Homework 10

6.3 $\preceq$ is transitive: For sets X, Y, Z

if $X \preceq Y$ and $Y \preceq Z$

There is an ~~in~~ injection $f: X \to Y$

There is an injection $g: Y \to Z$

Then $g \circ f: X \to Z$ is an injection. Therefore, $X \preceq Z$.

$\preceq$ is transitive.

$\preceq$ is reflexive: ~~If $X \preceq Y$ and $Y \preceq X$~~ For any set X

Let f be the identity function. $\forall x \in X$, $f(x) = x$.

Then, $f: X \to X$ is an injection.

Therefore, $X \preceq X$

6.5 $|P(\ulcorner n \urcorner)| = 2^n$ $P(X)$: power set of $X = \{Y \mid Y \subseteq X\}$

$\ulcorner n \urcorner = \{0, 1, 2, 3, \ldots, n-1\}$.

\# of $P(X)$ s.t. $|P(X)| = 0$ $1 = \binom{n}{0}$

$|P(X)| = 1$ $\binom{n}{1}$

$|P(X)| = 2$ $\binom{n}{2}$

$\vdots$

$|P(X)| = n$ $\binom{n}{n}$

So $|P(\ulcorner n \urcorner)| = \sum_{i=0}^{n} \binom{n}{i} = (1+1)^n = 2$

6.13 If $|Y| \le |X|$, then there is a surjection $f: X \to Y$.

Since $|Y| \le |X|$, we have $Y \preceq X$. There exists an injection g

$g: Y \to X$ Let $A = g(Y)$ and let Y_1 be an element in Y

Define $f(x) = \begin{cases} g^{-1}(x) & \text{if } x \in A \\ Y_1 & , x \notin A \end{cases}$ f is a surjection.

6.17 a) $f(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a}$ $f: (a,b) \to (c,d)$

f is a bijection, so (a,b) is bijective with (c,d)

b) $f(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a}$ $f: [a,b] \to [c,d]$ is a bijection

d) Since $c<d$, we can find e,f s.t. $c<e<f<d$

$\exists$ an bijection from $(a,b) \to (e,f)$

$\Rightarrow \exists$ an injection from $(a,b) \to [c,d]$ since $(e,f) \subset [c,d]$

Since $a<b$, we can find g,h s.t. $a<g<h<b$

$\exists$ an bijection from $[c,d] \to [g,h]$

$\Rightarrow \exists$ an injection from $[c,d] \to (a,b)$ since $[g,h] \subset (a,b)$

Therefore, (a,b) is bijective with the closed interval $[c,d]$.

e) $|[0,1]| = |\mathbb{R}|$ Since $[0,1] \subset \mathbb{R}$,

$\exists$ an injection from $[0,1]$ to $\mathbb{R}$ (*)

Let f be a function from $\mathbb{R}$ to $[-1,1]$ s.t.

$$f(x) = \begin{cases} \frac{x}{x+1} & \text{if } x \geq 0 \\ -\left[\frac{|x|}{|x|+1}\right] & \text{if } x < 0 \end{cases}$$

Then, f is an injection from $\mathbb{R}$ to $[-1,1]$

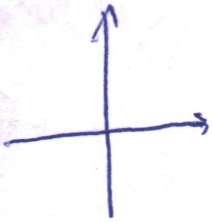
Since there is a bijection from $[-1,1]$ to $[0,1]$ by b), we know

$\exists$ a ~~bijection~~ injection from $\mathbb{R}$ to $[0,1]$ (**)

by (*) and (**), we know there is a bijection from $[0,1]$ to

So $|[0,1]| = |\mathbb{R}|$

3. Construct an explicit bijection between $(0,1)$ and $[0,1]$



$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{3} \\ \frac{1}{n-2} & \text{if } x = \frac{1}{n} \\ x & \text{otherwise} \end{cases}$$

$f : (0,1) \to [0,1]$

$n = 4, 5, 6, \dots\dots$

4. (a) The set of finite subsets of $\mathbb{N}$ has the same cardinality as $\underline{\mathbb{N}}$

(b) infinite subsets of $\mathbb{N}$ $\underline{P(\mathbb{N})}$

(c) finite sequence of elements of $\mathbb{N}$ $\underline{\mathbb{N}}$