Homework 7

4.9 Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

In 4.8 (HW6) we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. Thus,

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

4.10 Prove, for all $n \in \mathbb{N}^+$, $\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}$

Note: $\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1}$

When $n=1$, we have $\binom{2}{1} = 2 \geq \frac{2^{2\cdot 1-1}}{\sqrt{1}}$

Let $P(n)$ be the statement that $\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}$

If $P(n)$ is true, ~~then~~ and $n \in \mathbb{N}^+$, we have $\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}$

then $\binom{2(n+1)}{n+1} = \binom{2n+2}{n+1} = \frac{(2n+2)(2n+1)(2n)\cdots[2n+2-(n+1)+1]}{(n+1)\,n(n-1)\cdots 1}$

$$= \frac{(2n+2)(2n+1)(2n)(2n-1)\cdots(n+2)}{(n+1)\,n(n-1)\cdots 1} \frac{(n+1)}{(n+1)}$$

$$\geq \frac{(2n+2)(2n+1)}{(n+1)} \cdot \frac{2^{2n-1}}{\sqrt{n}} \cdot \frac{1}{n+1}$$

$$= \frac{2n+1}{n+1} \cdot \frac{2^{2n}}{\sqrt{n}} = \frac{2^{2n+1}}{\sqrt{n+1}} \frac{(2n+1)}{2\sqrt{n+1}\cdot\sqrt{n}}$$

Now, consider $\frac{2n+1}{2\sqrt{n^2+n}} = \frac{\sqrt{4n^2+4n+1}}{\sqrt{4n^2+4n}} > 1$

Thus $\binom{2(n+1)}{n+1} \geq \frac{2^{2n+1}}{\sqrt{n+1}}$. $P(n) \Rightarrow P(n+1)$

4.12 $F_1=1, F_2=1$, for $n\geq 3$, $F_n=F_{n-1}+F_{n-2}$

$$F_n=\frac{(1+\sqrt5)^n-(1-\sqrt5)^n}{2^n\sqrt5}$$

Let $P(n)$ be the statement that $F_n=\frac{(1+\sqrt5)^n-(1-\sqrt5)^n}{2^n\sqrt5}$ $n\in\mathbb{N}^+$

$P(1)$ is true and $P(2)$ is true

Assume that $P(k)$ is true for all $k\leq n$, we have

$F_k=\frac{(1+\sqrt5)^k-(1-\sqrt5)^k}{2^k\sqrt5}$, ~~then~~, for all $k\leq n$.

Then, $F_{n+1}=F_n+F_{n-1}$. By assumption, we have

$$F_{n+1}=\frac{(1+\sqrt5)^n-(1-\sqrt5)^n}{2^n\sqrt5}+\frac{(1+\sqrt5)^{n-1}-(1-\sqrt5)^{n-1}}{2^{n-1}\sqrt5}$$

$$=\frac{(1+\sqrt5)^n-(1-\sqrt5)^n+2(1+\sqrt5)^{n-1}-2(1-\sqrt5)^{n-1}}{2^n\sqrt5}$$

$$=\frac{(1+\sqrt5)^{n-1}(3+\sqrt5)-(1-\sqrt5)^{n-1}(3-\sqrt5)}{2^n\sqrt5}=\frac{(1+\sqrt5)^{n-1}(6+2\sqrt5)-(1-\sqrt5)^{n-1}(6\ldots}{2^{n+1}\sqrt5}$$

$$=\frac{(1+\sqrt5)^{n+1}-(1-\sqrt5)^{n+1}}{2^{n+1}\sqrt5}$$

$P(n)\Rightarrow P(n+1)$ ⌧

$6+2\sqrt5=(1+\sqrt5)^2$

$6-2\sqrt5=(1-\sqrt5)^2$

17.4 Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=9-x^2$. Find $f((-3,1])$ and $f^{-1}((-1,4\ldots$

$f((-3,1])=(0,9]$

$f^{-1}((-1,4))=$ ~~$(-7,9]$~~

$=(-\sqrt{10},-\sqrt5)\cup(\sqrt5,\sqrt{10})$

17.9 Let $f:\mathbb{Z}\to\mathbb{N}$ be defined by $f(n)=\begin{cases}-2n & n\le 0\\ 2n-1 & n>0\end{cases}$

Find $f(2\mathbb{Z})$ and prove it's correct.

$$f(2\mathbb{Z}) = \{0,4,8,12,16,\dots\}\cup\{3,7,11,15,\dots\}$$

$$= \{4k \mid k\geq 0, k\in\mathbb{Z}\}\cup\{4k-1 \mid k\geq 1, k\in\mathbb{Z}\}$$

It's correct because if $t\in f(2\mathbb{Z})$,

1) if t is odd, then by definition of f we have $t=2k-1$ for some $k>0$ and k is even.

Thus, $t=2\cdot(2c)-1=4c-1$ if we let ~~$k=2c$~~ $c=\frac{k}{2}$.

~~$t\in f(2\mathbb{Z})$~~

2) if t is even, by definition of f we have $t=-2\bar{k}$ for some $\bar{k}\leq 0$ and $\bar{k}$ is even. ~~Then, if we let $k=-2\bar{k}$,~~

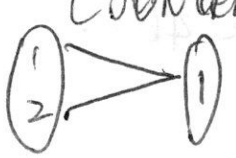
Then, if we let $k=-\frac{\bar{k}}{2}$, we have $t=-2\bar{k}=(-2)(-2k)=4k$ and $k\geq 0$. ~~Thus, $t\in f(2\mathbb{Z})$~~.

~~To conclude,~~ Therefore, $f(2\mathbb{Z})=\{4k \mid k\geq 0, k\in\mathbb{Z}\}\cup\{4k-1 \mid k\geq 1, k\in\mathbb{Z}\}$

17.19

(a) If A and B are subsets of X, then $f(A\setminus B)=f(A)\setminus f(B)$

COUNTER EXAMPLE

f: 

$A=\{1,2\}$

$B=\{1\}$

$f(A\setminus B)=f(\{2\})=\{1\}$

$f(A)\setminus f(B)=\emptyset\neq f(A\setminus B)$

(b) f is injective.

17.21 If f is onto, then, $\forall y\in B, \exists x\in A$ s.t. $y=f(x)$. $(*)$

$\{f^{-1}(\{b\}) : b\in B\}$ partitions the set A because if $a\in A$, then

$a\in\{f^{-1}(\{f(a)\})\}$. Therefore, $A\subset\{f^{-1}(\{b\}) ; b\in B\}$.

Also, if $t\in\{f^{-1}(\{b\}) ; b\in B\}$, $t\in A$ by $(*)$. Therefore, $\{f^{-1}(\{b\}) ; b\in B\}=A$

Therefore, it's a partition

For $f^{-1}(\{b_1\})$ and $f^{-1}(\{b_2\})$, $b_1\neq b_2$, $f^{-1}(\{b_1\})\cap f^{-1}(\{b_2\})=\emptyset$. Otherwise, f is not a function.

18.13

Show that if $n \in \mathbb{Z}^+$

$$2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

when $n=1$, $2(\sqrt{1+1} - 1) = 2\sqrt{2} - 2$,

and $2\sqrt{2} - 2 < 1 < 2$.

Let $p(n)$ be the statement that $2(\sqrt{n+1} - 1) < 1 + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$

$P(1)$ is true

If $P(k)$ is true, we have $(2\sqrt{k+1} - 1) < 1 + \dots + \frac{1}{\sqrt{k}} < 2\sqrt{k}$

then $1 + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+1} - 1 + \frac{1}{\sqrt{k+1}}$

$$= \frac{2k+2+1}{\sqrt{k+1}} - 1$$

Claim: $\frac{2k+3}{\sqrt{k+1}} > 2\sqrt{k+2}$ because $4k^2 + 12k + 9 > 4(k^2 + 3k + 2)$

so $1 + \dots + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2} - 1$

Also, $1 + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{2\sqrt{k^2+k} + 1}{\sqrt{k+1}}$

Claim: $\frac{2\sqrt{k^2+k} + 1}{\sqrt{k+1}} < 2\sqrt{k+1}$ because

$4(k^2+k) + 1 + 4\sqrt{k^2+k} < 4(k^2 + 2k + 1)$

$\cancel{1 + 4\sqrt{4k^2}}$

$1 + 4\sqrt{k^2+k} < 4k + 4$

$4\sqrt{k^2+k} < 4k + 3$

Thus, $P(k) \Rightarrow P(k+1)$

18.15:
Error: we can't assume that the term a*x is in q(x).

For instance, for the case when n=2, we have p(x) = ax(a1x+b1)=0.
Then, if p(c) = 0, we have ac*(a1c+b1)=0.
Now, if a1*c+b1=0 and a*c not zero, we cannot have q(x)=ax and say that q(c)=0. This is the error in the "not a proof."

18.21
Let P(n) be the statement that k is a prime or the product of prime numbers if 2 <= k <=n, k an integer.
P(2) is true because 2 is a prime.
Assume that P(n) is true, then, we know that for all k in[2,n], k is a prime or product of prime numbers.

Consider n+1, if n+1 is a prime, then P(n+1) is also true.
If n+1 is not a prime, then we have n = a*b for some integers a,b <= n . We used the induction hypothesis here so that a and b are primes or product of primes. As a result, since, n is a product of a and b, n is also a product of prime numbers. In this case P(n+1) is also true.

Thus, P(n) => P(n+1)