MATH V2000: Review for final, May 2016

Here are answers to some of the questions.

Question 1:

(ii) (8 points) Give a careful proof by induction on n that

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \geq 1$.

Base case: n = 1 gives $\frac{1}{2} = \frac{1}{2}$, which is clear.

Inductive step: Suppose that $\frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. We must prove that $\frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$.

But

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \quad by \quad bind \quad Hyp = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}$$

as required.

(iii) (10 points) Find integers m, n such that 14m + 13n = 7.

I seem to specialize in giving you ones of these you can do in your head! If m = 1 and n = -1 then $14 \cdot 1 + 13 \cdot (-1) = 1$. So take m = 7, n = -7. The general method is to use the Euclidean algorithm to find the gcd and then "back solve". There is a better example of this in question 7(ii).

Question 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. (iii) (8 points) Show that if $\lim_{x \to a} f(x) = L$ then f is bounded near a, i.e. there are constants C, M > 0 so that |f(x)| < M for all x such that 0 < |x - a| < C.

By definition of limit, if we take $\epsilon = 1$ there is $\delta > 0$ so that $0 < |x - a| < \delta$ implies |f(x) - L| < 1. But by the triangle inequality

$$|f(x)| \le |f(x) - L + L| \le |f(x) - L| + |L| \le 1 + |L|.$$

So we may take $C = \delta$ and M = 1 + |L|.

Notice that the question was slightly wrong – I wrote the condition on x as |x - a| < C instead of 0 < |x - a| < C.

Question 3: (iii) (7 points) If L is Dedekind cut, is the set $\{x^2 : x \in L\}$ a Dedekind cut? What about the set $\{0\} \cup \{\frac{1}{x} : x \in L \setminus 0\}$?

Neither of these need be a Dedekind cut. For example if $L = (-\infty, -1) \cap \mathbb{Q}$, then the set $\{x^2 : x \in L\}$ consists only of positive numbers and so fails condition (III): for example $4 \in \{x^2 : x \in L\}$ but -1 < 4 is not in this set.

Further with this L the set $\{0\} \cup \{\frac{1}{x} : x \in L \setminus 0\}$ is contained in the interval (-1,0) and does not contain -2.

(In fact there is no L for which these sets are Dedekind cuts.)

Question 4:

(i) (10 points) Let $m, n \in \mathbb{N}$. Prove by induction on m that if there is a bijection $\lceil m \rceil \rightarrow \lceil n \rceil$ then m = n.

This is bookwork – look at the proof on p 152-153 in DM.

(ii)(10 points) Are there surjections $\mathbb{R} \to \mathbb{N}$ or $\mathbb{N} \to \mathbb{R}$? Explain your answer.

There is a surjection $\mathbb{R} \to \mathbb{N}$ eg define $f(x) = |\lfloor x \rfloor|$, where $\lfloor x \rfloor|$ is the largest integer that is less than or equal to x.

There is no surjection $\mathbb{N} \to \mathbb{R}$ since \mathbb{R} is uncountable and so has cardinality strictly > that of $|\mathbb{N}|$.

One argument: Identify \mathbb{R} with the set of infinite decimals and suppose that $f : \mathbb{N} \to \mathbb{R}$ is any map. Then define the decimal $x := r_0 \cdot r_1 r_2 r_3 \dots$ as follows:

 $r_0 = 1$ if the number f(0) does not have 1 in the units place, $r_0 = 2$ otherwise,

 $r_1 = 1$ if the number f(1) does not have 1 in the first decimal place, $r_1 = 2$ otherwise, and in general

 $r_k = 1$ if the number f(k) does not have 1 in the kth decimal place, $r_k = 2$ otherwise, Then the real number x is not equal to f(k) for any $k \in \mathbb{N}$. Hence f is NOT surjective.

Question 5: Let X, Y, Z be any sets and $f : X \to Y$, $g : Y \to Z$ be functions. (ii) (5 points) Show that if $g \circ f$ is injective then so is f. – bookwork

- (iii) (5 points) If $g \circ f$ is surjective must g be surjective? YES you should give a proof. eg Let $z \in Z$. Since $g \circ f$ is surjective there is $x \in X$ such that $g \circ f(x) = g(f(x)) = z$. Therefore g(y) = z where y = f(x). Hence g is surjective
- (iv) (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function f(x) = x(x-1)(x-2); note that f(3) = 6.
 - What is $f^{-1}([0,6])$? $[0,1] \cup [2,3]$.
 - If A = [-1,0] and B = [2,3] what are $f(A) \cup f(B)$, $f(A) \cap f(B)$? $f(A) \cup f(B) = [-6,6]$ and $f(A) \cap f(B) = \{0\}$.
 - Find two distinct intervals C, D such that $\emptyset \neq f(C) \cap f(D) = f(C \cap D)$. Take C = [-1, 0] and $D = [0, \frac{1}{2}]$.

Question 6: (5 points each) Consider the statement

 $P: \quad \exists x \in \mathbb{R}, \text{ such that } \forall y \in \mathbb{R}, x^2 > y \Longrightarrow x > y.$

(i) Write down its negative in a form that does not involve any negations.

for all $x \in \mathbb{R}$, there is $y \in \mathbb{R}$ such that $x^2 > y$ and $x \leq y$.

(ii) Is P true or false ?

P is true: take x = 1. (or anything > 1)

(iv) Show that $\sqrt{3}$ is an irrational number.

Argue by contradiction: Assume $\frac{p}{q} = \sqrt{3}$ where $\frac{p}{q}$ is in lowest terms. Then $p^2 = 3q^2$. Hence (by Fund Thm of arithmetic) 3|p, i.e. p = 3k. Then $9k^2 = 3q^2$ so $3k^2 = q^2$. So 3 must also divide q, which is a contradiction. **Question 7:** (i) (10 points) Let $a, b \in \mathbb{N}$. Show that gcd(a, b) is the smallest positive element in the set $\{ma + nb \mid m, n \in \mathbb{Z}\}$. You may use without proof that if the positive integers a, b are relatively prime, then there are integers m, n such that ma + nb = 1, and that if a > b then gcd(a - b, b) = gcd(a, b). But prove all other results that you use.

Let g = gcd(a, b) and $s = \text{smallest positive element in the set } \{ma + nb \mid m, n \in \mathbb{Z}\}$. We must show $g \leq s$ and $s \leq g$.

Since g|a and g|b, g is a divisor of every element of the form ma + nb. Hence g|s, Thus $g \leq s$.

Next we show that $gcd(\frac{a}{g}, \frac{b}{g}) = 1$. But if not, they have a common divisor d > 1. But $d|\frac{a}{g}$ implies that dg|a. Similarly, $d|\frac{b}{g}$ implies that dg|b. Therefore dg divides both a, b. Since g is the largest common divisor and $dg \ge g$ this means that dg = g, i.e. d = 1. Therefore $gcd(\frac{a}{g}, \frac{b}{g}) = 1$.

This means that there are m, n so that $m\frac{a}{g} + n\frac{b}{g} = 1$. So, multiplying by g we get ma + nb = g. Therefore $g \in \{ma + nb \mid m, n \in \mathbb{Z}\}$. Therefore $s \leq g$ (since s is smallest positive element in this set.)

This completes the proof.

(ii) (10 points) Find $c = \gcd(3999, 1419)$ and find $m, n \in \mathbb{Z}$ so that 3999m + 1419n = c. gcd = 129. m = 5 and n = -14,

Question 8: Are the following true or false? Give reasons for your answers.

(i) (5 points) Define a relation on the subsets of X by saying $A \ R \ B \iff A \cap B \neq \emptyset$. This is an equivalence relation.

False This relation is not transitive in general: eg $A \cap C$ could be empty, but both these sets could have nonempty intersection with B. eg for intervals in \mathbb{R} : take A = [0, 1], B = [1, 2] and C = [2, 3].

(ii) (5 points) Define a relation R on pairs $(a, b) \in \mathbb{N}^+ \times \mathbb{N}^+$ by setting

 $(a,b)R(c,d) \iff ab \ge cd.$

This is an order relation. (Here $\mathbb{N}^+ = \{n \in \mathbb{N} : n > 0\}$.)

False This is not antisymmetric. eg (4,9)R(18,18) and (18,18)R(4.9) but $(4,9) \neq (8,18)$.

(iii) (5 points) Given any $m, n \in N$ with gcd(m, n) > 1, we can write n uniquely as a product ab where gcd(m, a) = 1 and gcd(m, b) > 1.

False; given m, n there might several decompositions of this kind. eg if m = 15, n = 66 we could take a = 3, b = 22 or a = 6, b = 11.

(iv) (5 points) If X is uncountable and $f : \mathbb{N} \to X$ is any map, then $|X \setminus f(\mathbb{N})| = |X|$. True.

Since there is an obvious injection $X \setminus f(\mathbb{N}) \to X$, by the Sch-Bern theorem we only need show that there is an injection $g: X \to X \setminus f(\mathbb{N})$.

First note that $X \setminus f(\mathbb{N})$ is infinite. (since otherwise X is the union of the two countable sets $f(\mathbb{N})$ and $X \setminus f(\mathbb{N})$.) Therefore there is an injection $h : \mathbb{N} \to X \setminus f(\mathbb{N})$. Now define $g : X \to X \setminus f(\mathbb{N})$ as follows:

- if $x \in X \setminus (f(\mathbb{N}) \cup h(\mathbb{N}))$, put g(x) = x;
- choose an injection ι from the subset $f(\mathbb{N}) \subset X$ onto the odd numbers in \mathbb{N} , and if $x \in f(\mathbb{N})$, define $g(x) = h(\iota(x))$;
- if $x \in h(\mathbb{N})$, define $g(x) = h(2h^{-1}(x))$.

Then this is injective since the elements of $f(\mathbb{N})$ map to images of odd numbers under h, while the elements of $h(\mathbb{N})$ map to images of even numbers under h.