1. (a) Prove by induction that, for all $n > 0$,
\[
\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}.
\]

Solution. Write $A(n) = \sum_{i=1}^{n} (-1)^i i^2$, $B(n) = (-1)^n \frac{n(n+1)}{2}$. Base case: for $n = 1$ one sees by inspection that $A(n) = -1$ and $B(n) = -1$.

Induction step: Suppose it is known for $n$, we prove it for $n + 1$. For all $n$ we have
\[
A(n + 1) = A(n) + (-1)^{n+1} (n + 1)^2;
\]
\[
B(n + 1) = B(n) + (-1)^{n+1} \frac{(n + 1)(n + 2)}{2} - (-1)^n \frac{n(n+1)}{2}
\]
\[
= B(n) + (-1)^{n+1} \left[ \frac{(n + 1)(n + 2)}{2} + \frac{(n + 1)(n)}{2} \right].
\]

Assuming $A(n) = B(n)$, it thus suffices to show that
\[
(-1)^{n+1} (n + 1)^2 = (-1)^{n+1} \left[ \frac{(n + 1)(n + 2)}{2} + \frac{(n + 1)(n)}{2} \right];
\]
in other words, that
\[
(n + 1)^2 = \frac{(n + 1)(n + 2) + (n)(n + 1)}{2}
\]
which is easy.

(b) For any $n \geq 1$ let $X_n = \{x \in \mathbb{N}, 1 \leq x \leq n\}$. We consider $X_n$ as a subset of $X_{n+1}$.

(i) Define an injective map $j : P(X_n) \hookrightarrow P(X_{n+1})$ and describe its image as a subset of $P(X_{n+1})$

(ii) Define a bijection between $j(P(X_n))$ and $P(X_{n+1}) \setminus j(P(X_n))$.

(iii) Prove by induction that $|P(X_n)| = 2^n$. 

Solution. (i) Since $X_n \subseteq X_{n+1}$, any subset of $X_n$ is a subset of $X_{n+1}$. This defines $j$. The image of $j$ is the set of subsets $S \subseteq X_{n+1}$ such that $n+1 \not\in S$.

(ii) By part (i), $P(X_{n+1}) \setminus j(P(X_n)) = \{ T \subseteq X_{n+1} \mid n + 1 \in T \}$. We define a bijective map

$$\alpha : j(P(X_n)) \to P(X_{n+1}) \setminus j(P(X_n)) \mid \alpha(S) = S \cup \{n + 1\}.$$ 

This is clearly injective and has inverse

$$\beta : P(X_{n+1}) \setminus j(P(X_n)) \to j(P(X_n)) \mid \beta(T) = T \setminus \{n + 1\}.$$ 

(iii) Base case: For $n = 1$, $P(X_1)$ has two elements: $X_1$ and $\emptyset$.

Induction step: Suppose we know that $|P(X_n)| = 2^n$. Since $j$ is injective, $|j(P(X_n))| = 2^n$. Now

$$|P(X_{n+1})| = |j(P(X_n))| + |P(X_{n+1}) \setminus j(P(X_n))| = 2|j(P(X_n))|$$

because the two subsets of $P(X_{n+1})$ are in bijection by part (ii). Thus

$$|P(X_{n+1})| = 2|j(P(X_n))| = 2 \cdot 2^n = 2^{n+1}$$

and this completes the induction step.

(c) Prove by induction that for all $n \geq 4$, $n! > 2^n$.

Solution. Base case: For $n = 4$, $n! = 24$, $2^4 = 16$.

Induction step: Suppose $n! > 2^n$. Then

$$(n + 1)! = (n + 1) \cdot n! > (n + 1) \cdot 2^n > 2 \cdot 2^n = 2^{n+1}$$

because $n + 1 > 2$.

(d) (Extra credit) Prove by induction that all natural numbers are interesting.

2. (a) Let $I = (-1, 1) \subset \mathbb{R}$ and let $f : I \to [1, \infty)$ be a continuous function. Prove carefully that

$$\lim_{n \to \infty} f((-1)^n \frac{1}{n}) = f(0).$$

Solution. Set $a_n = f((-1)^n \frac{1}{n})$, for $n \geq 1$, and set $L = f(0)$. Let $\varepsilon > 0$. Because $f$ is continuous at 0, there is $\delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |f(x) - L| = |f(x) - f(0)| < \varepsilon.$$ 

There exists an integer $N$ such that $N > \frac{1}{\delta}$. For $n \geq N$,

$$|(-1)^n \frac{1}{n} - 0| = \frac{1}{n} < \delta.$$ 

Thus for $n \geq N$,

$$|f(x) - L| < \varepsilon$$

which proves the claim.
(b) Can there be a continuous function \( g : I \to \mathbb{R} \) such that 
\[ g\left(\frac{1}{n}\right) = (-1)^n f\left(\frac{1}{n}\right) \]?

Explain your answer.

Solution. Suppose there were such a function. Then for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that
\[ |g(x) - g(0)| < \epsilon \quad \forall x \in (-\delta, \delta). \]
In particular, if \( n > \frac{1}{\delta} \) then
\[ |(-1)^n f\left(\frac{1}{n}\right) - g(0)| < \epsilon. \]

Take \( \epsilon = 1 \). Now for all \( n \), \( f\left(\frac{1}{n}\right) \geq 1 \) by hypothesis, so if \( n \) is even then \( (-1)^n f\left(\frac{1}{n}\right) \geq 1 \) but if \( n \) is odd then \( (-1)^n f\left(\frac{1}{n}\right) < -1 \). This means that if \( g(0) \geq 0 \) then \( |(-1)^n f\left(\frac{1}{n}\right) - g(0)| \geq 1 = \epsilon \) for all odd \( n \), which contradicts the above inequality; but if \( g(0) < 0 \) then \( |(-1)^n f\left(\frac{1}{n}\right) - g(0)| \geq 1 = \epsilon \) for all even \( n \). So there can be no such continuous \( g \).

3. (a) Let \( D \) denote the set of Dedekind cuts. Define the half-closed interval \([0, 1)\) and the open interval \((0, 1)\) explicitly as subsets of \( D \).

Solution. First, \((0, 1)\) is the set of \( L \in D \) satisfying the following three conditions: first
\[ \forall a \in \mathbb{Q} \ (a \leq 0) \Rightarrow a \in L; \]
second, there exists \( r \in \mathbb{Q} \), \( r < 1 \) such that \( r \notin L \); and finally, there exists \( a \in \mathbb{Q} \), \( a > 0 \) such that \( a \in L \).

Next, \([0, 1)\) is the set of \( L \in D \) that satisfy the first two conditions above but not the last.

(b) (This is a challenging problem, much more difficult than anything you are likely to see on the exam.) Let \( a < b \) be rational numbers, and let \( f : (a, b) \to \mathbb{R} \) be a continuous function. We suppose there are \( a', b' \in (a, b) \), \( a' < b' \), such that \( f(a') < 0 \) and \( f(b') > 0 \). Finally, we suppose \( f \) is strictly increasing: if \( x, y \in (a, b) \), \( x < y \Rightarrow f(x) < f(y) \).

(i) Let \( L \subset \mathbb{Q} \) be the set of all rational numbers in \((\infty, a]\), together with the set of all rational numbers \( x \in (a, b) \) such that \( f(x) < 0 \). Show that \( L \) is a Dedekind cut. (Hint: suppose \( L \) has a maximum, say \( x_0 \), and consider \( \epsilon = -\frac{f(x_0)}{2} \).)

(ii) Show that \( L \), viewed as a real number, belongs to \((a, b)\). Show that \( f(L) = 0 \).

(To be discussed during the review.)

(c) Extra practice: Work out the exercises 8.10, 8.11, 8.12, 8.13, 8.14, 8.15 in Dumas-McCarthy.

4. (a) Define surjective functions \( f : \mathbb{N} \to \mathbb{Z} \) and \( g : \mathbb{Z} \to \mathbb{N} \).
Solution. We can take $f(0) = 0$ and for $i > 0$ we take $f(2i - 1) = i$, $f(2i) = -i$. We can take $g(x) = x$ for $x > 0$ and $g(y) = 0$ for $y \leq 0$.

(b) Let $A$, $B$, and $C$ be the intervals in $\mathbb{R}$ given by $A = (0, 1]$, $B = [1, \infty)$, $C = [1, 2)$. (i) Construct bijections $f : A \to B$ and $g : A \to C$.

(ii) Construct a collection of sets $C_i, i \geq 1$, and bijections $g_i : C \to C_i$, $h : B \to \bigcup_{i \geq 1} C_i$

(iii) Show that $A$ has a bijection with a countable infinite union of copies of itself.

Solution. (i) Define $f(x) = \frac{1}{x}$, $g(x) = 2 - x$.

(ii) Define $C_i = [i, i + 1)$. Then $B = \bigcup C_i$ is a partition of $B$ so the function $h$ just takes $x$ to itself. Meanwhile, $g_i(x) = x + i - 1$.

(iii) Since $A$ is in bijection with $C$, and $C$ is in bijection with each $C_i$, we see that for each $i$ there is a bijection of $A$ with $C_i$ given by $g_i \circ g$. On the other hand, $f$ is a bijection of $A$ with $B$ which is in bijection, via $h$, with the infinite union of half-open intervals $C_i$, each of which is a copy of $A$.

5. (a) (i) Show that for all $x \in \mathbb{R}$, $|x^2 - 1| \leq x^2 + 1$.

(ii) Define $f : \mathbb{R} \to [-1, 1)$ by $f(x) = \frac{x^2 - 1}{x^2 + 1}$. Show that $f$ is continuous and surjective.

Solution. (i) Either $|x| \leq 1$ or $|x| > 1$. If $|x| \leq 1$ then $0 \leq x^2 \leq 1$ and so $0 \leq |x^2 - 1| \leq 1$, but $x^2 + 1 \geq 1$. On the other hand, if $|x| > 1$ then $|x^2 - 1| = x^2 - 1 < x^2 + 1$.

(ii) It follows from (i) that $|\frac{x^2 - 1}{x^2 + 1}| = \frac{|x^2 - 1|}{x^2 + 1} \leq 1$. Moreover, $f(x) = 1$ is impossible, so the image of $f$ is contained in the half-open interval $[-1, 1)$. It is continuous because it is the quotient of two continuous functions and the denominator never takes the value 0.

To show that $f$ is surjective, we let $a \in [-1, 1)$ and solve the equation

$$f(x) = \frac{x^2 - 1}{x^2 + 1} = a.$$ 

This gives us

$$x^2 - 1 = ax^2 + a; \ (1 - a)x^2 = a + 1$$

and thus $x = \pm \sqrt{\frac{a + 1}{1 - a}}$ is a solution, provided the expression on the right has a square root. First of all, note that the denominator never equals 0 because $1 \notin [-1, 1)$. Now the numerator is never negative if $a \geq -1$, so there is a real square root provided the denominator is positive; and this is true provided $a < 1$. Note that it is impossible for both numerator and denominator to be negative.

(b) (i) Does there exist an injective function $f : \mathbb{N} \to \mathbb{N}$ such that $(\forall i \in \mathbb{N})f(i) > f(i + 1)$? Explain.
(ii) Let $I$ denote the open interval $(0, 1) \subset \mathbb{R}$. Does there exist an injective function $f : \mathbb{N} \to I$ such that $(\forall i \in \mathbb{N}) f(i) > f(i + 1)$? Explain.

**Solution.** (i) There is no such function. Suppose there were such an $f$. Consider $N = f(0)$. It follows by induction that for all $i > 0$, $f(i) < f(0) = N$. Thus $f$ is an injective function from the infinite set $\mathbb{N}$ to the finite set $\{0, 1, \ldots , N\}$, which is impossible.

(ii) Yes. For example, we can take $f(n) = \frac{1}{2n+1}$.

(c) Let $Y$ be the set $\{1, 2, \ldots , m\}$. Prove by induction that the cardinality of the set of functions from $\{1, \ldots , n\}$ to $Y$ has cardinality $m^n$.

This has been done in class.

6. (a) Definitions: Define *bound variable*, *free variable*, *tautology*, *characteristic set*.

(b) Use truth tables to show that the following statement is not a tautology.

$$(P \land (\neg Q \lor R)) \iff \neg (R \Rightarrow (Q \land \neg P))$$

(One example suffices.)

**Solution.** If you substitute $T$ for $P$ and $F$ for $Q$ and $R$ you get $F$.

7. (a) Let $n > 0$ be a positive integer and let $\mathbb{Z}/n\mathbb{Z}$ be the set of congruence classes modulo $n$. Define a relation $R$ on $\mathbb{Z}/n\mathbb{Z}$:

$$aRb \iff ab \equiv 0 \pmod{n}.$$  

Is this relation reflexive, symmetric, or transitive for all $n$? For some $n$?

**Solution.** It is not reflexive for $n > 1$ (it is not true that $1R1$ if $n > 1$) but it is reflexive for $n = 1$. It is symmetric. Since it is not reflexive but $aR0$ for all $a$ it is also not transitive, again except when $n = 1$.

(b) Find $c = \gcd(3075, 3649)$ and find $m, n \in \mathbb{Z}$ such that $3075m + 3649n = c$.

**Solution.** The GCD is 41, computed as follows:

$$3649 - (3075 \times 1) = 574$$
$$3075 - (574 \times 5) = 205$$
$$574 - (205 \times 2) = 164$$
$$205 - (164 \times 1) = 41$$
$$164 - (41 \times 4) = 0$$

We work backwards from the next to last line:

$$41 = 205 - 1 \cdot 164 = 205 - 1 \cdot [574 - 2 \cdot 205] = -1 \cdot 574 + 3 \cdot 205.$$  

Continuing:

$$41 = -1 \cdot 574 + 3 \cdot [3075 - 5 \cdot 574] = 3 \cdot 3075 - 16 \cdot 574;$$
8. True or false? Justify your answer.
   (a) For (i) $S = \mathbb{R}$ and (ii) $S = \mathbb{N}$, determine the truth or falsity of the following sentence:
   \[(\forall x \in S)(\forall y \in S)x < y \Rightarrow (\exists z \in S)(x < z) \land (z < y).\]
   (i) True (take $z = \frac{x+y}{2}$); (ii) False: if $y = x + 1$ there is no natural number in between.

   (b) If $a, b \in \mathbb{N}$ then
   \[(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(\forall c \in \mathbb{N}) \ c|\gcd(a, b) \Rightarrow (\forall m \in \mathbb{N}) \ c|(ma - b).\]
   True.

   (c) Let $f : (0, 2) \to \mathbb{R}$ be a continuous function, where $(0, 2)$ is the open interval. Then there is a real number $C > 0$ and $\delta > 0$ such that, for all $x \in (1 - \delta, 1 + \delta)$, $|f(x)| < C$.
   True. Let $M = f(1)$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that
   \[|x - 1| < \delta \Rightarrow |f(x) - M| < \varepsilon\]
   and take $C = M + \varepsilon$.

   (d) Any compound statement is propositionally equivalent to one that contains only atomic statements and the propositional connectives $\neg$ and $\lor$.
   True: one can use De Morgan’s laws to replace occurrences of $\land$ by occurrences of $\neg$ and $\lor$: $P \land Q$ is equivalent to $\neg(\neg(P \lor Q))$. Similarly, $P \Rightarrow Q$ is equivalent to $Q \lor \neg P$. 