MATH V2000: Review for final, May 2016

There will be eight questions each worth 20 points, of which you should do six. (There is no advantage to answering more questions; I will grade only 6 of them.) Write complete proofs justifying all statements. The syllabus is

Ch 1.1-1.6

Ch 2.1-2.3; and 2.5

all of Ch 3

Ch 4.1-4.2

Ch 5.1, 5.2, 5.3 to middle of p 141 in the on line second edition (same as for midterm II)

Ch 6.1 - 6.4 (but not algebraic and transcendental numbers)

Ch 7.1 - 7.3

Ch 8.2 - 8.5 (sec 8.9, 8.10 – I did similar but not identical things in class; you are expected to know what I did in class and what is on the handout sheets, rather than what is in the text book – but you might find this interesting)

The exam is cumulative, with topics pretty evenly spread out.

I will expect to you to know the key definitions, and proofs of the key results, and will ask you about some of them.

As usual, we write $\mathbb{R}$ for the real numbers, $\mathbb{Q}$ for the rational numbers, $\mathbb{Z}$ for the integers and $\mathbb{N} = \{0, 1, 2, \dots\}$ for the natural numbers. Also $\lceil n \rceil = \{0, 1, \dots, n-1\}$.

Question 1: (i) (2 points) State the Principle of Mathematical induction.

(ii) (8 points) Give a careful proof by induction on n that

$$\tfrac{1}{2} + \tfrac{1}{2\cdot 3} + \cdots + \tfrac{1}{n(n+1)} = \tfrac{n}{n+1}$$

for all $n \geq 1$.

(iii) (10 points) Find integers m, n such that $14m + 13n = 7$.

Question 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $L \in \mathbb{R}$.

(i) (2 points) What does it mean to say $\lim_{x\to a} f(x) = L$?

(ii) (10 points) Show from this definition that $\lim_{x\to 1}(2x^2 + x) = 3$.

(iii) (8 points) Show that if $\lim_{x\to a} f(x) = L$ then f is bounded near a, i.e. there are constants $C, M > 0$ so that $|f(x)| < M$ for all x such that $|x - a| < C$.

Question 3: (i) (3 points) Define a Dedekind cut.

(ii) (10 points) Show that if L, M are Dedekind cuts, so is their sum $L + M = \{a + b \mid a \in L, b \in M\}$.

(iii) (7 points) If L is Dedekind cut, is the set $\{x^2 : x \in L\}$ a Dedekind cut? What about the set $\{0\} \cup \{\frac{1}{x} : x \in L \smallsetminus 0\}$?

Question 4:
(i) (10 points) Let $m, n \in \mathbb{N}$. Prove by induction on m that if there is a bijection $\lceil m \rceil \to \lceil n \rceil$ then $m = n$.
(ii)(10 points) Are there surjections $\mathbb{R} \to \mathbb{N}$ or $\mathbb{N} \to \mathbb{R}$? Explain your answer.

Question 5: Let X, Y, Z be any sets and $f : X \to Y$, $g : Y \to Z$ be functions.
(i) (2 points) What does it mean to say that f is injective, surjective?
(ii) (5 points) Show that if $g \circ f$ is injective then so is f.
(iii) (5 points) If $g \circ f$ is surjective must g be surjective?
(iv) (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x(x-1)(x-2)$; note that $f(3) = 6$.

- Sketch the graph of f.
- What is $f^{-1}([0, 6])$?
- If $A = [-1, 0]$ and $B = [2, 3]$ what are $f(A) \cup f(B)$, $f(A) \cap f(B)$?
- Find two distinct intervals C, D such that $\emptyset \neq f(C) \cap f(D) = f(C \cap D)$.

Question 6: (5 points each) Consider the statement

$$P : \quad \exists x \in \mathbb{R}, \text{ such that } \forall y \in \mathbb{R}, x^2 > y \Longrightarrow x > y.$$

(i) Write down its negative in a form that does not involve any negations.
(ii) Is P true or false ?
(iii) Show (using truth tables or otherwise) that the statement $P \vee (P \Rightarrow Q)$ is a tautology.
(iv) Show that $\sqrt{3}$ is an irrational number.

Question 7: (i) (10 points) Let $a, b \in \mathbb{N}$. Show that $\gcd(a, b)$ is the smallest positive element in the set $\{ma + nb \mid m, n \in \mathbb{Z}\}$. You may use without proof that if the positive integers a, b are relatively prime, then there are integers m, n such that $ma + nb = 1$, and that if $a > b$ then $\gcd(a - b, b) = \gcd(a, b)$. But prove all other results that you use.

(ii) (10 points) Find $c = \gcd(3999, 1419)$ and find $m, n \in \mathbb{Z}$ so that $3999m + 1419n = c$.

Question 8: Are the following true or false? Give reasons for your answers.

(i) (5 points) Define a relation on the subsets of X by saying $A \ R \ B \iff A \cap B \neq \emptyset$. This is an equivalence relation.

(ii) (5 points) Define a relation R on pairs $(a, b) \in \mathbb{N}^+ \times \mathbb{N}^+$ by setting

$$(a, b) R (c, d) \Longleftrightarrow ab \geq cd.$$

This is an order relation. (Here $\mathbb{N}^+ = \{n \in \mathbb{N} : n > 0\}$.)

(iii) (5 points) Given any $m, n \in N$ with $\gcd(m, n) > 1$, we can write n uniquely as a product ab where $\gcd(m, a) = 1$ and $\gcd(m, b) > 1$.

(iv) (5 points) If X is uncountable and $f : \mathbb{N} \to X$ is any map, then $|X \smallsetminus f(\mathbb{N})| = |X|$.