Speaker: John Roberts

Title: Arithmetic dynamics of iterated maps: order, chaos and symmetry and their arithmetic signatures

Abstract: The study of iterated maps over real or complex domains in n-dimensions is an important topic in dynamical systems. It has informed notions of order (integrability) and disorder (chaos) in dynamics. Polynomial, rational and piecewise affine maps are important examples of algebraic maps which can be represented over any mathematical field in which their coefficients can be so represented. The intersection of iterated maps and number theory is loosely labelled arithmetic dynamics.

Whereas reduction of polynomials over finite fields has been an important part of number theory for hundreds of years, the reduction of the dynamics of maps is relatively new. A big advantage is that over a finite field, the dynamics of an algebraic map can be described exactly and in full by a complete decomposition of the finite phase space into its constituent orbits. The main dynamical questions relate to (periodic) orbit statistics (their number, length and distribution).

In recent years, Franco Vivaldi (London) and I have studied the implications upon reduction of key dynamical structures like integrability and symmetry. Broadly speaking, we ask: If an algebraic map of the real or complex numbers has a structural property expressible in algebraic terms, does the inherited dynamics of this map over a finite field leave a signature of the property in the finite phase space? Our results suggest that there are different characteristic distributions of periods or orbit lengths over finite fields for reductions of maps, dependent upon the algebraic property present but with universal (i.e. map-independent) aspects. In flavour, this resembles the universal distributions found for different ensembles of random matrices which are otherwise independent of the matrix details.

Turning to piecewise affine maps of the plane or torus with appropriate coefficients, we can find the co-existence of ordered and chaotic behaviour if we restrict ourselves to the rational numbers. We show that we can define arithmetic exponents (akin to Lyapunov exponents in real dynamics) to try and distinguish between order and chaos.

I will review our work in these areas and point to some of the related number theoretic, statistical and combinatoric topics that arise, e.g. translations on elliptic surfaces over finite fields and the Hasse-Weil bounds, periodic orbits of toral automorphisms, generalised Lucas sequences, the expected statistics of random and restricted permutations.