Review for Midterm II, spring 2016: solutions

Ex. 1. (i) - (iii)
Solutions: omitted (definition or done in class).

Ex. 1 (iv): Show that limits are unique.

Solution: Suppose \( \lim_{x \to a} f(x) = M \) and \( \lim_{x \to a} f(x) = N \). We need to show that \( M = N \). Instead we will show that, for any \( \varepsilon > 0 \), \( |M - N| < \varepsilon \). Thus the difference between them is smaller than every positive number, and this implies that they must be equal.

By hypothesis, for any \( \varepsilon > 0 \), there exists \( \delta_1 \) such that

\[
0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - M| < \varepsilon/2
\]

and

\[
0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x) - N| < \varepsilon/2
\]

Let \( \delta = \min(\delta_1, \delta_2) \). Thus

\[
0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - M| < \varepsilon/2 \text{ AND } |f(x) - N| < \varepsilon/2.
\]

It follows that

\[
0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - M| + |f(x) - N| < \varepsilon.
\]

Thus by the triangle inequality

\[
|M - N| = |M - f(x) + f(x) - N| \leq |f(x) - M| + |f(x) - N| < \varepsilon,
\]

which is what we wanted to prove.

Ex. 2.
Omitted because we didn’t cover limits of sequences.

Ex. 3
Solutions: We omit 3. (i) which is a definition.

(ii) We prove

\[
S(k) = 1 + 5 + \cdots + (4k + 1) = (k + 1)(2k + 1)
\]

The case \( k = 0 \) is obvious. Suppose we know it for \( k \). Then

\[
S(k+1) = S(k) + 4(k+1) + 1 = (k+1)(2k+1) + 4(k+1) + 1 = (k+2)(2k+3)
\]

as one verifies by simple algebra.

(iii) Define the Fibonacci sequence by \( F_1 = 1, F_2 = 1, F_3 = 2, F_{n+1} = F_n + F_{n-1} \). Prove by induction that for all \( k \geq 1 \), \( F_{5k} \) is divisible by 5.
Proof by induction: First $F_4 = 2 + 1 = 3$, $F_5 = 3 + 2 = 5$ so it’s true for $k = 1$.

Now for general $n$ we know We know that

$$F_{n+5} = F_{n+4} + F_{n+3} = F_{n+3} + F_{n+2} + F_{n+3}.$$

Substituting $F_{n+3} = F_{n+2} + F_{n+1}$ we find

$$F_{n+5} = 3F_{n+2} + 2F_{n+1}.$$

Substituting $F_{n+2} = F_n + F_{n+1}$ we find

$$F_{n+5} = 3(F_n + F_{n+1}) + 2F_{n+1} \equiv 3F_n \pmod{5}.$$

Now suppose $F_{5k}$ is divisible by 5; then

$$F_{5(k+1)} = F_{5k+5} \equiv 3F_{5k} \pmod{5}$$

is also divisible by 5.

Prove by induction on $n$ that when $x > 0$ we have

$$(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2.$$

Proof: It’s clearly true for $n = 1$. Suppose it’s true for $n$. Then

$$(1 + x)^{n+1} = (1 + x)(1 + x)^n \geq (1 + x)[1 + nx + \frac{n(n-1)}{2}x^2]$$

and when we work out the right-hand side we find this is

$$1 + (n+1)x + \frac{n(n-1)}{2}x^2 + nx^2 + \frac{n(n-1)}{2}x^2 \geq 1 + (n+1)x + \left[\frac{n(n-1)}{2} + n\right]x^2$$

and it is now obvious that it’s true for $n + 1$.

Ex. 4 (i) Omitted

(ii) Find integers $m$, $n$ such that $14m + 13n = 7$.

Solution: Obviously $14 \cdot 1 + 13 \cdot (-1) = 1$. Multiply both sides by 7 to find the solution $m = 7, n = -7$.

(iii) Find the simplest proof of the fact that if we define $\gcd(a, b)$ to be the largest integer that divides both $a$ and $b$, then if $s \mid a$ and $s \mid b$ then $s$ divides the gcd of $a$ and $b$.

This is a somewhat ambiguous question: what is the “simplest” proof? Probably the proof uses the fact that if we let $c$ be the largest integer dividing both $a$ and $b$, then there are integers $m$ and $n$ such that

$$c = ma + nb.$$

It’s clear that if $s$ divides $a$ and $b$ then $s$ divides $ma + nb$, and therefore divides $c$. The problem with the wording is that it’s not clear whether or not the proof includes the proof that $c$ can be written in the indicated way.

If we write $a = ic$ and $b = jc$ then one can prove that $i$ and $j$ have no common factor – otherwise $a$ and $b$ would have a common divisor larger
than $c$ (this fact also requires proof, but it is easy). Then Bezout’s lemma implies that there are $m$ and $n$ such that $mi + nj = 1$. An easy argument (that nevertheless needs to be written down) then implies $ma + nb = c$.

I’m not sure whether or not this was the expected answer.

Ex. 5. (i) False: This is not even a linear ordering. There is no order relation between $(1,2)$ and $(2,1)$.

(ii) False: If $L = \lim_{x \to 0} f(x)$ exists then for $n > \frac{1}{\delta}$ we must have $|f(\frac{1}{n}) - L| < \varepsilon$ when $\delta$ and $\varepsilon$ are given by the usual conditions. In particular, $f(\frac{1}{n}) \leq L + 1$ for sufficiently large $n$, but this is contradicted by the assumption.

(iii) Not covered in class. (But the claim is true.)