Homework 9

5.2 Prove 5.14 and $|c|-|d| \le |c+d|$

Let c, d be real numbers, Then $|c+d| \le |c|+|d|$

Proof: If $c \geq 0$ and $d \geq 0$, we have $c+d \geq 0$, so $c=|c|$, $d=|d|$, $|c+d| = |c+d|$

Therefore, $|c+d| = c+d = |c|+|d|$. If $c<0$ and $d<0$, we have $|-c-d| \le |-c| + |-d|$ $\Rightarrow |c+d| \le |c|+|d|$

~~If $c \geq 0$ and $d<0$, WLOG~~

If $cd \le 0$, WLOG we can assume $d<0, c \geq 0$.

Then, $c+d \le c-d = |c|+|d|$

if $c+d \geq 0$, then $|c+d| = c+d \le |c|+|d|$

if $c+d<0$, then $-c-d>0$. $0<-c-d \le c-d = |c-d| \le |c|+|-d| = |c|+|d|$

$\Rightarrow |c+d| = |-c-d| \le |c|+|d|$

To conclude, $|c+d| \le |c|+|d|$. As a result,

$|c+d-d| \le |c+d| + |-d| = |c+d|+|d|$

$\Rightarrow |c|-|d| \le |c+d|$

5.3 $\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|$ by 5.2 and Induction

5.6 $f(x) = \ln|x|$, $g(x) = -\ln|x|$. They don't have limits at 0

but $f(x)+g(x) = \ln|x| - \ln|x| = 0$ has limit at 0.

$f-g$ can't have a limit at a. Otherwise,

$f = \frac{f-g}{2} + \frac{f+g}{2}$ has a limit at a, a contradiction.

5.11 If f has a limit at a point a. Then, we can denote the limit by L. $\forall \varepsilon>0, \exists \delta>0$ s.t. $|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$ (*)

The right limit of f at a is also L because by (*), $\forall \varepsilon>0, \exists \delta$ s.t. $|x^+-a|<\delta \Rightarrow |f(x^+)-L|<\varepsilon$

By the same argument, f has a left limit at a and it's also L.
$\Leftarrow$ Now, suppose f has left and right limits at a and they're both L.
we have $\forall \varepsilon > 0, \exists \delta_1 > 0$ s.t. $|x^+ - a| < \delta \Rightarrow |f(x^+) - L| < \varepsilon$

$\forall \varepsilon > 0, \exists \delta_2 > 0$ s.t. $|x^- - a| < \delta \Rightarrow |f(x^-) - L| < \varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$. Then,

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. f has limit at a and it's L.

5.17 $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{2^n\sqrt{5}}{2^{n+1}\sqrt{5}} \cdot \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} = \lim_{n\to\infty} \frac{1}{2}\left[\frac{(1+\sqrt{5})^n + \sqrt{5}(1+\sqrt{5})^n - (1-\sqrt{5})^n + \sqrt{5}(1-\sqrt{5})^n}{(1+\sqrt{5})^n - (1-\sqrt{5})^n}\right]$

$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ $\qquad = \frac{\sqrt{5}+1}{2}$ $\qquad n > 4$

5.18 How large must n be to ensure F_{n+1}/F_n is within 10^{-1} of the limit

$$\left|\frac{F_{n+1}}{F_n} - \frac{\sqrt{5}+1}{2}\right| = \left|\frac{\sqrt{5}}{2} \cdot \frac{2}{\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^n - 1}\right| = \left|\frac{\sqrt{5}}{\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^n - 1}\right| < 10^{-k}$$

10^{-2} $(n > 6)$

10^{-k} $\qquad \Rightarrow 10^k < \left|\frac{\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^n - 1}{\sqrt{5}}\right| \Rightarrow \sqrt{5} \cdot 10^k < \left|\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^n - 1\right|$

$$n \approx \frac{\lg(\sqrt{5} \cdot 10^k + 1)}{\lg\left|\frac{1+\sqrt{5}}{1-\sqrt{5}}\right|}$$

3. (i) $\lim_{x\to 3} 4x - 2 = 10$ $\qquad \forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{4}$. then, if $|x-3| < \delta$,

$|4x - 12| < 4\delta = \varepsilon$

so ~~4x-10~~ $|(4x-2) - 10| < \varepsilon$

So $\lim_{x\to 3} 4x - 2 = 10$

(ii) $\lim_{x\to 2} 4x - 3 \neq 6$; Let $\varepsilon = \frac{1}{2}$, then $\forall \delta > 0$, if $|x - 2| < \delta$

1) if $2+\delta > 2.1$, then $|2.1-2|<\delta$.

$|2.1\times 4-3-6| = 0.6 > \frac{1}{2} = \varepsilon$

2) if $2+\delta < 2.1$, then $|2+\frac{\delta}{2}-2| = \frac{\delta}{2} < \delta$.

$\delta < 0.1$

$|(2+\frac{\delta}{2})\times 4-3-6| = |2\delta-1| > 0.8 > 0.5 = \varepsilon$

Thus, $\lim_{x\to 2} 4x-3 \neq 6$

4. (a) U be the set of rational numbers of the form $\frac{m}{n}$, with $\gcd(m,n)=1$, $n<N$ and $\alpha \le \frac{m}{n} \le \beta$. (α,β) is a small finite interval around a.

a is an irrational number, N is a positive integer

U is a finite set. ~~a is irrational [illegible]~~

numbers of the form $\frac{m}{n}$: when $n=1$, number of integers m s.t. $\alpha \le m \le \beta$ is bounded

~~or $n=-1$, number of m s.t. $\beta \le m \le \alpha$ is boun~~

when $n=2$, number of m s.t. $2\alpha \le m \le 2\beta$ bounded

⋮

when $n=N-1$, number of m s.t. $(N-1)\alpha \le m \le (N-1)\beta$ is bounded

Therefore, # of $\frac{m}{n}$ s.t. $\alpha \le \frac{m}{n} \le \beta$ is bounded and U is a finite set.

5.20 $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{n} & x \in \mathbb{Q} \setminus \{0\} \\ 1 & x = 0 \end{cases}, \quad x = \frac{m}{n},\ \gcd(m,n) = 1,\ n > 0.$$

ϕ is continuous at every irrational number and discontinuous at every rational number.

Let t be an irrational number.

Then, $\phi(t) = 0$. $\forall \varepsilon > 0$, let N be an integer s.t. $N > 0$ and $N > \frac{1}{\varepsilon}$

By the result of the previous problem, we know that for small interval (α, β) around t, the set $U = \{ \frac{m}{n} \mid \alpha \le \frac{m}{n} \le \beta,\ \gcd(m,n) = 1,\ n < N \}$ is finite.

Then, let $\delta = \frac{1}{3} \min \{ |t - \frac{m}{n}| \mid \frac{m}{n} \in U \}$

when $|x - t| < \delta$, if
- x is irrational, $|\phi(x) - \phi(t)| = |0 - 0| = 0 < \varepsilon$
- x is rational, $\phi(x) < \frac{1}{N} < \varepsilon \Rightarrow |\phi(x) - \phi(t)| < \varepsilon$

Thus, ϕ is continuous at every irrational number.

ϕ is discontinuous at 0. let $\varepsilon = \frac{1}{2}$, for any $\delta > 0$, ~~if $|x-0|<\delta$~~,
we can find a rational number $0 < x < \delta$ s.t. $x = \min(\frac{1}{10000}, \frac{1}{10^k})$
where $\frac{1}{10^k} < \delta$, $k \in \mathbb{N}^+$

Then, $|\phi(x) - \phi(0)| = |f(x) - 1| > \varepsilon = \frac{1}{2}$. ϕ is discontinuous at 0.

For ~~other~~ rational number $k = \frac{m}{n}$, if we let $\varepsilon = \frac{1}{2n}$, then, for any $\delta > 0$
we can find an irrational number t s.t. $|t - k| < \delta$.

Then, $|\phi(t) - \phi(k)| = |0 - \frac{1}{n}| > \frac{1}{2n} = \varepsilon$. Thus, ϕ is discontinuous at all rational numbers.

(b) Let I be the interval $(-1, 1)$, and define the function $f : I \setminus 0 \to \mathbb{R}$ by

$$f(x) = \frac{(1+x)^2 - 1}{x}.$$

Use the definition of limit to show that

$$\lim_{x \to 0} f(x) = 2.$$

(c) Use the definition of limit to show that

$$\lim_{x \to 4} \frac{2}{x} \neq 2.$$

(b)

For any e>0, let d = e/2
If lx-0l<d and x doesn't equal to 0, we have

lf(x)-2l = l (x^2+2x+1-1)/x - 2 l = l x^2/x l = lxl<d = e/2 <e

Therefore, lim f(x) when x goes to 0 is 2.