Homework 5

3.18 Proof: If the last two digits of a number is divisible by 4, suppose the number is ~~positive~~ not negative
then, let us denote the number by k, $k = 100 \cdot h_1 + 10h_2 + h_3$,
where $h_1, h_2, h_3 \in (\mathbb{Z} \cap [0,9])$, $h_1 \geq 0$ and $h_1 \in \mathbb{Z}$.

we have $10h_2 + h_3 = 4 \cdot m$ for some $m \in \mathbb{Z}^+$.

Thus, $k = 100h_1 + 4m = 4(25h_1 + m)$. Therefore, k is also divisible by 4.

If the number k is divisible by 4, we have

$k = 100h_1 + 10h_2 + h_3 = 4n$ for some ~~$n \in \mathbb{Z}$~~ $n \in \mathbb{N}$

and $10h_2 + h_3 = 4n - 100h_1 = 4(n - 25h_1)$

Thus, its last 2 digits are divisible by 4.

To conclude, a number is divisible by 4 iff its last 2 digits are.

If the number is negative, we can use the proof for $(-k)$ and then number k is still divisible by 4 iff ...

3.19 ~~If~~ Let us denote the number by $k = 1000a_1 + 100a_2 + 10a_3 + a_4$, $(k \geq 0)$
where $a_1 \in \mathbb{N}$, $a_2, a_3, a_4 \in [0,9] \cap \mathbb{Z}$.

If k is divisible by 8, we have $k = 1000a_1 + 100a_2 + 10a_3 + a_4 = 8m$ for some $m \in \mathbb{N}$.

Thus, $100a_2 + 10a_3 + a_4 = 8m - 1000a_1 = 8(m - 125a_1)$.

It's last 3 digits are divisible by 8.

If its last 3 digits are divisible by 8, we have:

$100a_2 + 10a_3 + a_4 = 8n$ for some $n \in \mathbb{N}$

then $k = 1000a_1 + 8n = 8(125a_1 + n)$ is divisible by 8.

If $k < 0$, then $(-k)$ is divisible by 8 iff its last 3 digits are

$\Rightarrow$ k is divisible by 8 iff its last 3 digits are.

3.20 Let the number k be denoted in this way: (firstly discuss the case when $k \geq 0$)

$k = a_1 + 10a_2 + 100a_3 + \dots + 10^{n-1}a_n + 10^n a_{n+1}$, where $a_1, \dots, a_n \in [0,9] \cap \mathbb{Z}$, $a_{n+1} \geq 0$

If k is divisible by 2^n, we have $k = m \cdot 2^n$ $m \in \mathbb{N}$

and $a_1 + \dots + 10^{n-1}a_n = m \cdot 2^n - 10^n a_{n+1} = 2^n(m - 5^n a_{n+1})$ is divisible by 2^n.

If k's last n digits are divisible by 2^n, we have for some $h \in \mathbb{N}$

$a_1 + \dots + 10^{n-1}a_n = h \cdot 2^n$. Thus, $k = h \cdot 2^n + 10^n a_{n+1} = 2^n(h + 5^n a_{n+1})$

Therefore, k is also divisible by 2^n.

If $k < 0$, then $(-k) > 0$.

$(-k)$ is divisible by 2^n iff its last three digits are

$\Rightarrow$ k is divisible by 2^n iff its last 3 digits are

Exercise 3.23 Show that every interval contains rational and irrational numbers.

Suppose that K is a subinterval of the form [a, b] of an arbitrary interval.
Then, let c = 1/(b-a) > 0.
Therefore, we can always find some rational number d such that 0 < c < d.
As a result, 0 < 1/d < 1/c = (b-a) (*)

For any real number r, there exists some integers n and n+1 such that n <= r < n+1
Therefore, we know n < ad < n+1 or ad = n for some integer n.

i) n/d < a < (n+1)/d
We claim that (n+1)/d is a rational number in [a, b] because
(n+1)/d - a < (n+1)/d-n/d = 1/d < 1/c = (b-a)

ii) if ad = n, then, a = n/d is rational in [a, b].

Therefore, every interval contains rational numbers.

To find an irrational number in K:
pi = 3.141592653...... is an irrational number
Consider the interval [a-pi, b-pi], we can find some rational number t in this interval.
Then, t + pi is irrational and it is in [a,b] Thus, we proved 3.23.