Analytic Number Theory
Homework #3
(due Tuesday, March 25, 2014)

Problem 1: By the functional equation for
\[ \xi(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s), \]
the function \( s(1-s)\xi(s) \) can be regarded as an entire function of \( s^2 - s \); what is the order of this function? Use this to obtain the alternative infinite product
\[ \xi(s) = \frac{\xi(1/2)}{4(s - s^2)} \prod_{\rho} \left( 1 - \left( \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}} \right)^2 \right) \]
the product extending over zeros \( \rho \) of \( \xi(s) \) whose imaginary part is positive. [This symmetrical form eliminates the exponential factors \( e^{A+Bs} \) and \( e^{s/\rho} \) occurring in the usual Hadamard factorization of \( \xi(s) \).]

Problem 2: Calculate the mean value
\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 dt \]
provided \( \sigma > 1 \).

Problem 3: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function \( \chi : \mathbb{Z} \to \mathbb{C} \) satisfying \( \chi(n + 15) = \chi(n) \) for all \( n \in \mathbb{Z} \).

Problem 4: Let \( q \) be an integer which has the property that every Dirichlet character \( \chi \) (mod \( q \)) is real valued (takes on only the values 0, \( \pm 1 \)). Show that \( q \) must divide 24.

Problem 5: Use the inequality \( 3 + 4 \cos \theta + \cos 2\theta \geq 0 \) to give an alternative proof that \( L(1, \chi) \neq 0 \) when \( \chi \) is a complex Dirichlet character (a character such that \( \chi \neq \chi \)).