Homework 6

2 $(\forall n \in \mathbb{N})\ 2^n > n$.

Let $P(k)$ be the statement that $2^k > k$, ~~$k \in \mathbb{N}$~~.

Since $2^0 = 1 > 0$ and $2^1 = 2 > 1$, $2^2 = 4 > 2$, $P(0)$ and $P(1)$ $P(2)$ are true.

f we assume that $P(k)$ is true, and $\left(\begin{matrix} k \in \mathbb{N} \\ k \geq 2 \end{matrix}\right)$, then,

e have $2^k > k$ and $k \in \mathbb{N}$

Therefore, $2 \cdot 2^k > 2k \Rightarrow 2^{k+1} > 2k$. since $k \geq 2$, we have $2k > k+1$. Therefore, $2^{k+1} > 2k > k+1$, $P(k+1)$ is true.

Thus, If $k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$. We prove that $(\forall n \in \mathbb{N})\ 2^n > n$.

4 $(1+x)^n \geq 1+nx\ \forall n \in \mathbb{N}^+\ \forall x \in (-1,\infty)$

Let $p(n)$ be the statement that $\forall x \in (-1,\infty)$, $(1+x)^n \geq 1+nx$

$p(1)$ is true because $(1+x)^1 = 1+x \geq 1+x$

If $P(k)$ is true and $k \in \mathbb{N}^+$, we have $(1+x)^k \geq 1+kx$. Since $x \in (-1,\infty)$, $(1+x) \in (0,\infty)$

Then, $(1+x)^{k+1} \geq (1+kx)(1+x) = 1+kx+x+kx^2 = 1+(k+1)x+kx^2$

Since $k \geq 0$ and $x^2 \geq 0$, we have $kx^2 \geq 0$.

Thus, $(1+x)^{k+1} \geq 1+(k+1)x+kx^2 \geq 1+(k+1)x$. $P(k+1)$ is true.

herefore, $P(k) \Rightarrow P(k+1)$ and thus $\forall n \in \mathbb{N}^+\ \forall x \in (1,\infty)$ $(1+x)^n \geq 1+nx$.

.7 Induction on n: Let $P(n)$ be the statement that

$\leq k \leq n-1$, $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ $P(2)$ is true because $\binom{2}{1} = \binom{1}{1} + \binom{1}{0} = 2 = \frac{2!}{(2-1)!1!}$

f ~~P(h)~~ is true, we have $\binom{h}{k} = \frac{h!}{(h-k)!k!}\ \forall 1 \leq k \leq h-1$

$P(h)$ and $h \geq 2$

Then, $\binom{h+1}{k} = \binom{h}{k} + \binom{h}{k-1} = \frac{h!}{(h-k)!k!} + \frac{h!}{[h-(k-1)]!(k-1)!}$

$= \frac{(h-k+1)h! + k \cdot h!}{(h-k+1)!k!} = \frac{(h+1)!}{(h+1-k)!k!}$ Thus, $P(h) \Rightarrow P(h+1)$.

prove that $\forall 1 \leq k \leq n-1$, $\binom{n}{k} = \frac{n!}{\ldots}$

4.8 $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$ $\forall n\in\mathbb{N}$ let $P(h)$ be the statement that $(x+y)^h=\sum_{k=0}^{h}\binom{h}{k}x^{h-k}y^k$

If $n=0$, $(x+y)^0=1=\sum_{k=0}^{0}\binom{0}{k}x^{0-k}y^k$

Thus, $p(0)$ is true,

Suppose $P(h)$ is true and $h\in\mathbb{N}$, then, we have $(x+y)^h=\sum_{k=0}^{h}\binom{h}{k}x^{h-k}y^k$

$$\text{Thus, } (x+y)^{h+1}=(x+y)\left[\sum_{k=0}^{h}\binom{h}{k}x^{h-k}y^k\right]=\sum_{k=0}^{h}\binom{h}{k}x^{h+1-k}y^k+\sum_{k=0}^{h}\binom{h}{k}x^{h-k}y^{k+1}$$

$$=\left[\sum_{k=1}^{h}\binom{h}{k}x^{(h+1)-k}y^k\right]+\left(x^{h+1}\right)+\left[\sum_{k=0}^{h-1}\binom{h}{k}x^{h-k}y^{k+1}\right]+\left(y^{h+1}\right)$$

let $t=k+1$

$$=\left(x^{h+1}\right)+\left(y^{h+1}\right)+\sum_{k=1}^{h}\binom{h}{k}x^{h+1-k}y^k+\sum_{t=1}^{h}\binom{h}{t-1}x^{h+1-t}y^t$$

$$=x^{h+1}+y^{h+1}+\sum_{k=1}^{h}\binom{h+1}{k}x^{h+1-k}y^k=\sum_{k=0}^{h+1}\binom{h+1}{k}x^{h+1-k}y^k$$

Therefore, $P(h)\Rightarrow p(h+1)$, and $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$, $n\in\mathbb{N}$

~~8.3 (a)~~ $\left|\prod_{k=1}^{n}x_k\right|=\prod_{k=1}^{n}|x_k|$ and (b) $\left|\sum_{k=1}^{n}x_k\right|\leq\sum_{k=1}^{n}|x_k|$

(a) when $n=1$, $\left|\prod_{k=1}^{1}x_k\right|=|x_k|=\prod_{k=1}^{1}|x_k|$ $(*)$

Let $p(h)$ be the statement that $\left|\prod_{k=1}^{h}x_k\right|=\prod_{k=1}^{h}|x_k|$.

$P(1)$ is true by $(*)$. Suppose $P(h)$ is true, then

$$\left|\prod_{k=1}^{h+1}x_k\right|=\left|\left(\prod_{k=1}^{h}x_k\right)\cdot x_{h+1}\right|=\left|\prod_{k=1}^{h}x_k\right|\cdot|x_{h+1}|=\left[\prod_{k=1}^{h}|x_k|\right]|x_{h+1}|$$
$$=\prod_{k=1}^{h+1}|x_k|.$$

Thus, $P(h)\Rightarrow P(h+1)$. $\forall n\in\mathbb{Z}^+$, $\left|\prod_{k=1}^{n}x_k\right|=\prod_{k=1}^{n}|x_k|$

(b) $P(h)$: $\left|\sum_{k=1}^{n}x_k\right|\leq\sum_{k=1}^{n}|x_k|$ $\qquad P(1)$ is true,

If $P(n)$ is true, then $\left|\sum_{k=1}^{n+1}x_k\right|=\left|\sum_{k=1}^{n}x_k+x_{n+1}\right|\leq\left|\sum_{k=1}^{n}x_k\right|+|x_{n+1}|$

$$\leq\sum_{k=1}^{n}|x_k|+|x_{n+1}|=\sum_{k=1}^{n+1}|x_k|$$

$P(n)\Rightarrow P(n+1)$. ※.

~~8.4~~

18.3 Let ~~P(n)~~ $P(n)$ be the statement that $1^3 + \dots + n^3 = (1+2+\dots+n)^2$

$P(1)$ is true. Suppose that $P(k)$ is true. We have $\sum_{i=1}^{k} i^3 = \left(\sum_{i=1}^{k} i\right)^2$

Then, $1^3+2^3+\dots+k^3+(k+1)^3$

$$= (1+2+\dots+k)^2 + (k+1)^3$$
$$= (1+2+\dots+k)^2 + (k+1)^2(k+1) = (1+2+\dots+k)^2 + (k+1)^2 + k(k+1)^2$$

Since $1+2+\dots+k = \frac{(1+k)k}{2}$, we have

$$(k+1)^2 k = 2(1+2+\dots+k)(k+1)$$

Therefore, $1^3+\dots+(k+1)^3 = (1+2+\dots+k)^2 + (k+1)^2 + 2\cdot(k+1)\cdot(1+2+\dots+k)$
$$= (1+2+\dots+k+1)^2$$

$P(k) \Rightarrow P(k+1)$ ※.

18.4 $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ Let $r^0 + r^1 + \dots + r^{n-1} = c$

Then, $cr = r^1 + r^2 + \dots + r^{n-1} + r^n$.

$$cr - c = r^n - r^0 = r^n - 1 \Rightarrow c = \frac{r^n - 1}{r-1} = \frac{1-r^n}{1-r}$$

18.7 $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$

Let $P(n)$ be the statement that $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$

$P(1)$ is true since $\frac{1}{1} + \frac{1}{2} \geq 1 + \frac{1}{2}$

Suppose that $P(k)$ is true, then we have $\frac{1}{1} + \dots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$

Then, $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^k} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k}{2} + \underbrace{\left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}\right)}_{(2^k \text{ terms})}$

$$\geq 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} = 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$$

Thus, $P(k) \Rightarrow P(k+1)$ and $\frac{1}{1} + \dots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ ※.