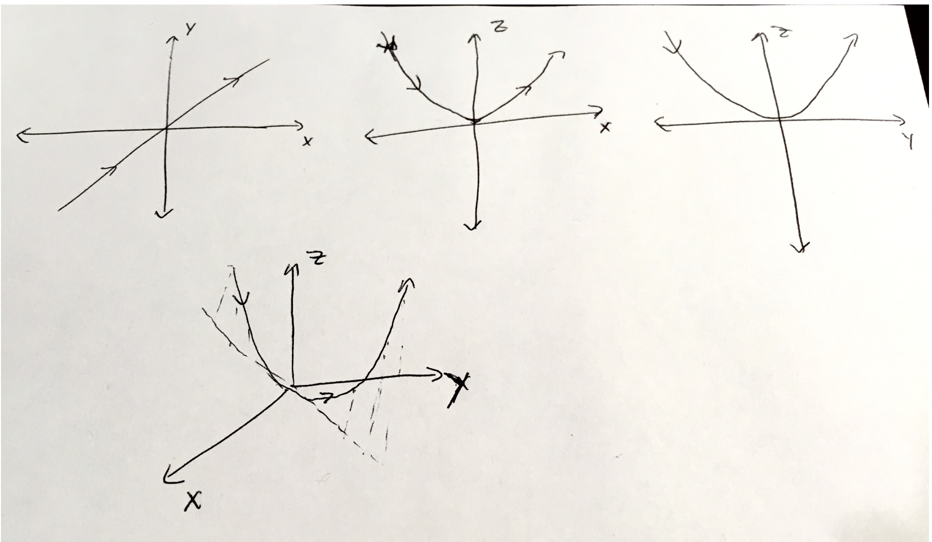
HOMEWORK 6 SOLUTIONS

FALL 2018

UN1201: CALCULUS III, SECTIONS 6 & 7

Problem 13.1.16.



Problem 13.1.21.

II: expanding spiral, circular in projection to xz-plane

Problem 13.1.22.

VI: helix around z-axis with maximum z value 1

Problem 13.1.23.

V: unbounded and parabolic in projection to xz-plane

Date: October 28, 2018.

Problem 13.1.24.

I: sinusoidal along z, circular in projection to xy-plane

Problem 13.1.25.

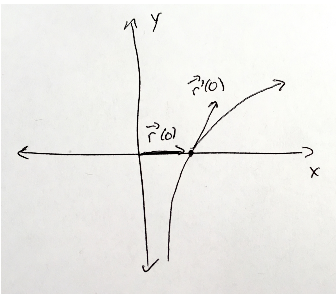
IV: helix around z-axis with unbounded z

Problem 13.1.26.

III: $x + y = 1$ with $x, y \geq 0$

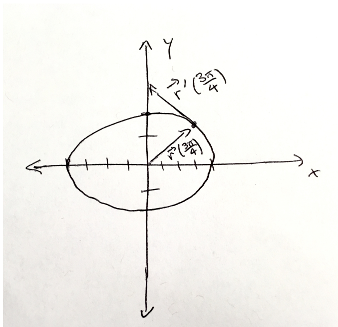
Problem 13.2.6.

We have$\vec{r}(t) = (e^t, 2t)$ and $\vec{r}'(t) = (e^t, 2)$, so $\vec{r}(0) = (1, 0)$ and $\vec{r}'(0) = (1, 2)$. When graphing the function, observe that $x = e^{\frac{y}{2}}$.



Problem 13.2.7.

We have $\vec{r}(t) = (4\sin(t), -2\cos(t))$ and $\vec{r}'(t) = (4\cos(t), 2\sin(t))$, so $\vec{r}(\frac{3\pi}{4}) = (2\sqrt{2}, \sqrt{2})$ and $\vec{r}'(\frac{3\pi}{4}) = (-2\sqrt{2}, \sqrt{2})$. When graphing the function, observe that $\frac{x^2}{16} + \frac{y^2}{4} = 1$.



Problem 13.2.10.

Take the derivative component-wise, so $\vec{r}'(t) = (-e^{-t}, t - 3t^2, \frac{1}{t})$.

Problem 13.2.11.

Take the derivative component-wise, so $\vec{r}'(t) = (2t, -2t\sin(t^2), \frac{1}{t})$.

Problem 13.2.12.

Take the derivative component-wise, so $\vec{r}'(t) = (-\frac{1}{(1+t)^2}, \frac{1}{(1+t)^2}, \frac{t^2+2t}{(1+t)^2})$. Be sure to simplify!

Problem 13.2.16.

We have $\vec{r}'(t) = \frac{d}{dt}(t\vec{a}) \times (\vec{b}+t\vec{c}) + t\vec{a} \times \frac{d}{dt}(\vec{b}+t\vec{c}) = \vec{a} \times (\vec{b}+t\vec{c}) + t\vec{a} \times \vec{c} = \vec{a} \times (\vec{b}+2t\vec{c})$.

Problem 13.2.21.

Since $\vec{r}(t) = (t, t^2, t^3)$, we have $\vec{r}'(t) = (1, 2t, 3t^2)$ and $\vec{r}''(t) = (1, 2, 6t)$. Then $\vec{r}'(1) = (1, 2, 3)$, $\vec{T}(1) = \frac{\vec{r}'(1)}{||\vec{r}'(1)||} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$, and $\vec{r}'(t) \times \vec{r}''(t) = (6t^2, -6t, 2)$.

Problem 13.2.26.

We have $\vec{r}'(t) = (\frac{t}{\sqrt{t^2+3}}, \frac{2t}{t^2+3}, 1)$ so $\vec{r}'(1) = (\frac{1}{2}, \frac{1}{2}, 1)$. Furthermore, $\vec{r}(1) = (2, \ln(4), 1)$ so the tangent line at $(2, \ln(4), 1)$, i.e. $t = 1$, is $\ell : (\frac{t}{2} + 2, \frac{t}{2} + \ln(4), t + 1)$.

Problem 13.2.27.

If we let $x = 5\cos(t)$, $y = 5\sin(t)$, then we have $25\sin^2(t) + z^2 = 20$, so $z = \pm\sqrt{20 - 25\sin^2(t)}$. Then writing $\vec{r}(t) = (5\cos(t), 5\sin(t), \sqrt{20 - 25\sin^2(t)})$, we have $\vec{r}'(t) = (-5\sin(t), 5\cos(t), -\frac{50\sin(t)\cos(t)}{\sqrt{20-25\sin^2(t)}})$. At point $\vec{r}(t_0) = (3, 4, 2)$, we have that $3 = 5\cos(t_0)$ and $4 = 5\sin(t_0)$, so $\cos(t_0) = \frac{3}{5}$ and $\sin(t_0) = \frac{4}{5}$. Then $\vec{r}'(t_0) = (-4, 3, -6)$. Then ℓ is given by $(3 - 4t, 4 + 3t, 2 - 6t)$.

Note that we can use any multiple of $(-4, 3, -6)$ for the coefficient of t. For instance, if we set $t := x$, then we would arrive at $\ell : (3 + t, 4 - \frac{3}{4}t, 2 + \frac{3}{2}t)$. If we set $t := y$, then we would have $\ell : (3 - \frac{4}{3}t, 4 + t, 2 - 2t)$.

Problem 13.2.34.

Calculating the intersection of $\vec{r_1}(t)$ and $\vec{r_2}(s)$, we see that $t = 3 - s$ and $3 + t^2 = s^2$, so $s^2 = 3 + (3 - s)^2 = 12 - 6s + s^2$ from which we can deduce $s = 2$ and $t = 1$. Then $\vec{r_1}(t)$ and $\vec{r_2}(s)$ intersect at $\vec{r_1}(1) = \vec{r_2}(2) = (1, 0, 4)$.

We can calculate $\vec{r_1}'(t) = (1, -1, 2t)$ so $\vec{r_1}'(1) = (1, -1, 2)$. Similarly, $\vec{r_2}'(s) = (-1, 1, 2s)$ so $\vec{r_2}'(2) = (-1, 1, 4)$. Then

$$\theta = \cos^{-1}\left(\frac{\vec{r_1}'(1) \cdot \vec{r_2}'(2)}{|\vec{r_1}'(1)|\,|\vec{r_2}'(2)|}\right) = \cos^{-1}\left(\frac{6}{\sqrt{6}\sqrt{18}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

This is approximately $54.7°$, but the expression $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is the "real" (exact) answer!

Problem 13.2.37.

We just evaluate the integral component-wise, so we have

$$\left(\int_0^1 \frac{1}{t+1}\,dt, \int_0^1 \frac{1}{t^2+1}\,dt, \int_0^1 \frac{t}{t^2+1}\,dt\right) = \left(\ln|t+1| \,|_0^1, \tan^{-1}(t) \,|_0^1, \frac{1}{2}\ln|t^2+1| \,|_0^1\right) = \left(\ln(2), \frac{\pi}{4}, \frac{1}{2}\ln(2)\right).$$

Problem 13.2.42.

Integrating component-wise, we have $\vec{r}(t) = (\frac{t^2}{2} + A, e^t + C, e^t(t-1) + C)$ and $\vec{r}(0) = (1, 1, 1)$. Then we must have $A = 1, B = 0, C = 2$, i.e. $\vec{r}(t) = (\frac{t^2}{2} + 1, e^t, e^t(t-1) + 2)$.

Problem 13.3.1.

First, compute $\vec{r}'(t) = (1, -3\sin(t), 3\cos(t))$. Then the length of the curve from $t = -5$ to $t = 5$ is given by

$$\begin{aligned}
L &= \int_{-5}^{5} ||\vec{r}'(t)||\,dt \\
&= \int_{-5}^{5} \sqrt{1^2 + 9\sin^2(t) + 9\cos^2(t)}\,dt \\
&= \int_{-5}^{5} \sqrt{1 + 9(\sin^2(t) + \cos^2(t))}\,dt \\
&= \int_{-5}^{5} \sqrt{10}\,dt \\
&= \sqrt{10}t \,|_{-5}^{5} \\
&= 10\sqrt{10}.
\end{aligned}$$

Problem 13.3.4.

First, compute $\vec{r}'(t) = (-\sin(t), \cos(t), -\tan(t))$. Then the length of the curve from $t = 0$ to $t = \frac{\pi}{4}$ is given by

$$\begin{aligned}
L &= \int_0^{\frac{\pi}{4}} ||\vec{r}'(t)||\, dt \\
&= \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 + \cos^2(t) + \tan^2(t)}dt \\
&= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(t)}dt \\
&= \int_0^{\frac{\pi}{4}} |\sec(t)|dt \\
&= \ln|\sec(t) + \tan(t)\ |_0^{\frac{\pi}{4}} \\
&= \ln(\sqrt{2} + 1).
\end{aligned}$$

Problem 13.3.11.

First, to parametrize the curve let us set $x = t$. Then $y = \frac{x^2}{2} = \frac{t^2}{2}$ and $z = \frac{xy}{3} = \frac{t^3}{6}$. Then $\vec{r}'(t) = (1, t, \frac{t^2}{2})$, so the length of the curve from $t = 0$ to $t = 6$ (when $\vec{r}(t) = (6, 18, 36)$) is

$$\begin{aligned}
L &= \int_0^6 ||\vec{r}'(t)||\, dt \\
&= \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}}dt \\
&= \int_0^6 \sqrt{(1 + \frac{t^2}{2})^2}dt \\
&= \int_0^6 (1 + \frac{t^2}{2})dt \\
&= (t + \frac{t^3}{6})\ |_0^6 \\
&= 42.
\end{aligned}$$

Problem 13.3.13.

Calculate $\vec{r}'(t) = (-1, 4, 3)$. The point P has $4 = x = 5 - t$, so $t = 1$. Then

$$s(t) = \int_1^t ||\vec{r}'(\tau)||\, d\tau = \int_1^t \sqrt{26}d\tau = \sqrt{26}\tau\ |_1^t = \sqrt{26}(t - 1).$$

Solving $s(t) = \sqrt{26}(t - 1)$ for t, we have $t = \frac{s}{\sqrt{26}} + 1$. Then $\vec{r}(s) = (5 - \frac{s}{\sqrt{26}} - 1, 4\frac{s}{\sqrt{26}} + 4 - 3, 3\frac{s}{\sqrt{26}} + 3) = (4 - \frac{s}{\sqrt{26}}, 4\frac{s}{\sqrt{26}} + 1, 3\frac{s}{\sqrt{26}} + 3)$.

Then at $s = 4$, $\vec{r}(4) = (4 - \frac{4}{\sqrt{26}}, \frac{16}{\sqrt{26}} + 1, \frac{12}{\sqrt{26}} + 3)$.