INTRODUCTION TO HIGHER MATHEMATICS V2000

Practice Final, December 2016

No use of written or printed materials, nor of electronic devices, is permitted during the exam. The exam will be graded on a curve. There will be eight questions and you should do six of them; there will be no extra credit for partial answers to the remaining questions.

All necessary steps should be clearly stated in proofs, and explanations should be provided where appropriate. Make your explanations as complete as possible.

1. (a) Prove by induction that, for all $n > 0$,

$$\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}.$$

(b) For any $n \geq 1$ let $X_n = \{x \in \mathbb{N}, 1 \leq x \leq n\}$. We consider X_n as a subset of X_{n+1}.

(i) Define an injective map $j : P(X_n) \hookrightarrow P(X_{n+1})$ and describe its image as a subset of $P(X_{n+1})$

(ii) Define a bijection between $j(P(X_n))$ and $P(X_{n+1}) \setminus j(P(X_n))$.

(iii) Prove by induction that $|P(X_n)| = 2^n$.

(c) Prove by induction that for all $n \geq 4$, $n! > 2^n$.

(d) (Extra credit) Prove by induction that all natural numbers are interesting.

2. (a) Let $I = (-1, 1) \subset \mathbb{R}$ and let $f : I \to [1, \infty)$ be a continuous function. Prove carefully that

$$\lim_{n\to\infty} f((-1)^n \frac{1}{n}) = f(0).$$

(b) Can there be a continuous function $g : I \to \mathbb{R}$ such that

$$g(\frac{1}{n}) = (-1)^n f(\frac{1}{n})?$$

Explain your answer.

3. (a) Let $\mathcal{D}$ denote the set of Dedekind cuts. Define the half-closed interval $[0, 1)$ and the open interval $(0, 1)$ explicitly as subsets of $\mathcal{D}$.

(b) (This is a challenging problem, much more difficult than anything you are likely to see on the exam.) Let $a < b$ be rational numbers, and let $f : (a, b) \to \mathbb{R}$ be a continuous function. We suppose there are $a', b' \in (a, b)$, $a' < b'$, such that $f(a') < 0$ and $f(b') > 0$. Finally, we suppose f is *strictly increasing*: if $x, y \in (a, b)$, $x < y \Rightarrow f(x) < f(y)$.

(i) Let $L \subset \mathbb{Q}$ be the set of all rational numbers in $(\infty, a]$, together with the set of all rational numbers $x \in (a, b)$ such that $f(x) < 0$. Show that L is a Dedekind cut. (Hint: suppose L has a maximum, say x_0, and consider $\varepsilon = -\frac{f(x_0)}{2}$.)

(ii) Show that L, viewed as a real number, belongs to (a, b). Show that $f(L) = 0$.

(c) Extra practice: Work out the exercises 8.10, 8.11, 8.12, 8.13, 8.14, 8.15 in Dumas-McCarthy.

4. (a) Define surjective functions $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{N}$.

(b) Let A, B, and C be the intervals in $\mathbb{R}$ given by $A = (0, 1]$, $B = [1, \infty)$, $C = [1, 2)$. (i) Construct bijections $f : A \to B$ and $g : A \to C$.

(ii) Construct a collection of sets $C_i, i \geq 1$, and bijections

$$g_i : C \to C_i, \;\; h : B \to \;\cup_{i \geq 1} C_i$$

(iii) Show that A has a bijection with a countable infinite union of copies of itself.

5. (a) (i) Show that for all $x \in \mathbb{R}$, $|x^2 - 1| \leq x^2 + 1$.
(ii) Define $f : \mathbb{R} \to [-1, 1)$ by $f(x) = \frac{x^2-1}{x^2+1}$. Show that f is continuous and surjective.

(b) (i) Does there exist an injective function $f : \mathbb{N} \to \mathbb{N}$ such that $(\forall i \in \mathbb{N}) f(i) > f(i + 1)$? Explain.

(ii) Let I denote the open interval $(0, 1) \subset \mathbb{R}$Does there exist an injective function $f : \mathbb{N} \to I$ such that $(\forall i \in \mathbb{N}) f(i) > f(i + 1)$? Explain.

(c) Let Y be the set $\{1, 2, \ldots, m\}$. Prove by induction that the cardinality of the set of functions from $\{1, \ldots, n\}$ to Y has cardinality m^n.

6. (a) Definitions: Define *bound variable*, *free variable*, *tautology*, *characteristic set*.

(b) Use truth tables to show that the following statement is not a tautology.

$$(P \wedge (\neg Q \vee R)) \Leftrightarrow \neg(R \Rightarrow (Q \wedge \neg P)))$$

(One example suffices.)

7. (a) Let $n > 0$ be a positive integer and let $\mathbb{Z}/n\mathbb{Z}$ be the set of congruence classes modulo n. Define a relation R on $\mathbb{Z}/n\mathbb{Z}$:

$$aRb \Leftrightarrow ab \equiv 0 \pmod{n}.$$

Is this relation reflexive, symmetric, or transitive for all n? For some n?

(b) Find $c = \gcd(3075, 3649)$ and find $m, n \in \mathbb{Z}$ such that $3075m + 3649n = c$.

8. True or false? Justify your answer.

(a) For (i) $S = \mathbb{R}$ and (ii) $S = \mathbb{N}$, determine the truth or falsity of the following sentence:

$$(\forall x \in S)(\forall y \in S) x < y \Rightarrow (\exists z \in S)(x < z) \wedge (z < y).$$

(b) If $a, b \in \mathbb{N}$ then

$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(\forall c \in \mathbb{N})\ c|\gcd(a,b) \Rightarrow (\forall m \in \mathbb{N})\ c|(ma - b).$$

(c) Let $f : (0, 2) \to \mathbb{R}$ be a continuous function, where $(0, 2)$ is the open interval. Then there is a real number $C > 0$ and $\delta > 0$ such that, for all $x \in (1 - \delta, 1 + \delta)$, $|f(x)| < C$.

(d) Any compound statement is propositionally equivalent to one that contains only atomic statements and the propositional connectives $\neg$ and $\vee$.