$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\mathbf{j}$$

$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j} = \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1, 0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ ($k$ an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

**17.** (a) $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \Rightarrow \mathbf{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1+9\sin^2 t + 9\cos^2 t} = \sqrt{10}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$ or $\left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{10}}\sqrt{0 + 9\cos^2 t + 9\sin^2 t} = \frac{3}{\sqrt{10}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{10}}{3/\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle = \langle 0, -\cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$

**18.** (a) $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle \Rightarrow$

$\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\cos^2 t + \sin^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$ [since $t > 0$]. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \Rightarrow$

$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$

**19.** (a) $\mathbf{r}(t) = \left\langle \sqrt{2}\,t, e^t, e^{-t}\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t + e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle = \dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle$ $\left[\text{after multiplying by } \frac{e^t}{e^t}\right]$ and

$$\mathbf{T}'(t) = \frac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle$$

$$= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle\right] = \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle$$

Then

$$|\mathbf{T}'(t)| = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1 + e^{2t}\right)^2} = \frac{\sqrt{2}\,e^t(1 + e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}$$

Therefore

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\,\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1-e^{2t}),\, 2e^{2t},\, 2e^{2t}\right\rangle$$

$$= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1-e^{2t}),\, 2e^{2t},\, 2e^{2t}\right\rangle = \frac{1}{e^{2t}+1}\left\langle 1-e^{2t},\, \sqrt{2}\,e^t,\, \sqrt{2}\,e^t\right\rangle$$

(b) $$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\frac{1}{e^t+e^{-t}} = \frac{\sqrt{2}\,e^t}{e^{3t}+2e^t+e^{-t}} = \frac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1} = \frac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$$

20. (a) $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, t, 2t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+5t^2}}\langle 1, t, 2t\rangle.$$

$$\mathbf{T}'(t) = \frac{-5t}{(1+5t^2)^{3/2}}\langle 1, t, 2t\rangle + \frac{1}{\sqrt{1+5t^2}}\langle 0, 1, 2\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(1+5t^2)^{3/2}}\left(\langle -5t, -5t^2, -10t^2\rangle + \langle 0, 1+5t^2, 2+10t^2\rangle\right) = \frac{1}{(1+5t^2)^{3/2}}\langle -5t, 1, 2\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+1+4} = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+5} = \frac{\sqrt{5}\,\sqrt{5t^2+1}}{(1+5t^2)^{3/2}} = \frac{\sqrt{5}}{1+5t^2}$$

Thus $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1+5t^2}{\sqrt{5}}\cdot\frac{1}{(1+5t^2)^{3/2}}\langle -5t, 1, 2\rangle = \frac{1}{\sqrt{5+25t^2}}\langle -5t, 1, 2\rangle.$$

(b) $$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{5}/(1+5t^2)}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

21. $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 3t^2\,\mathbf{j} + 2t\,\mathbf{k},\;\; \mathbf{r}''(t) = 6t\,\mathbf{j} + 2\,\mathbf{k},\;\; |\mathbf{r}'(t)| = \sqrt{0^2+(3t^2)^2+(2t)^2} = \sqrt{9t^4+4t^2},$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = -6t^2\,\mathbf{i},\;\; |\mathbf{r}'(t)\times\mathbf{r}''(t)| = 6t^2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{6t^2}{\left(\sqrt{9t^4+4t^2}\right)^3} = \dfrac{6t^2}{(9t^4+4t^2)^{3/2}}.$

22. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + e^t\,\mathbf{k},\;\; \mathbf{r}''(t) = 2\,\mathbf{j} + e^t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{1^2+(2t)^2+(e^t)^2} = \sqrt{1+4t^2+e^{2t}},\;\; \mathbf{r}'(t)\times\mathbf{r}''(t) = (2t-2)e^t\,\mathbf{i} - e^t\,\mathbf{j} + 2\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{[(2t-2)e^t]^2+(-e^t)^2+2^2} = \sqrt{(2t-2)^2e^{2t}+e^{2t}+4} = \sqrt{(4t^2-8t+5)e^{2t}+4}.$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{\left(\sqrt{1+4t^2+e^{2t}}\right)^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{(1+4t^2+e^{2t})^{3/2}}.$

23. $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2\sqrt{6}\,t\,\mathbf{i} + 2\,\mathbf{j} + 6t^2\,\mathbf{k},\;\; \mathbf{r}''(t) = 2\sqrt{6}\,\mathbf{i} + 12t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = \sqrt{4(3t^2+1)^2} = 2(3t^2+1),$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = 24t\,\mathbf{i} - 12\sqrt{6}\,t^2\,\mathbf{j} - 4\sqrt{6}\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{96(3t^2+1)^2} = 4\sqrt{6}\,(3t^2+1).$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{4\sqrt{6}\,(3t^2+1)}{8(3t^2+1)^3} = \dfrac{\sqrt{6}}{2(3t^2+1)^2}.$

and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} \dfrac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

31. Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1+(y'(x))^2]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}} = e^x(1+e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = e^x(1+e^{2x})^{-3/2} + e^x\left(-\tfrac{3}{2}\right)(1+e^{2x})^{-5/2}(2e^{2x}) = e^x\frac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}} = e^x\frac{1-2e^{2x}}{(1+e^{2x})^{5/2}}.$$

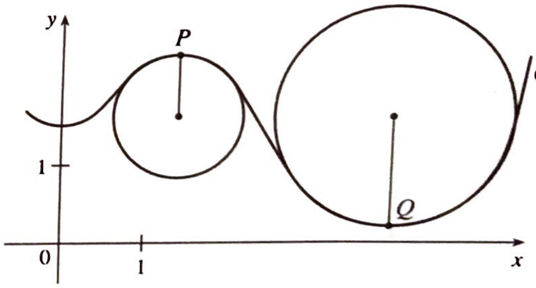
$\kappa'(x) = 0$ when $1-2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1-2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1-2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} e^x(1+e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

32. We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2$, $a > 0$. Then by Equation 11, $\kappa(x) = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1+(2ax)^2]^{3/2}} = \dfrac{2a}{(1+4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

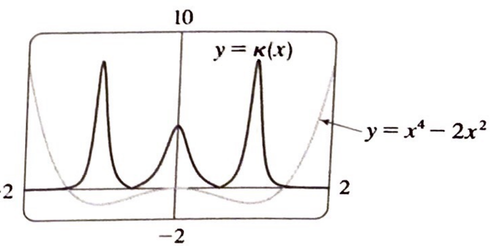
33. (a) $C$ appears to be changing direction more quickly at $P$ than $Q$, so we would expect the curvature to be greater at $P$.

(b) First we sketch approximate osculating circles at $P$ and $Q$. Using the axes scale as a guide, we measure the radius of the osculating circle at $P$ to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \Rightarrow \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at $Q$ to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.




34. $y = x^4 - 2x^2 \Rightarrow y' = 4x^3 - 4x$, $y'' = 12x^2 - 4$, and

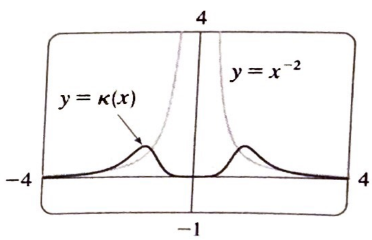
$\kappa(x) = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \dfrac{|12x^2-4|}{\left[1+(4x^3-4x)^2\right]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of $y = x^4 - 2x^2$ appears to be bending most sharply at the origin and near $x = \pm 1$.




35. $y = x^{-2} \Rightarrow y' = -2x^{-3}$, $y'' = 6x^{-4}$, and

$\kappa(x) = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \dfrac{|6x^{-4}|}{\left[1+(-2x^{-3})^2\right]^{3/2}} = \dfrac{6}{x^4\,(1+4x^{-6})^{3/2}}$.

The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^{-2}$ increases asymptotically at the origin from both directions, and so its graph has very little bend there. [Note that $\kappa(0)$ is undefined.]



**44.** $x = a\cos\omega t \;\Rightarrow\; \dot{x} = -a\omega\sin\omega t \;\Rightarrow\; \ddot{x} = -a\omega^2\cos\omega t,$

$y = b\sin\omega t \;\Rightarrow\; \dot{y} = b\omega\cos\omega t \;\Rightarrow\; \ddot{y} = -b\omega^2\sin\omega t.$ Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{|(-a\omega\sin\omega t)(-b\omega^2\sin\omega t) - (b\omega\cos\omega t)(-a\omega^2\cos\omega t)|}{[(-a\omega\sin\omega t)^2 + (b\omega\cos\omega t)^2]^{3/2}}$$
$$= \frac{|ab\omega^3\sin^2\omega t + ab\omega^3\cos^2\omega t|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}} = \frac{|ab\omega^3|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}}$$

**45.** $x = e^t\cos t \;\Rightarrow\; \dot{x} = e^t(\cos t - \sin t) \;\Rightarrow\; \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t\sin t,$

$y = e^t\sin t \;\Rightarrow\; \dot{y} = e^t(\cos t + \sin t) \;\Rightarrow\; \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t.$ Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{|e^t(\cos t - \sin t)(2e^t\cos t) - e^t(\cos t + \sin t)(-2e^t\sin t)|}{([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2)^{3/2}}$$
$$= \frac{|2e^{2t}(\cos^2 t - \sin t\cos t + \sin t\cos t + \sin^2 t)|}{[e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\cos t\sin t + \sin^2 t)]^{3/2}} = \frac{|2e^{2t}(1)|}{[e^{2t}(1+1)]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}$$

**46.** $f(x) = e^{cx},\; f'(x) = ce^{cx},\; f''(x) = c^2e^{cx}.$ Using Formula 11 we have

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|c^2e^{cx}|}{[1 + (ce^{cx})^2]^{3/2}} = \frac{c^2e^{cx}}{(1 + c^2e^{2cx})^{3/2}}$$ so the curvature at $x = 0$ is

$\kappa(0) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. To determine the maximum value for $\kappa(0)$, let $f(c) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. Then

$$f'(c) = \frac{2c\cdot(1 + c^2)^{3/2} - c^2\cdot\frac{3}{2}(1 + c^2)^{1/2}(2c)}{[(1 + c^2)^{3/2}]^2} = \frac{(1 + c^2)^{1/2}\left[2c(1 + c^2) - 3c^3\right]}{(1 + c^2)^3} = \frac{(2c - c^3)}{(1 + c^2)^{5/2}}.$$ We have a critical

number when $2c - c^3 = 0 \;\Rightarrow\; c(2 - c^2) = 0 \;\Rightarrow\; c = 0$ or $c = \pm\sqrt{2}$. $f'(c)$ is positive for $c < -\sqrt{2}$, $0 < c < \sqrt{2}$

and negative elsewhere, so $f$ achieves its maximum value when $c = \sqrt{2}$ or $-\sqrt{2}$. In either case, $\kappa(0) = \dfrac{2}{3^{3/2}}$, so the members

of the family with the largest value of $\kappa(0)$ are $f(x) = e^{\sqrt{2}x}$ and $f(x) = e^{-\sqrt{2}x}$.

**47.** $(1, \frac{2}{3}, 1)$ corresponds to $t = 1$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$.

$$\mathbf{T}'(t) = -4t(2t^2 + 1)^{-2}\langle 2t, 2t^2, 1\rangle + (2t^2 + 1)^{-1}\langle 2, 4t, 0\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$
$$= (2t^2 + 1)^{-2}\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\rangle = 2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\rangle = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle$.

**48.** $(1, 0, 0)$ corresponds to $t = 0$. $\mathbf{r}(t) = \langle \cos t, \sin t, \ln\cos t\rangle$, and in Exercise 4 we found that $\mathbf{r}'(t) = \langle -\sin t, \cos t, -\tan t\rangle$

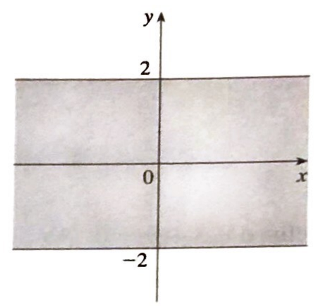
and $|\mathbf{r}'(t)| = |\sec t|$. Here we can assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and then $\sec t > 0 \;\Rightarrow\; |\mathbf{r}'(t)| = \sec t$.

9. (a) $g(2,-1) = \cos(2 + 2(-1)) = \cos(0) = 1$

(b) $x + 2y$ is defined for all choices of values for $x$ and $y$ and the cosine function is defined for all input values, so the domain of $g$ is $\mathbb{R}^2$.

(c) The range of the cosine function is $[-1, 1]$ and $x + 2y$ generates all possible input values for the cosine function, so the range of $\cos(x + 2y)$ is $[-1, 1]$.

10. (a) $F(3,1) = 1 + \sqrt{4 - 1^2} = 1 + \sqrt{3}$




(b) $\sqrt{4 - y^2}$ is defined only when $4 - y^2 \geq 0$, or $y^2 \leq 4 \quad \Leftrightarrow$ $-2 \leq y \leq 2$. So the domain of $F$ is $\{(x, y) \mid -2 \leq y \leq 2\}$.

(c) We know $0 \leq \sqrt{4 - y^2} \leq 2$ so $1 \leq 1 + \sqrt{4 - y^2} \leq 3$. Thus the range of $F$ is $[1, 3]$.

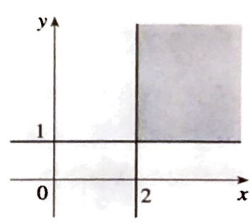
11. (a) $f(1,1,1) = \sqrt{1} + \sqrt{1} + \sqrt{1} + \ln(4 - 1^2 - 1^2 - 1^2) = 3 + \ln 1 = 3$

(b) $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ are defined only when $x \geq 0$, $y \geq 0$, $z \geq 0$, and $\ln(4 - x^2 - y^2 - z^2)$ is defined when $4 - x^2 - y^2 - z^2 > 0 \quad \Leftrightarrow \quad x^2 + y^2 + z^2 < 4$, thus the domain is $\{(x, y, z) \mid x^2 + y^2 + z^2 < 4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$, the portion of the interior of a sphere of radius 2, centered at the origin, that is in the first octant.

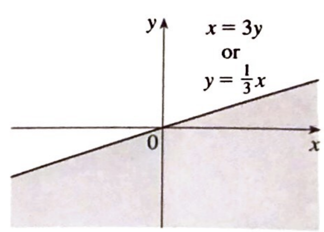
12. (a) $g(1,2,3) = 1^3 \cdot 2^2 \cdot 3\sqrt{10 - 1 - 2 - 3} = 12\sqrt{4} = 24$

(b) $g$ is defined only when $10 - x - y - z \geq 0 \quad \Leftrightarrow \quad z \leq 10 - x - y$, so the domain is $\{(x, y, z) \mid z \leq 10 - x - y\}$, the points on or below the plane $x + y + z = 10$.

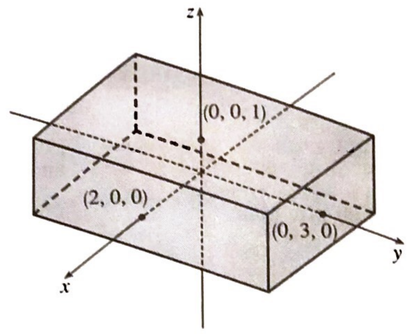
13. $\sqrt{x - 2}$ is defined only when $x - 2 \geq 0$, or $x \geq 2$, and $\sqrt{y - 1}$ is defined only when $y - 1 \geq 0$, or $y \geq 1$. So the domain of $f$ is $\{(x, y) \mid x \geq 2,\ y \geq 1\}$.



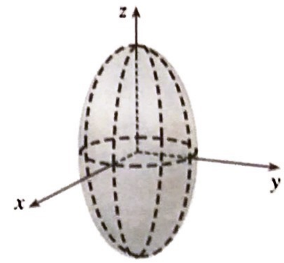

14. $\sqrt[4]{x - 3y}$ is defined only when $x - 3y \geq 0$, or $x \geq 3y$. So the domain of $f$ is $\{(x, y) \mid x \geq 3y\}$ or equivalently $\{(x, y) \mid y \leq \frac{1}{3}x\}$.



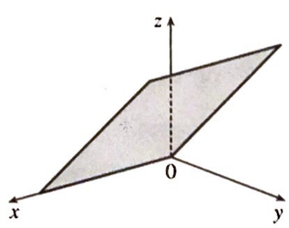
**21.** $f$ is defined only when $4 - x^2 \ge 0 \;\Leftrightarrow\; -2 \le x \le 2$ and $9 - y^2 \ge 0 \;\Leftrightarrow\; -3 \le y \le 3$ and $1 - z^2 \ge 0 \;\Leftrightarrow\; -1 \le z \le 1$. Thus the domain of $f$ is $\{(x, y, z) \mid -2 \le x \le 2,\ -3 \le y \le 3,\ -1 \le z \le 1\}$, a solid rectangular box with vertices $(\pm 2, \pm 3, \pm 1)$ (all combinations).



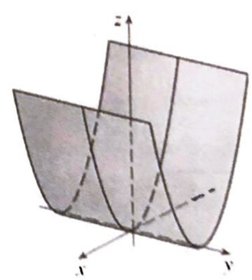

**22.** $f$ is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0 \;\Rightarrow\; \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1$. Thus, $D = \left\{(x, y, z) \,\middle|\, \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} < 1\right\}$, that is, the points inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.



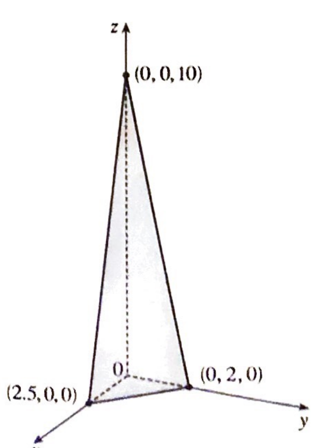

**23.** The graph of $f$ has equation $z = y$, a plane which intersects the $yz$-plane in the line $z = y$, $x = 0$. The portion of this plane in the first octant is shown.



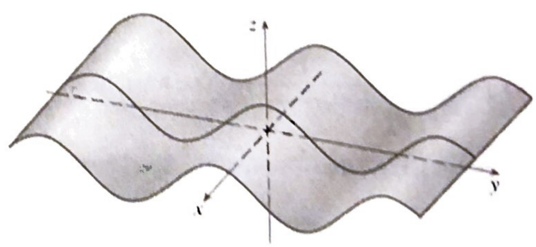

**24.** The graph of $f$ has equation $z = x^2$, a parabolic cylinder.



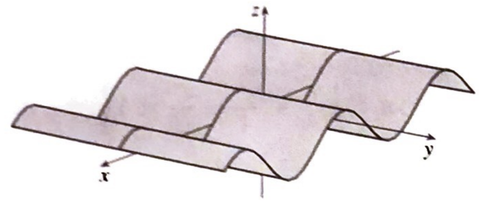

**25.** $z = 10 - 4x - 5y$ or $4x + 5y + z = 10$, a plane with intercepts 2.5, 2, and 10.



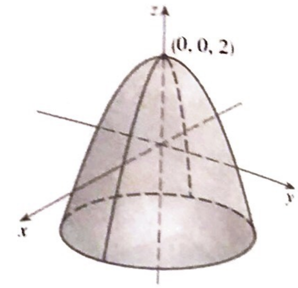

**26.** $z = \cos y$, a cylinder.



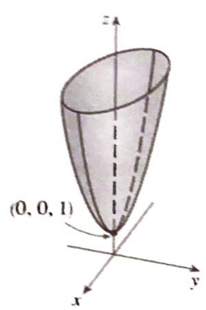
**27.** $z = \sin x$, a cylinder.



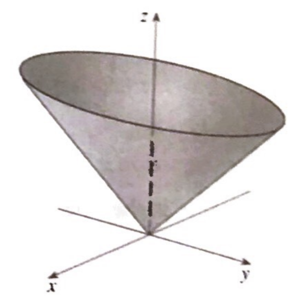

**28.** $z = 2 - x^2 - y^2$, a circular paraboloid opening downward with vertex at $(0, 0, 2)$.



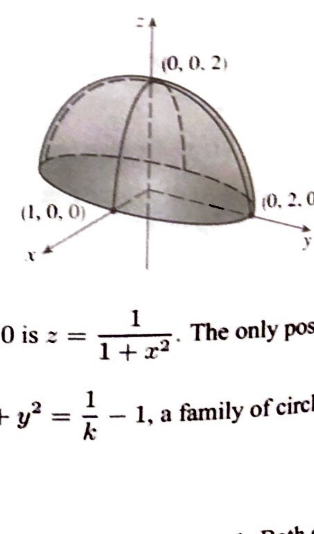

**29.** $z = x^2 + 4y^2 + 1$, an elliptic paraboloid opening upward with vertex at $(0, 0, 1)$.




**30.** $z = \sqrt{4x^2 + y^2}$ so $4x^2 + y^2 = z^2$ and $z \geq 0$, the top half of an elliptic cone.




**31.** $z = \sqrt{4 - 4x^2 - y^2}$ so $4x^2 + y^2 + z^2 = 4$ or $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$ and $z \geq 0$, the top half of an ellipsoid.

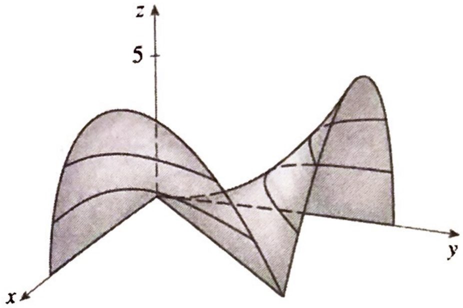



**32.** (a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and the trace in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. The only possibility is graph III. Notice also that the level curves of $f$ are $\dfrac{1}{1 + x^2 + y^2} = k \Leftrightarrow x^2 + y^2 = \dfrac{1}{k} - 1$, a family of circles for $k < 1$.

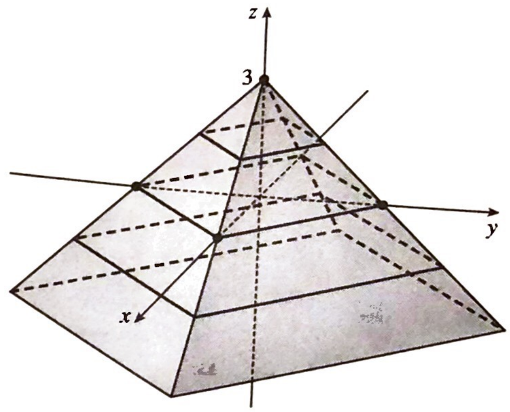
(b) $f(x, y) = \dfrac{1}{1 + x^2 y^2}$. The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x, y) = \ln(x^2 + y^2)$. The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of $f$ are $\ln(x^2 + y^2) = k \Leftrightarrow x^2 + y^2 = e^k$, a family of circles. In addition, $f$ is large negative when $x^2 + y^2$ is small, so this is graph IV.

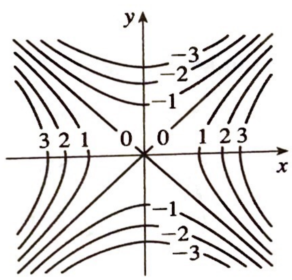
43.



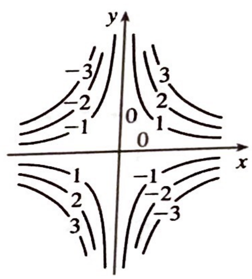

44.



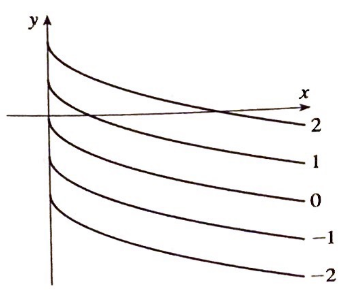

45. The level curves are $x^2 - y^2 = k$. When $k = 0$ the level curve is the pair of lines $y = \pm x$, and when $k \neq 0$ the level curves are a family of hyperbolas (oriented differently for $k > 0$ than for $k < 0$).



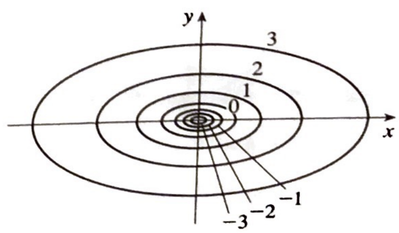

46. The level curves are $xy = k$ or $y = k/x$. When $k \neq 0$ the level curves are a family of hyperbolas. When $k = 0$ the level curve is the pair of lines $x = 0$, $y = 0$.



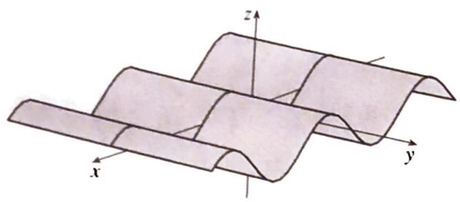

47. The level curves are $\sqrt{x} + y = k$ or $y = -\sqrt{x} + k$, a family of vertical translations of the graph of the root function $y = -\sqrt{x}$.



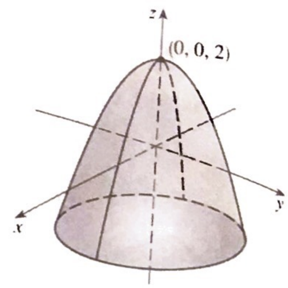

48. The level curves are $\ln(x^2 + 4y^2) = k$ or $x^2 + 4y^2 = e^k$, a family of ellipses.



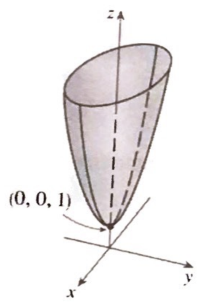
**27.** $z = \sin x$, a cylinder.



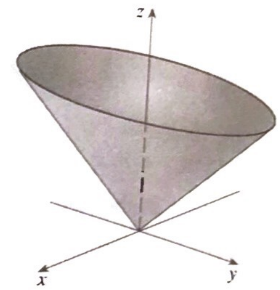

**28.** $z = 2 - x^2 - y^2$, a circular paraboloid opening downward with vertex at $(0, 0, 2)$.



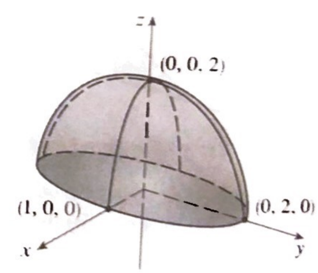

**29.** $z = x^2 + 4y^2 + 1$, an elliptic paraboloid opening upward with vertex at $(0, 0, 1)$.




**30.** $z = \sqrt{4x^2 + y^2}$ so $4x^2 + y^2 = z^2$ and $z \geq 0$, the top half of an elliptic cone.




**31.** $z = \sqrt{4 - 4x^2 - y^2}$ so $4x^2 + y^2 + z^2 = 4$ or $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$ and $z \geq 0$, the top half of an ellipsoid.




**32.** (a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and the trace in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. The only possibility is graph III. Notice also that the level curves of $f$ are $\dfrac{1}{1 + x^2 + y^2} = k \Leftrightarrow x^2 + y^2 = \dfrac{1}{k} - 1$, a family of circles for $k < 1$.

(b) $f(x, y) = \dfrac{1}{1 + x^2y^2}$. The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x, y) = \ln(x^2 + y^2)$. The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of $f$ are $\ln(x^2 + y^2) = k \Leftrightarrow x^2 + y^2 = e^k$, a family of circles. In addition, $f$ is large negative when $x^2 + y^2$ is small, so this is graph IV.

(d) $f(x,y) = \cos\sqrt{x^2+y^2}$. The trace in $x=0$ is $z = \cos\sqrt{y^2} = \cos|y| = \cos y$, and the trace in $y=0$ is $z = \cos\sqrt{x^2} = \cos|x| = \cos x$. Notice also that the level curve $f(x,y)=0$ is $\cos\sqrt{x^2+y^2}=0$ $\Leftrightarrow$ $x^2+y^2 = \left(\frac{\pi}{2}+n\pi\right)^2$, a family of circles, so this is graph V.

(e) $f(x,y) = |xy|$. The trace in $x=0$ is $z=0$, and the trace in $y=0$ is $z=0$, so it must be graph VI.

(f) $f(x,y) = \cos(xy)$. The trace in $x=0$ is $z = \cos 0 = 1$, and the trace in $y=0$ is $z=1$. As mentioned in part (b), these traces match both graphs I and II. Here $z$ can be negative, so the graph is II. (Also notice that the trace in $x=1$ is $z = \cos y$, and the trace in $y=1$ is $z = \cos x$.)

33. The point $(-3,3)$ lies between the level curves with $z$-values 50 and 60. Since the point is a little closer to the level curve with $z=60$, we estimate that $f(-3,3) \approx 56$. The point $(3,-2)$ appears to be just about halfway between the level curves with $z$-values 30 and 40, so we estimate $f(3,-2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

34. (a) $C$ (Chicago) lies between level curves with pressures 1012 and 1016 mb, and since $C$ appears to be located about one-fourth the distance from the 1012 mb isobar to the 1016 mb isobar, we estimate the pressure at Chicago to be about 1013 mb. $N$ lies very close to a level curve with pressure 1012 mb so we estimate the pressure at Nashville to be approximately 1012 mb. $S$ appears to be just about halfway between level curves with pressures 1008 and 1012 mb, so we estimate the pressure at San Francisco to be about 1010 mb. $V$ lies close to a level curve with pressure 1016 mb but we can't see a level curve to its left so it is more difficult to make an accurate estimate. There are lower pressures to the right of $V$ and $V$ is a short distance to the left of the level curve with pressure 1016 mb, so we might estimate that the pressure at Vancouver is about 1017 mb.

(b) Winds are stronger where the isobars are closer together (see Figure 13), and the level curves are closer near $S$ than at the other locations, so the winds were strongest at San Francisco.

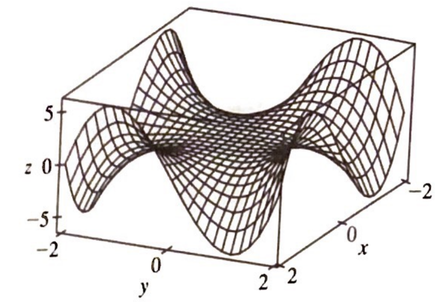
35. The point $(160, 10)$, corresponding to day 160 and a depth of 10 m, lies between the isothermals with temperature values of 8 and 12°C. Since the point appears to be located about three-fourths the distance from the 8°C isothermal to the 12°C isothermal, we estimate the temperature at that point to be approximately 11°C. The point $(180, 5)$ lies between the 16 and 20°C isothermals, very close to the 20°C level curve, so we estimate the temperature there to be about 19.5°C.

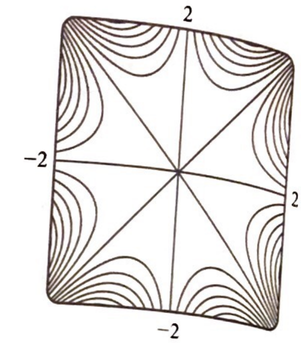
36. If we start at the origin and move along the $x$-axis, for example, the $z$-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has $z$-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

37. Near $A$, the level curves are very close together, indicating that the terrain is quite steep. At $B$, the level curves are much farther apart, so we would expect the terrain to be much less steep than near $A$, perhaps almost flat.

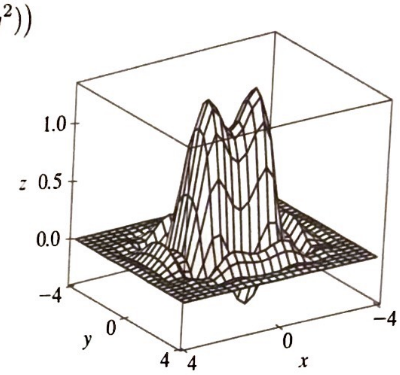
**58.** $f(x,y) = xy^3 - yx^3$

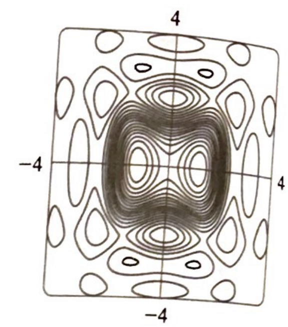
The traces parallel to either the $yz$-plane or the $xz$-plane are cubic curves.



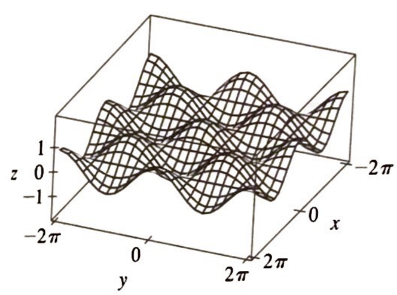

**59.** $f(x,y) = e^{-(x^2+y^2)/3}\left(\sin(x^2) + \cos(y^2)\right)$

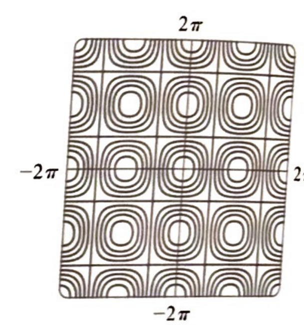



**60.** $f(x,y) = \cos x \, \cos y$

The traces parallel to either the $yz$- or $xz$-plane are cosine curves with amplitudes that vary from 0 to 1.




**61.** $z = \sin(xy)$ (a) C (b) II

Reasons: This function is periodic in both $x$ and $y$, and the function is the same when $x$ is interchanged with $y$, so its graph is symmetric about the plane $y = x$. In addition, the function is 0 along the $x$- and $y$-axes. These conditions are satisfied only by C and II.

**62.** $z = e^x \cos y$ (a) A (b) IV

Reasons: This function is periodic in $y$ but not $x$, a condition satisfied only by A and IV. Also, note that traces in $x = k$ are cosine curves with amplitude that increases as $x$ increases.

**63.** $z = \sin(x - y)$ (a) F (b) I

Reasons: This function is periodic in both $x$ and $y$ but is constant along the lines $y = x + k$, a condition satisfied only by F and I.

**64.** $z = \sin x - \sin y$ (a) E (b) III

Reasons: This function is periodic in both $x$ and $y$, but unlike the function in Exercise 63, it is not constant along lines such as $y = x + \pi$, so the contour map is III. Also notice that traces in $y = k$ are vertically shifted copies of the sine wave $z = \sin x$, so the graph must be E.

65. $z = (1 - x^2)(1 - y^2)$ (a) B (b) VI

Reasons: This function is 0 along the lines $x = \pm 1$ and $y = \pm 1$. The only contour map in which this could occur is VI. Also note that the trace in the $xz$-plane is the parabola $z = 1 - x^2$ and the trace in the $yz$-plane is the parabola $z = 1 - y^2$, so the graph is B.

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$ (a) D (b) V

Reasons: This function is not periodic, ruling out the graphs in A, C, E, and F. Also, the values of $z$ approach 0 as we use points farther from the origin. The only graph that shows this behavior is D, which corresponds to V.

67. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5 \rangle$.

68. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

69. Equations for the level surfaces are $k = y^2 + z^2$. For $k > 0$, we have a family of circular cylinders with axis the $x$-axis and radius $\sqrt{k}$. When $k = 0$ the level surface is the $x$-axis. (There are no level surfaces for $k < 0$.)

70. Equations for the level surfaces are $x^2 - y^2 - z^2 = k$. For $k = 0$, the equation becomes $y^2 + z^2 = x^2$ and the surface is a right circular cone with vertex the origin and axis the $x$-axis. For $k > 0$, we have a family of hyperboloids of two sheets with axis the $x$-axis, and for $k < 0$, we have a family of hyperboloids of one sheet with axis the $x$-axis.

71. (a) The graph of $g$ is the graph of $f$ shifted upward 2 units.

(b) The graph of $g$ is the graph of $f$ stretched vertically by a factor of 2.

(c) The graph of $g$ is the graph of $f$ reflected about the $xy$-plane.

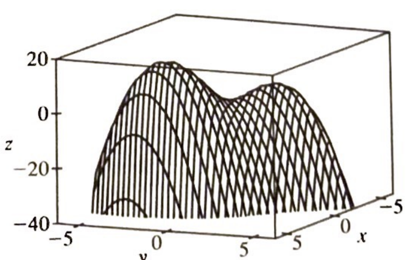
(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of $f$ reflected about the $xy$-plane and then shifted upward 2 units.

72. (a) The graph of $g$ is the graph of $f$ shifted 2 units in the positive $x$-direction.

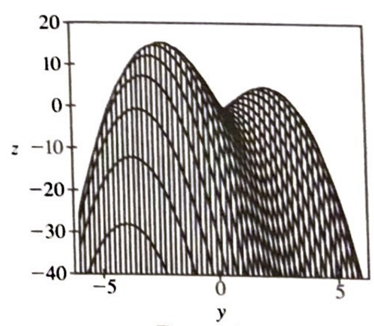
(b) The graph of $g$ is the graph of $f$ shifted 2 units in the negative $y$-direction.

(c) The graph of $g$ is the graph of $f$ shifted 3 units in the negative $x$-direction and 4 units in the positive $y$-direction.

73. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



Three-dimensional view



Front view

[continued]