpha w + c\alpha z = A(\alpha)w + cA(\alpha)z$.

It follows that this map is also a linear map on $\mathbb{C}$ as a real vector space, by restricting $c \in \mathbb{R}$.

Now assume that $T : \mathbb{C} \to \mathbb{C}$ is a linear transformation over $\mathbb{C}$. Then for all $z \in \mathbb{C}$, $T(z) = zT(1)$. If $\alpha = T(1)$, then $T = A(\alpha)$.

As a counter example complex conjugation $x + iy \mapsto x - iy$ is a linear transformation of $\mathbb{C}$ as a real vector space that is not equal to $A(\alpha)$ for any $\alpha \in \mathbb{C}$.

Let $\alpha = c + di \neq 0$. In the usual basis $A(\alpha) = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. The change of basis matrix to the basis $\{1, w\}$ is $S = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$. In the new basis $A(\alpha) = S \begin{pmatrix} c & -d \\ d & c \end{pmatrix} S^{-1}$. Since associated matrices have the same determinant $\det A(\alpha) = c^2 + d^2 = |\alpha|^2 > 0$.

Problem 2.

From linear algebra the composition of linear maps is linear, thus $GL_K(L)$ is closed under composition. Composition of maps is associative. By definition every element in the space has an inverse. The trivial map is the identity. Thus $GL_K(L)$ is a group under composition.

It is easy to see that $GL_L(L) \subset GL_K(L)$ is a subgroup. First of all any $L-$linear map is also $K - linear$ thus it is a subset. Seconde the composition of two $L-$linear maps is also $L-linear$ thus it is also a subgroup.

Assume that $GL_L(L) = GL_K(L)$. Assume that $[L : K] = n > 1$ and let $\{e_i\}$ be a $K-$basis of L. Then a linear transformation of L is represented as a matrix in the chosen basis. As in problem 1 we can see that the linear transformations in $GL_L(L)$ correspond to multiplication by a non zero element of L. Consider the transformation corresponding to the following matrix

$$\begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & I_{n-2} \end{pmatrix}$$

Above I_{n-2} is the $(n-2) \times (n-2)$ identity matrix (if $n = 2$ we just ignore it). This transformation adds e_1 to e_2 and fixes all other vectors (if they exist). It is easy to see that such transformation cannot correspond to multiplication by $l \in L$.

Let $\sigma : L \to L$ be a homomorphism in H. Since $ker\sigma$ is an ideal it must be injective. Let $G \subset H$ is the subset of isomorphisms that are trivial on K. By definition of ring isomorphisms $\sigma(s + t) = \sigma(s) + \sigma(t)$ for all $s, t \in L$. Also if $\sigma \in G$, $\sigma(ks) = \sigma(k)\sigma(s) = k\sigma(s)$ for all $k \in K$ and $s \in L$. Thus $\sigma \in GL_K(L)$.