4.9 Prove \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

In 4.8 we have \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\). Thus,

\[(n+6) \quad 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}\]

4.10 Prove, for all \(n \in \mathbb{N}^+\), \(\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}\)

Note: \(\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}\)

When \(n=1\), we have \(\binom{2}{1} = 2 \geq \frac{2^{1-1}}{\sqrt{1}} = 1\)

Let \(P(n)\) be the statement that \(\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}\)

If \(P(n)\) is true, and \(n \in \mathbb{N}^+\), we have \(\binom{2n}{n} \geq \frac{2^{2n-1}}{\sqrt{n}}\)

then \(\left(\frac{2(n+2)}{n+1}\right) = \left(\frac{2n+2}{n+1}\right)\left(\frac{2n+1}{n}\right)\left(\frac{2n}{n-1}\right)\cdots\left(\frac{2}{1}\right) = \frac{(2n+2)(2n+1)(2n)(2n-1)\cdots(2)(1)}{(n+1)n(n-1)\cdots1}\)

\[\geq \frac{(2n+2)(2n+1)}{(n+1)} \cdot \frac{2^{2n-1}}{\sqrt{n}} \cdot \frac{1}{n+1}\]

\[\geq \frac{2n+1}{n+1} \cdot \frac{2^{2n-1}}{\sqrt{n}} \cdot \frac{1}{n+1}\]

\[= \frac{2n+1}{n+1} \cdot \frac{2^{2n-1}}{\sqrt{n}} = \frac{2^{2n+1}}{\sqrt{n+1} \cdot 2^{n+1} \sqrt{n}} \geq 1\]

Now, consider \(\frac{2n+1}{2 \sqrt{n^2+n}} = \sqrt{\frac{4n^2+4n+1}{4n^2+4n}} > 1\)

Thus, \(\left(\frac{2(n+1)}{n+1}\right) \geq \frac{2^{2n-1}}{\sqrt{n+1}} \quad P(n) \Rightarrow P(n+1)\)
4.12 \( F_n = 1, F_2 = 1, \) for \( n \geq 3 \), \( F_n = F_{n-1} + F_{n-2} \)
\[
F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}
\]

Let \( P(n) \) be the statement that \( F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} \) for all \( n \in \mathbb{N}^+ \)

\( P(1) \) is true and \( P(2) \) is true

Assume that \( P(k) \) is true, we have
\[
F_k = \frac{(1 + \sqrt{5})^k - (1 - \sqrt{5})^k}{2^k \sqrt{5}}, \quad \text{then, for all } k \leq n.
\]

Then, \( F_{n+1} = F_n + F_{n-1} \). By assumption, we have
\[
F_{n+1} = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} + \frac{(1 + \sqrt{5})^{n-1} - (1 - \sqrt{5})^{n-1}}{2^{n-1} \sqrt{5}}
\]
\[
= \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} + \frac{2(1 + \sqrt{5})^{n-1} - 2(1 - \sqrt{5})^{n-1}}{2^n \sqrt{5}}
\]
\[
= \frac{(1 + \sqrt{5})^n}{2^n \sqrt{5}} \cdot (3 + \sqrt{5}) - \frac{(1 - \sqrt{5})^n}{2^n \sqrt{5}} \cdot (3 - \sqrt{5}) = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1} \sqrt{5}}
\]

\( P(n) \Rightarrow P(n+1) \)

17.4 Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = 9 + x^2 \). Find \( f(3, 1) \) and \( f^{-1}(114) \)

\( f(-3, 1) = (0, 9) \)

\( f^{-1}(114) = (-\sqrt{10}, -\sqrt{5}) \cup (\sqrt{5}, \sqrt{10}) \)
17.9 Let \( f: \mathbb{Z} \rightarrow \mathbb{N} \) be defined by \( f(n) = \begin{cases} -2n & n \leq 0 \\ 2n - 1 & n > 0 \end{cases} \)

Find \( f(2\mathbb{Z}) \) and prove it is correct.

\( f(2\mathbb{Z}) = \{0, 4, 8, 12, 16, \ldots \} \cup \{3, 7, 11, 15, \ldots \} = \{4k | k \geq 0, k \in \mathbb{Z}\} \cup \{4k - 1 | k > 1, k \in \mathbb{Z}\} \)

It's correct because if \( t \in f(2\mathbb{Z}) \),

1) if \( t \) is odd, then by definition of \( f \) we have \( t = 2k - 1 \) for some \( k > 0 \) and \( k \) is even.

Thus, \( t = 2 \cdot 2c - 1 = 4c - 1 \) if we let \( c = \frac{k}{2} \).

2) if \( t \) is even, by definition of \( f \) we have \( t = -2k \) for some \( k \leq 0 \) and \( k \) is even. Then, if we let \( k = -\frac{t}{2} \), we have \( t = -2k = (2)(-2k) = 4k \) and \( k > 0 \). Thus, \( t \in f(2\mathbb{Z}) \).

Therefore, \( f(2\mathbb{Z}) = \{4k | k \geq 0, k \in \mathbb{Z}\} \cup \{4k - 1 | k > 1, k \in \mathbb{Z}\} \).

17.19

(a) If \( A \) and \( B \) are subsets of \( X \), then \( f(A \cap B) = f(A) \cap f(B) \)

\[ f(A \cap B) = \{13\} \]

(b) \( f \) is injective.

Therefore, \( f \) is a function.
18.15:
Error: we can't assume that the term \( a^x \) is in \( q(x) \).

For instance, for the case when \( n=2 \), we have \( p(x) = ax(a_1x+b_1) = 0 \).
Then, if \( p(c) = 0 \), we have \( ac(a_1c+b_1) = 0 \).
Now, if \( a_1c+b_1 \neq 0 \) and \( ac \neq 0 \), we cannot have \( q(x) = ax \) and say that \( q(c) = 0 \). This is the error in the "not a proof."

18.21
Let \( P(n) \) be the statement that \( k \) is a prime or the product of prime numbers if \( 2 \leq k \leq n \), \( k \) an integer.

Assume that \( P(n) \) is true, then, we know that for all \( k \) in \( [2, n] \), \( k \) is a prime or product of prime numbers.

Consider \( n+1 \), if \( n+1 \) is a prime, then \( P(n+1) \) is also true.

If \( n+1 \) is not a prime, then we have \( n = a*b \) for some integers \( a, b \leq n \). We used the induction hypothesis here so that \( a \) and \( b \) are primes or product of primes. As a result, since, \( n \) is a product of \( a \) and \( b \), \( n \) is also a product of prime numbers. In this case \( P(n+1) \) is also true.

Thus, \( P(n) \implies P(n+1) \)