MATH V2000: Review for final, May 2016

Here are answers to some of the questions.

Question 1:

(ii) (8 points) Give a careful proof by induction on \( n \) that

\[
\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}
\]

for all \( n \geq 1 \).

Base case: \( n = 1 \) gives \( \frac{1}{2} = \frac{1}{2} \), which is clear.

Inductive step: Suppose that

\[
\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]

We must prove that

\[
\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.
\]

But

\[
\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2) + 1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}
\]

as required.

(iii) (10 points) Find integers \( m, n \) such that \( 14m + 13n = 7 \).

I seem to specialize in giving you ones of these you can do in your head! If \( m = 1 \) and \( n = -1 \) then \( 14 \cdot 1 + 13 \cdot (-1) = 1 \). So take \( m = 7, n = -7 \). The general method is to use the Euclidean algorithm to find the gcd and then "back solve". There is a better example of this in question 7(ii).

Question 2: Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a function and \( L \in \mathbb{R} \).

(iii) (8 points) Show that if \( \lim_{x \to a} f(x) = L \) then \( f \) is bounded near \( a \), i.e. there are constants \( C, M > 0 \) so that \( |f(x)| < M \) for all \( x \) such that \( 0 < |x - a| < C \).

By definition of limit, if we take \( \epsilon = 1 \) there is \( \delta > 0 \) so that \( 0 < |x - a| < \delta \) implies \( |f(x) - L| < 1 \). But by the triangle inequality

\[
|f(x)| \leq |f(x) - L + L| \leq |f(x) - L| + |L| \leq 1 + |L|.
\]

So we may take \( C = \delta \) and \( M = 1 + |L| \).

Notice that the question was slightly wrong – I wrote the condition on \( x \) as \( |x - a| < C \) instead of \( 0 < |x - a| < C \).

Question 3: (iii) (7 points) If \( L \) is Dedekind cut, is the set \( \{ x^2 : x \in L \} \) a Dedekind cut?

What about the set \( \{ 0 \} \cup \{ \frac{1}{x} : x \in L \setminus \{0\} \} \)?

Neither of these need be a Dedekind cut. For example if \( L = (-\infty, -1) \cap \mathbb{Q} \), then the set \( \{ x^2 : x \in L \} \) consists only of positive numbers and so fails condition (III); for example \( 4 \in \{ x^2 : x \in L \} \) but \( -1 < 4 \) is not in this set.

Further with this \( L \) the set \( \{ 0 \} \cup \{ \frac{1}{x} : x \in L \setminus \{0\} \} \) is contained in the interval \((-1,0)\) and does not contain \(-2\).

(In fact there is no \( L \) for which these sets are Dedekind cuts.)
Question 4:
(i) (10 points) Let \( m, n \in \mathbb{N} \). Prove by induction on \( m \) that if there is a bijection \([m] \to [n]\) then \( m = n \).

This is bookwork – look at the proof on p 152-153 in DM.

(ii) (10 points) Are there surjections \( \mathbb{R} \to \mathbb{N} \) or \( \mathbb{N} \to \mathbb{R} \)? Explain your answer.

There is a surjection \( \mathbb{R} \to \mathbb{N} \) eg define \( f(x) = |\lfloor x \rfloor| \), where \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \).

There is no surjection \( \mathbb{N} \to \mathbb{R} \) since \( \mathbb{R} \) is uncountable and so has cardinality strictly > that of \( |\mathbb{N}| \).

One argument: Identify \( \mathbb{R} \) with the set of infinite decimals and suppose that \( f : \mathbb{N} \to \mathbb{R} \) is any map. Then define the decimal \( x := r_0 \cdot r_1 r_2 r_3 \ldots \) as follows:
\[
r_0 = 1 \text{ if the number } f(0) \text{ does not have 1 in the units place},
\]
\[
r_1 = 1 \text{ if the number } f(1) \text{ does not have 1 in the first decimal place},
\]
and in general
\[
r_k = 1 \text{ if the number } f(k) \text{ does not have 1 in the } k\text{th decimal place},
\]

Then the real number \( x \) is not equal to \( f(k) \) for any \( k \in \mathbb{N} \). Hence \( f \) is NOT surjective.

Question 5: Let \( X, Y, Z \) be any sets and \( f : X \to Y, g : Y \to Z \) be functions.

(ii) (5 points) Show that if \( g \circ f \) is injective then so is \( f \). – bookwork

(iii) (5 points) If \( g \circ f \) is surjective must \( g \) be surjective? YES – you should give a proof.

eg Let \( z \in Z \). Since \( g \circ f \) is surjective there is \( x \in X \) such that \( g \circ f(x) = g(f(x)) = z \).

Therefore \( g(y) = z \) where \( y = f(x) \). Hence \( g \) is surjective.

(iv) (8 points) Let \( f : \mathbb{R} \to \mathbb{R} \) be the function \( f(x) = x(x-1)(x-2) \); note that \( f(3) = 6 \).

- What is \( f^{-1}([0, 6]) \)? \([0, 1] \cup [2, 3]\).
- If \( A = [-1, 0] \) and \( B = [2, 3] \) what are \( f(A) \cup f(B) \), \( f(A) \cap f(B) \)?
  \( f(A) \cup f(B) = [-6, 6] \) and \( f(A) \cap f(B) = \{0\} \).
- Find two distinct intervals \( C, D \) such that \( \emptyset \neq f(C) \cap f(D) = f(C \cap D) \).
  Take \( C = [-1, 0] \) and \( D = [0, \frac{1}{2}] \).

Question 6: (5 points each) Consider the statement
\[
P : \exists x \in \mathbb{R}, \text{ such that } \forall y \in \mathbb{R}, x^2 > y \implies x > y.
\]

(i) Write down its negative in a form that does not involve any negations.

for all \( x \in \mathbb{R} \), there is \( y \in \mathbb{R} \) such that \( x^2 > y \) and \( x \leq y \).

(ii) Is \( P \) true or false?

\( P \) is true: take \( x = 1 \). (or anything > 1)

(iii) \( \sqrt{3} \) is an irrational number.

Argue by contradiction: Assume \( \frac{p}{q} = \sqrt{3} \) where \( \frac{p}{q} \) is in lowest terms. Thrn \( p^2 = 3q^2 \).

Hence (by Fund Thm of arithmetic) \( 3|p \), i.e. \( p = 3k \). Then \( 9k^2 = 3q^2 \) so \( 3k^2 = q^2 \). So 3 must also divide \( q \), which is a contradiction.
**Question 7:** (i) (10 points) Let $a, b \in \mathbb{N}$. Show that $\gcd(a, b)$ is the smallest positive element in the set \( \{ma + nb \mid m, n \in \mathbb{Z}\} \). You may use without proof that if the positive integers $a, b$ are relatively prime, then there are integers $m, n$ such that $ma + nb = 1$, and that if $a > b$ then $\gcd(a - b, b) = \gcd(a, b)$. But prove all other results that you use.

Let $g = \gcd(a, b)$ and $s$是最小的正元素在集合 $\{ma + nb \mid m, n \in \mathbb{Z}\}$. We must show $g \leq s$ and $s \leq g$.

Since $g | a$ and $g | b$, $g$ is a divisor of every element of the form $ma + nb$. Hence $g | s$, Thus $g \leq s$.

Next we show that $\gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1$. But if not, they have a common divisor $d > 1$. But $d | \frac{a}{g}$ implies that $dg | a$. Similarly, $d | \frac{b}{g}$ implies that $dg | b$. Therefore $dg$ divides both $a, b$. Since $g$ is the largest common divisor and $dg \geq g$ this means that $dg = g$, i.e. $d = 1$. Therefore $\gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1$.

This means that there are $m, n$ so that $m \frac{a}{g} + n \frac{b}{g} = 1$. So, multiplying by $g$ we get $ma + nb = g$. Therefore $g \in \{ma + nb \mid m, n \in \mathbb{Z}\}$. Therefore $s \leq g$ (since $s$ is smallest positive element in this set.)

This completes the proof.

(ii) (10 points) Find $c = \gcd(3999, 1419)$ and find $m, n \in \mathbb{Z}$ so that $3999m + 1419n = c$.

\[ \gcd = 129. \quad m = 5 \text{ and } n = -14, \]

**Question 8:** Are the following true or false? Give reasons for your answers.

(i) (5 points) Define a relation on the subsets of $X$ by saying $A \mathbin{R} B \iff A \cap B \neq \emptyset$.

This is an equivalence relation.

False This relation is not transitive in general: eg $A \cap C$ could be empty, but both these sets could have nonempty intersection with $B$. eg for intervals in $\mathbb{R}$:

- take $A = [0, 1], B = [1, 2]$ and $C = [2, 3]$.

(ii) (5 points) Define a relation $R$ on pairs $(a, b) \in \mathbb{N}^+ \times \mathbb{N}^+$ by setting

\[(a, b)R(c, d) \iff ab \geq cd.\]

This is an order relation. (Here $\mathbb{N}^+ = \{n \in \mathbb{N} : n > 0\}$.)

False This is not antisymmetric. eg $(4, 9)R(18, 18)$ and $(18, 18)R(4, 9)$ but $(4, 9) \neq (8, 18)$.

(iii) (5 points) Given any $m, n \in \mathbb{N}$ with $\gcd(m, n) > 1$, we can write $n$ uniquely as a product $ab$ where $\gcd(m, a) = 1$ and $\gcd(m, b) > 1$.

False; given $m, n$ there might several decompositions of this kind. eg if $m = 15, n = 66$ we could take $a = 3, b = 22$ or $a = 6, b = 11$.

(iv) (5 points) If $X$ is uncountable and $f : \mathbb{N} \to X$ is any map, then $|X \setminus f(\mathbb{N})| = |X|$. True.

Since there is an obvious injection $X \setminus f(\mathbb{N}) \to X$, by the Sch-Bern theorem we only need show that there is an injection $g : X \to X \setminus f(\mathbb{N})$. 

First note that $X\setminus f(\mathbb{N})$ is infinite. (since otherwise $X$ is the union of the two countable sets $f(\mathbb{N})$ and $X\setminus f(\mathbb{N})$.) Therefore there is an injection $h : \mathbb{N} \to X\setminus f(\mathbb{N})$.

Now define $g : X \to X\setminus f(\mathbb{N})$ as follows:
- if $x \in X\setminus(f(\mathbb{N}) \cup h(\mathbb{N}))$, put $g(x) = x$;
- choose an injection $\iota$ from the subset $f(\mathbb{N}) \subset X$ onto the odd numbers in $\mathbb{N}$, and if $x \in f(\mathbb{N})$, define $g(x) = h(\iota(x))$;
- if $x \in h(\mathbb{N})$, define $g(x) = h(2h^{-1}(x))$.

Then this is injective since the elements of $f(\mathbb{N})$ map to images of odd numbers under $h$, while the elements of $h(\mathbb{N})$ map to images of even numbers under $h$. 