1. Notes on Dedekind cuts

Definition 1.1. *A subset $L \subset \mathbb{Q}$ of the rationals is called a* **Dedekind cut** *if*
(I) *L is proper (i.e. $L \neq \emptyset, L \neq \mathbb{Q}$);*
(II) *L has no maximal element;*
(III) *for all elements $a, b \in \mathbb{Q}$ with $a < b$, $\quad b \in L \Longrightarrow a \in L$.*

Example 1.2. (i) If $a \in \mathbb{Q}$, the open interval $L_a := (-\infty, a) \cap \mathbb{Q}$ is a Dedekind cut that we take to represent the rational number a.
(ii) Let $r_1 \leq r_2 \leq r_3 \leq \dots$ be any non decreasing sequence of rational numbers such that
a) the sequence is bounded, i.e. $\exists M \in \mathbb{Q}$ s.t. $r_n < M, \ \forall n \in \mathbb{N}^+$;
(b) the sequence is not eventually constant, i.e. for all n_1 there is $n_2 > n_1$ with $r_{n_2} > r_{n_1}$. Then

$$L := \bigcup_{n \geq 1} (-\infty, r_n)$$

is a Dedekind cut. We need (a) for condition (I) and (b) for condition (II). Writing out a precise proof is on your HW for this week.

As the next lemma shows, there are many other ways to define a Dedekind cut.

Lemma 1.3. *Let $M = \{x \in \mathbb{Q} \,|\, x \leq 0 \text{ or } x^2 < 2\}$. Then M is a Dedekind cut.*

Proof. M satisfies (I) because $0 \in M$ (so $M \neq \emptyset$) and $3 \notin M$ because $3 > 0$ and $3^2 > 2$.

To see that M satisfies (III) suppose that $a < b$ and that $b \in M$. We must show that $a \in M$. If $a \leq 0$ then $a \in M$ by the definition of M. So suppose that $a > 0$. Then $b > 0$ as well and also $b^2 < 2$. Since $0 < a < b$, we find $a^2 < b^2$. Therefore $a^2 < 2$ and so $a \in M$. Thus (III) holds.

To see that M satisfies (II), we must show that each element $a \in M$ is not a maximal element, i.e. there is $a' \in M$ with $a' > a$.

So consider $a \in M$. If $a \leq 1$ then we may take $a' = 5/4$.

Therefore we may suppose that $a > 1$. Since $(3/2)^2 > 2$, we know that $a < 3/2$. We want to show that there is a rational number $y > 0$ such that $(a + y)^2 < 2$. (For then we take $a' = a + y$.) Let us assume that $y = 1/n$ for some integer n. Thus we need

$$a^2 + 2ay + y^2 < 2, \quad \text{or} \quad a^2 + 2\tfrac{a}{n} + \tfrac{1}{n^2} < 2.$$

I simplified this inequality by noticing that since $a \geq 1$ and $n \geq 1$ we have $\frac{1}{n^2} \leq \frac{1}{n} \leq \frac{a}{n}$. Therefore

$$a^2 + 2\tfrac{a}{n} + \tfrac{1}{n^2} < a^2 + 2\tfrac{a}{n} + \tfrac{a}{n} = a^2 + 3\tfrac{a}{n}$$

so that it suffices to find n so that $a^2 + 3\frac{a}{n} < 2$. But this we can immediately solve:
we need $3\frac{a}{n} < 2 - a^2$, i.e. $n > 3\frac{a}{2-a^2}$.

Notice, that just as when we find $\delta = \delta(\epsilon)$ when proving continuity, there are many ways to do this estimate. But I think that what I did above is one of the most direct arguments. $\square$

Note: The last argument above is a version of the proof I gave in class that
the set $S := \{a \in \mathbb{Q} \mid a > 0 \text{ and } a^2 < 2\}$ has no least upper bound in $\mathbb{Q}$.
Here are the details of that argument.

Step 1: Suppose that $r \in \mathbb{Q}$ has the property that $r > 0$ and $r^2 > 2$. Then I claim that r is an upper bound for S but is not the least upper bound since we can always find a smaller upper bound. Here are the steps:

(i) *to see that r is an upper bound*:
If not there is $a \in S$ such that $a \geq r$ then $2 > a^2 \geq r^2 > 2$ which is impossible.
(ii) *to see that there is a smaller upper bound than r*:
We look for $x < r$ such that $x^2 > 2$, where n is a large integer. (The is essentially the same calculation as above)

If $r > 2$ this is clear: just take $x = 2$. Otherwise we look for x of the form $x = r - \frac{1}{n}$. Then we want

$$x^2 = r^2 - 2\frac{r}{n} + \frac{1}{n^2} > 2, \quad i.e.\ 2\frac{r}{n} - \frac{1}{n^2} < r^2 - 2.$$

But

$$\begin{aligned} 2\frac{r}{n} - \frac{1}{n^2} &< 4\frac{1}{n} - \frac{1}{n^2} \quad \text{since } r < 2 \\ &< 4\frac{1}{n} - \frac{1}{n} = \frac{3}{n}, \quad \text{since } \frac{1}{n^2} \leq \frac{1}{n}. \end{aligned}$$

Therefore it suffices to find n so that $\frac{1}{n} < r^2 - 2$, i.e. we need $n > \frac{1}{r^2-2}$.

Step 2: Suppose that $r \in \mathbb{Q}$ has the property that $r > 0$ and $r^2 < 2$. Then I claim that there is $x \in S$ such that $x > r$ so that r is not an upper bound for S.

Now we look for x of the form $r + \frac{1}{n}$. The argument that we can choose suiatble n is given in the proof of the lemma above.

Here is a useful result about Dedekind cuts.

Lemma 1.4. *Let L be a Dedekind cut and $u \notin L$. Then u is an upper bound for L, i.e. every $a \in L$ satisfies $a < u$.*

Proof. Let $a \in L$. Then $a \neq u$ because $u \notin L$ and $a \in L$. If $a > u$ then $u \in L$ by (III), which is also impossible. Hence $a < u$. □

Now let us define arithmetic operations on Dedekind cuts. We define addition here; one case of the product is on the HW.

Proposition 1.5. *Given Dedekind cuts L, M define the subset $L + M$ of $\mathbb{Q}$ by*

$$L + M = \{a + b \,|a \in L,\, b \in M\}.$$

Then $L + M$ is a Dedekind cut.

Proof. I will divide this into lots of little steps.
Step 1: $L + M \neq \emptyset$:
Proof. There is $a_0 \in L, b_0 \in M$, which implies that $a_0 + b_0 \in L + M$.
Step 2: $L + M \neq \mathbb{Q}$.
Proof. Since L, M satisfy (I) there are elements u, v such that

$$u \notin L, \qquad v \notin M.$$

We show that $u + v \notin L + M$ by contradiction.

If $u + v \in L + M$ there are $a \in L, b \in M$ so that $u + v = a + b$. But Lemma **??** shows that $a < u$ (since $a \in L, u \notin L$. Similarly $b < v$. Therefore $a + b < u + v$, which is impossible.

Steps 1 and 2 show that (I) holds for $L + M$. The next two steps show that it satisfies the other conditions.

Step 3: *$L+M$ has no maximal element.*
Proof. Let $a+b \in L+M$. Since L has no maximal element there is $x \in L$ such that $a < x$. Then $x+b \in L+M$ and $a+b < x+b$. Therefore $a+b$ is not maximal in $L+M$. Since this holds for all $a+b \in L+M$, the set $L+M$ has no maximal element.

Step 4: *$L+M$ satisfies condition* (III).
Proof. Suppose $x < y$ where $y \in L+M$. Hence we can write $y = a+b$. Then $x = a+b-(y-x) = a-(y-x)+b = a'+b$ where $a' = a-(y-x) < a$. Since $a \in L$ we know $a' \in L$ since L satisfies (III). Hence we may write $x = a'+b$ as the sum of an element in L and an element in M. Therefore $x \in L+M$, as required. □

With this notion of $+$ the zero element is $L_0 := (-\infty, 0)$. In other words, I claim that:

Lemma 1.6. *For any Dedekind cut M we have $M + L_0 = M$, where $L_0 = (-\infty, 0)$.*

Proof. We must show $L_0 + M \subset M$ and $M \subset L_0 + M$.
Proof that $L_0 + M \subset M$.
Since $\{L_0 + M = \{a+b : a \in L_0, b \in M\}$ we must show that every element of the form $a+b$ where $a \in L_0, b \in M$ lies in M. But $a \in L_0$ implies $a < 0$. Hence $a+b < b$. Hence $a+b \in M$ by condition (III) for M.

Proof that $M \subset L_0 + M$.
Given any $m \in M$ we must find $a \in L_0, b \in M$ such that $m = a+b$. Notice that $a < 0$ so that we must have $b > m$. But there is $b > m \in M$ by condition 2. for a Dedekind cut. Therefore pick such b and then define $a := m-b$. Then $a \in \mathbb{Q}$ and $a < 0$ so that $a \in L_0$. Hence $m = (m-b)+b = a+b \in L_0 + M$, as required. □

The Order relation We define $L \le M$ if $L \subseteq M$. It is immediate that this is an order relation. Moreover, given any Dedekind cuts L, M we have either $L \subseteq M$ or $M \subseteq L$ (on HW). Finally notice that every set $S := \{L_s \mid s \in S\}$ that is bounded above has a least upper bound U; namely

$$U := \cup_{s \in S} L_s.$$

Why do we require that a Dedekind cut has no maximal element?
A Dedekind cut is the left part of a partition of $\mathbb{Q}$ into two pieces. Each rational number gives two possible partitions

$$(-\infty, a)_{\mathbb{Q}} \cup [a, \infty)_{\mathbb{Q}}, \quad \text{and} \quad (-\infty, a]_{\mathbb{Q}} \cup (a, \infty)_{\mathbb{Q}},$$

(where I wrote $(a,b)_{\mathbb{Q}}$ to denote $(a,b) \cap \mathbb{Q}$). We need to choose one of these – either $(-\infty, a)_{\mathbb{Q}}$ or $(-\infty, a]_{\mathbb{Q}}$. Since there are partitions (such as those given by $\sqrt{2}$) that have no maximal element, it is most consistent to choose $(-\infty, a)_{\mathbb{Q}}$.
For those of you who are interested, here is how you define the negative $-L$ of a Dedekind cut L.
The operation of multiplication by -1 reverses order and hence interchanges the two halves of the partitions, taking left to right and vice versa. The basic idea is that the negative $-L$ of the cut L should consist of the negatives of the right partition: i.e. $if L = (-\infty, a)_{\mathbb{Q}}$ then $-L$ should be $\{-x \mid x \in (a, \infty)_{\mathbb{Q}}\}$. But the condition $x \in (a, \infty)$ is NOT simply $x \notin (-\infty, a)$ since $a \notin (-\infty, a)$. So if we start from L we have to make a more complicated definition that avoids this problem with endpoints.

Given L define $R_L := \{u \in \mathbb{Q} \mid \exists v \in \mathbb{Q} v < u \text{ such that } v \notin L\}$.
One can prove the following: (i) if $L = (-\infty, a)$ for some $a \in \mathbb{Q}$ then $R_L = (a, \infty)$.

(ii) $-R_L := \{-u \mid u \in R\}$ is a Dedekind cut.

(iii) $L + (-R_L) = L_0 (:= (-\infty, 0)$. In other words, $-R_L$ represents the negative of L.
Note that the definition of R_L has to be so complicated in order that its negative has no maximal element.