

## MATH UN1201: Solutions to HW3

- 12.5.1 (a) True; each of the first 2 lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, hence the 2 lines, are parallel.
- (b) False; for example, the x- and y- axes are both perpendicular to the z-axis, yet they are not parallel.
- (c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.
- (d) False; for example, the xy- and yz- planes are not parallel, yet they are both perpendicular to the xz-plane.
- (e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane  $z = 1$ .
- (f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.
- (g) False; the planes  $y = 1$  and  $z = 1$  are not parallel, yet they are both perpendicular to the x-axis.
- (h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.
- (i) True;
- (j) False; they can be skew.
- (k) True; consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel.

12.5.2 For this line, we have  $\mathbf{r}_0 = 6i - 5j + 2k$  and  $\mathbf{v} = i + 3j - \frac{2}{3}k$ , so a vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (6 + t)i + (-5 + 3t)j + (2 - \frac{2}{3}t)k,$$

and the parametric equations are

$$x = 6 + t, y = -5 + 3t, z = 2 - \frac{2}{3}t$$

12.5.4 This line has the same direction as the given line,  $\mathbf{v} = 2i - 3j + 9k$ . Here  $\mathbf{r}_0 = 14j - 10k$ , so a vector equation is  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = 2ti + (14 - 3t)j + (-10 + 9t)k$  and parametric equations are

$$x = 2t, y = 14 - 3t, z = -10 + 9t.$$

12.5.10  $\mathbf{v} = (i + j) \times (j + k) = i - j + k$  is the direction of the line perpendicular to both  $i + j$  and  $j + k$ . With  $P_0 = (2, 1, 0)$ , parametric equations are

$$x = 2 + t, y = 1 - t, z = t,$$

and symmetric equations are

$$x - 2 = 1 - y = z$$

12.5.14 Direction vectors of the lines are  $\mathbf{v}_1 = \langle 3, -3, 1 \rangle$  and  $\mathbf{v}_2 = \langle 1, -4, -12 \rangle$ . Since  $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 \neq 0$ , the vectors and thus the lines are not perpendicular.

12.5.18 The line segment from  $\mathbf{r}_0$  to  $\mathbf{r}_1$  has vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 = (-2i + 18j + 31k) + t(13i - 22j + 17k), \quad 0 \leq t \leq 1.$$

The corresponding parametric equations are:

$$x = -2 + 13t, y = 18 - 22t, z = 31 + 17t, \quad 0 \leq t \leq 1.$$

12.5.19 Since the direction vectors  $\langle 2, -1, 3 \rangle$  and  $\langle 4, -2, 5 \rangle$  are not scalar multiples of each other, the lines are not parallel. For the lines to intersect, we must be able to find a value of  $t$  and a value of  $s$  that produce the same point. Thus we need to satisfy the following three equations:

$$3 + 2t = 1 + 4s, 4 - t = 3 - 2s, 1 + 3t = 4 + 5s.$$

Solving the last two equations we get  $t = 1, s = 0$  and checking, we see that these values don't satisfy the first equation. Thus the lines are not parallel and don't intersect, so they must be skew lines.

12.5.22 The direction vectors  $\langle 1, -1, 3 \rangle$  and  $\langle 2, -2, 7 \rangle$  are not parallel, so neither are the lines. Parametric equations for the lines are:

$$L_1 : x = t, y = 1 - t, z = 2 + 3t$$

$$L_2 : x = 2 + 2s, y = 3 - 2s, z = 7s.$$

For the lines to intersect, the following equations must be satisfied simultaneously:

$$t = 2 + 2s, 1 - t = 3 - 2s, 2 + 3t = 7s.$$

Solving the last two equations gives  $t = -10$  and  $s = -4$  and checking, we see these values don't satisfy the third equation, so the lines must be skew.

12.5.26 Since the line is perpendicular to the plane, its direction vector  $\langle 3, -1, 4 \rangle$  is a normal vector to the plane. The point  $(2, 0, 1)$  is on the plane, so the equation of the plane is

$$\begin{aligned} 3(x - 2) + (-1)(y - 0) + 4(z - 1) &= 0 \\ 3x - y + 4z &= 10 \end{aligned}$$

12.5.34 The vectors  $\mathbf{a} = \langle -2 - 3, -2 - 0, 3 - (-1) \rangle = \langle -5, -2, 4 \rangle$  and  $\mathbf{b} = \langle 7 - 3, 1 - 0, -4 - (-1) \rangle = \langle 4, 1, -3 \rangle$  lie in the plane, so a normal vector to the plane is

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 2, 1, 3 \rangle,$$

and an equation of the plane is

$$\begin{aligned} 2(x - 3) + 1(y - 0) + 3(z - (-1)) &= 0 \\ 2x + y + 3z &= 3. \end{aligned}$$

12.5.40  $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$  and  $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$ . Setting  $z = 0$ , it is easy to see that  $(1, 3, 0)$  is a point on the line of intersection of  $x - z = 1$  and  $y + 2z = 3$ . The direction of this line is  $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$ . A second vector parallel to the desired plane is  $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$ , since it is perpendicular to  $x + y - 2z = 1$ . Therefore, a normal vector of the plane in question is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 3, 3, 3 \rangle.$$

Taking  $(x_0, y_0, z_0) = (1, 3, 0)$ , the equation we are looking for is

$$\begin{aligned} 3(x - 1) + 3(y - 3) + 3z &= 0 \\ x + y + z &= 4. \end{aligned}$$

12.5.53 Normal vectors for the planes are  $\mathbf{n}_1 = \langle 1, 2, -1 \rangle$  and  $\mathbf{n}_2 = \langle 2, -2, 1 \rangle$ . The normals are not parallel, so neither are the planes. Furthermore,  $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 \neq 0$ , so the planes are not perpendicular. The angle between the planes is the same as the angle between the normals, given by

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = -\frac{1}{\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right).$$

12.5.55 The planes are  $2x - 3y - z = 0$  and  $4x - 6y - 2z = 3$  with normal vectors  $\mathbf{n}_1 = \langle 2, -3, -1 \rangle$  and  $\mathbf{n}_2 = \langle 4, -6, -2 \rangle$ . Since  $\mathbf{n}_1 = 2\mathbf{n}_2$ , the normals, and thus the planes, are parallel.

12.5.64 (a) For the lines to intersect, we must be able to find  $t, s$  to satisfy all three equations:

$$1 + t = 2 - s, 1 - t = s, 2t = 2.$$

From the last two we get  $t = 1, s = 0$ . These numbers satisfy the first equation as well, so the lines intersect at the point  $P_0 = (2, 0, 2)$

(b) The direction vectors of the lines are  $\langle 1, -1, 2 \rangle$  and  $\langle -1, 1, 0 \rangle$  so a normal vector for the plane is their cross product,  $\langle 2, 2, 0 \rangle$ . The plane also contains the point  $(2, 0, 2)$ , so an equation for the plane is

$$\begin{aligned} 2(x - 2) + 2(y - 0) + 0(z - 2) &= 0 \\ x + y &= 2. \end{aligned}$$

12.5.65 Two vectors which are perpendicular to the required line are the normal of the given plane,  $\langle 1, 1, 1 \rangle$  and a direction vector of the given line,  $\langle 1, -1, 2 \rangle$ . So a direction for the required line is the cross product of those two vectors,  $\langle 3, -1, -2 \rangle$ . Thus  $L$  is given by

$$\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$$

$$x = 3t, y = 1 - t, z = 2 - 2t.$$

12.5.77 A vector which is perpendicular to both of the lines is

$$\mathbf{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 1, -2, 1 \rangle.$$

Pick any point on each of the lines, say  $(-2, -2, -2)$  and  $(-2, -2, -3)$ , and form the vector  $\mathbf{b} = \langle 0, 0, 1 \rangle$  connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of  $\mathbf{b}$  along  $\mathbf{n}$ , that is:

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{6}}.$$