

THE FIRST COEFFICIENT OF LANGLANDS EISENSTEIN SERIES FOR $SL(n, \mathbb{Z})$

DORIAN GOLDFELD, ERIC STADE, AND MICHAEL WOODBURY

Dedicated to Henryk Iwaniec on the occasion of his 75th birthday

ABSTRACT. Fourier coefficients of Eisenstein series figure prominently in the study of automorphic L-functions via the Langlands-Shahidi method, and in various other aspects of the theory of automorphic forms and representations.

In this paper, we define Langlands Eisenstein series for $SL(n, \mathbb{Z})$ in an elementary manner, and then determine the first Fourier coefficient of these series in a very explicit form. Our proofs and derivations are short and simple, and use the Borel Eisenstein series as a template to determine the first Fourier coefficient of other Langlands Eisenstein series.

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1. Introduction

The classical upper half-plane is the set of all complex numbers

$$\mathfrak{h}^2 := \{x + iy \mid x \in \mathbb{R}, y > 0\},$$

which can also be realized, in group theoretic terms, by the Iwasawa decomposition (see [Gol06]) as

$$\mathfrak{h}^2 = GL(2, \mathbb{R}) / (O(2, \mathbb{R}) \cdot \mathbb{R}^\times) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}, y > 0 \right\}.$$

For $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2$ and $s \in \mathbb{C}$ we define the power function

$$(1.1) \quad I_s(g) := y^s.$$

Let $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}$. Then the non-holomorphic Eisenstein series for $SL(2, \mathbb{Z})$ is defined for $\operatorname{Re}(s) > 1$ by the convergent series

$$E(g, s) := \sum_{\gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z})} \frac{1}{2} \cdot I_s(\gamma g).$$

Dorian Goldfeld is partially supported by Simons Collaboration Grant Number 567168.

For $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2$ (with y fixed), the Eisenstein series $E(g, s)$ has a Fourier expansion in the x -variable given by

$$(1.2) \quad \boxed{E(g, s) = \underbrace{y^s + \phi(s)y^{1-s}}_{\text{constant term}} + \underbrace{\frac{2}{\pi^{-s}\Gamma(s)\zeta(2s)}}_{\text{first Fourier coeff.}} \sum_{n \neq 0} \underbrace{\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}}_{\text{Hecke eigenvalue}} W_n(y) \cdot e^{2\pi i n x}}$$

where

$$\phi(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})\zeta(2s - 1)}{\Gamma(s)\zeta(2s)}, \quad \sigma_s(n) = \sum_{\substack{d|n \\ d>0}} d^s, \quad W_n(y) = \frac{\sqrt{y}}{2} \int_0^\infty e^{-\frac{\pi|n|y}{2}(u+\frac{1}{u})} u^s \frac{du}{u}.$$

The Fourier expansion (1.2) is one of the most important in the theory of modular forms. We have singled out the “*constant term*,” the “*first Fourier coefficient*,” and the “*Hecke eigenvalue*,” which have each played a significant role in the history of the subject.

Let F be a number field with associated adèle ring \mathbb{A}_F . The constant term of the Fourier expansion of Langlands Eisenstein series for a quasi-split group over \mathbb{A}_F has been known for a long time (see [Lan76], [Lan71], [GS88]). The so called “*Langlands-Shahidi method*” (first introduced in [Sha81]) is a method to compute local coefficients for generic representations of reductive groups. In the case of Eisenstein series Shahidi uses the Casselman-Shalika formula for Whittaker functions to express the the first coefficient as a product of L-functions (see [Sha85], [Sha90]). This gave a new proof of the analytic continuation and functional equation of Rankin-Selberg L-functions since they occur in the non-constant term of certain Eisenstein series.

The Langlands-Shahidi method of studying L-functions by way of Eisenstein series has numerous applications. For example, Kim and Shahidi apply this method to the analysis of $GL(2) \times GL(3)$ tensor product representations [KS02b], and to the symmetric cube representation on $GL(2)$ [KS99] [KS02b], deriving functoriality results in both cases. Further, from the symmetric cube result, they are able to advance the state of the art concerning the Ramanujan-Petersson and Selberg conjectures for $GL(2)$, obtaining an upper bound of $5/34$ for Hecke eigenvalues of $GL(2)$ Maass forms, over any number field and at any prime (finite or infinite).

In additional work, Kim [Kim03] uses the Langlands-Shahidi method to obtain functoriality results concerning exterior square representations on $GL(4)$, and symmetric fourth power representations on $GL(2)$. As a consequence of the latter result, Kim and Sarnak [Kim03, Appendix 2] obtain a lower bound $\lambda_1 \geq 975/4096 \approx 0.238$ for the first eigenvalue of the Laplacian, acting on the corresponding hyperbolic space. Moreover, in [KS02a], Kim and Shahidi prove a criterion for cuspidality of the $GL(2)$ symmetric fourth power representation, and deduce from this a number of results towards the Ramanujan-Petersson and Sato-Tate conjectures.

In further work, Kim [Kim08] applies the Langlands-Shahidi method to exceptional groups. In this context, various other types of L functions arise, and a number of results concerning the holomorphy of these L functions follow.

There are numerous other applications and potential applications, some of which are discussed in [Kim03]. In sum, information concerning Fourier coefficients of Eisenstein series is central to the Langlands-Shahidi method, which has proved a powerful tool in the theory of

automorphic forms and representations, and has strong potential for relevance to additional Langlands functoriality and related results.

The main goal of this paper is to first define Langlands Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$ in an elementary manner, and then determine the first Fourier coefficient of the Langlands Eisenstein series in a very explicit form. This result is stated in Theorem 4.12 which is the main theorem of this paper. The proof of this theorem is also short and simple (following the methods introduced in [GMW21]) using the Borel Eisenstein series as a template to determine the first Fourier coefficient of other Eisenstein series.

2. Basic functions on the generalized upper half plane \mathfrak{h}^n

For an integer $n \geq 2$, let $U_n(\mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ denote the group of upper triangular unipotent matrices and let $O(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ denote the group of real orthogonal matrices.

Definition 2.1 (Generalized upper half plane). We define the generalized upper half plane as

$$\mathfrak{h}^n := \mathrm{GL}(n, \mathbb{R}) / (O(n, \mathbb{R}) \cdot \mathbb{R}^\times).$$

By the Iwasawa decomposition of $\mathrm{GL}(n)$ (see [Gol06]) every element of \mathfrak{h}^n has a coset representative of the form $g = xy$ where

$$(2.2) \quad x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in U(n, \mathbb{R}), \quad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $y_i > 0$ for each $1 \leq i \leq n-1$. The group $\mathrm{GL}(n, \mathbb{R})$ acts as a group of transformations on \mathfrak{h}^n by left multiplication.

Definition 2.3 (Character of $U_n(\mathbb{R})$). Let $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. For an element $x \in U_n(\mathbb{R})$ of the form

$$(2.4) \quad x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix},$$

we define the character

$$(2.5) \quad \psi_M(x) := m_1 x_{1,2} + m_2 x_{2,3} + \cdots + m_{n-1} x_{n-1,n}.$$

Next, we generalize the power function (1.1) which was used to construct the Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$.

Definition 2.6 (Power function). Fix an integer $n \geq 2$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ with $\alpha_1 + \cdots + \alpha_n = 0$. Let $\rho = (\rho_1, \dots, \rho_n)$, where $\rho_i = \frac{n+1}{2} - i$ for $i = 1, 2, \dots, n$. We define a power function on $xy \in \mathfrak{h}^n$ by

$$(2.7) \quad I_n(xy, \alpha) = \prod_{i=1}^n d_i^{\alpha_i + \rho_i} = \prod_{i=1}^{n-1} y_i^{\alpha_1 + \cdots + \alpha_{n-i} + \rho_1 + \cdots + \rho_{n-i}},$$

where $d_i = \prod_{j \leq n-i} y_j$ is the j -th diagonal entry of the matrix $g = xy$ as above.

Definition 2.8 (Weyl group). Let $W_n \cong S_n$ denote the Weyl group of $\mathrm{GL}(n, \mathbb{R})$. We consider it as the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of permutation matrices, i.e., matrices that have exactly one 1 in each row/column and all zeros otherwise. The *long element* of W_n is $w_{\mathrm{long}} := \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix}$.

Definition 2.9 (Jacquet’s Whittaker function). Let $g \in \mathrm{GL}(n, \mathbb{R})$ with $n \geq 2$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$ with $\alpha_1 + \dots + \alpha_n = 0$. We define the completed Whittaker function $W_{n,\alpha}^\pm : \mathrm{GL}(n, \mathbb{R}) / (\mathrm{O}(n, \mathbb{R}) \cdot \mathbb{R}^\times) \rightarrow \mathbb{C}$ by the integral

$$W_{n,\alpha}^\pm(g) := \prod_{1 \leq j < k \leq n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \cdot \int_{U_4(\mathbb{R})} I_n(w_{\mathrm{long}}ug, \alpha) \overline{\psi_{1,\dots,1,\pm 1}(u)} du,$$

which converges absolutely if $\mathrm{Re}(\alpha_i - \alpha_{i+1}) > 0$ for $1 \leq i \leq n-1$ (cf. [GMW21]), and has meromorphic continuation to all $\alpha \in \mathbb{C}^n$ satisfying $\alpha_1 + \dots + \alpha_n = 0$.

Remark 2.10. With the additional Gamma factors included in this definition (which can be considered as a “*completed*” Whittaker function) there are $n!$ functional equations which is equivalent to the fact that the Whittaker function is invariant under all permutations of $\alpha_1, \alpha_2, \dots, \alpha_n$. Moreover, even though the integral (without the normalizing factor) often vanishes identically as a function of α , this normalization never does.

If g is a diagonal matrix in $\mathrm{GL}(n, \mathbb{R})$ then the value of $W_{n,\alpha}^\pm(g)$ is independent of sign, so we drop the \pm . We also drop the \pm if the sign is $+1$.

3. The Borel Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$

The Borel parabolic subgroup \mathcal{B} for $\mathrm{GL}(n, \mathbb{R})$ is given by

$$\mathcal{B} = \left\{ \left(\begin{array}{cccc} * & * & \cdots & * \\ & * & \cdots & * \\ & & \ddots & \vdots \\ & & & * \end{array} \right) \in \mathrm{GL}(n, \mathbb{R}) \right\}.$$

More general parabolic subgroups are defined in Definition 4.3, but the Borel is the minimal parabolic subgroup. The Borel Eisenstein series $E_{\mathcal{B}}(g, s)$ of $\mathrm{SL}(n, \mathbb{Z})$ is a complex valued function of the variables $g \in \mathrm{GL}(n, \mathbb{R})$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ where $\alpha_1 + \dots + \alpha_n = 0$. For $\Gamma_n := \mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{Re}(\alpha_i) - \mathrm{Re}(\alpha_{i+1}) > 1$, ($i = 1, \dots, n-1$), it is defined by the absolutely convergent series

$$(3.1) \quad E_{\mathcal{B}}(g, \alpha) := \sum_{(\mathcal{B} \cap \Gamma_n) \backslash \Gamma_n} I_n(\gamma g, \alpha).$$

Proposition 3.2. (The M^{th} Fourier-Whittaker coefficient of $E_{\mathcal{B}}$) Define the vector $M := (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}_+^{n-1}$ and the matrix

$$M^* := \begin{pmatrix} m_1 m_2 \cdots m_{n-1} & & & \\ & \ddots & & \\ & & m_1 m_2 & \\ & & & m_1 \\ & & & & 1 \end{pmatrix}.$$

Then the M^{th} term in the Fourier-Whittaker expansion of E_B (see [Gol06]) is given by

$$\int_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} E_B(ug, \alpha) \overline{\psi_M(u)} du = \frac{A_{E_B}(M, \alpha)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}} W_{n, \alpha}(M^* g),$$

where $A_{E_B}(M, \alpha) = A_{E_B}((1, \dots, 1), \alpha) \cdot \lambda_{E_B}(M, \alpha)$, and

$$(3.3) \quad \lambda_{E_B}((m, 1, \dots, 1), \alpha) = \sum_{\substack{c_1, \dots, c_n \in \mathbb{Z}_+ \\ c_1 c_2 \cdots c_n = m}} c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_n^{\alpha_n}, \quad (m \in \mathbb{Z}_+)$$

is the $(m, 1, \dots, 1)^{\text{th}}$ (or more informally the m^{th}) Hecke eigenvalue of E_B .

Proof. See [Gol06]. □

Proposition 3.4. (The first Fourier coefficient of E_B) We have

$$A_{E_B}((1, \dots, 1), \alpha) = c_0 \prod_{1 \leq j < k \leq n} \zeta^*(1 + \alpha_j - \alpha_k)^{-1}$$

for some constant $c_0 \neq 0$ (depending only on n), and $\zeta^*(w) = \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right) \zeta(w)$ is the completed Riemann ζ -function.

Proof. See [GMW21]. □

4. Eisenstein series twisted by Maass cusp forms of lower rank

Definition 4.1 (Langlands parameters). Let $n \geq 2$. A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ is termed a Langlands parameter if $\alpha_1 + \dots + \alpha_n = 0$.

Definition 4.2 (Maass cusp forms). Fix $n \geq 2$. A Maass cusp form with Langlands parameter $\alpha \in \mathbb{C}^n$ for $\text{SL}(n, \mathbb{Z})$ is a smooth function $\phi : \mathfrak{h}^n \rightarrow \mathbb{C}$ which satisfies $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \text{SL}(n, \mathbb{Z})$, $g \in \mathfrak{h}^n$. In addition ϕ is square integrable and has the same eigenvalues under the action of the algebra of $\text{GL}(n, \mathbb{R})$ invariant differential operators on \mathfrak{h}^n as the power function $I_n(*, \alpha)$. The Laplace eigenvalue of ϕ is given by

$$\frac{n^3 - n}{24} - \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}{2}.$$

See Section 6 in [Mil02]. The Maass cusp form ϕ is said to be tempered at ∞ if the Langlands parameters $\alpha_1, \dots, \alpha_n$ are all pure imaginary.

Definition 4.3. (Parabolic Subgroup) For $n \geq 2$ and $1 \leq r \leq n$, consider a partition of n given by $n = n_1 + \cdots + n_r$ with positive integers n_1, \dots, n_r . We define the standard parabolic subgroup

$$\mathcal{P} := \mathcal{P}_{n_1, n_2, \dots, n_r} := \left\{ \begin{pmatrix} \text{GL}(n_1) & * & \cdots & * \\ 0 & \text{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{GL}(n_r) \end{pmatrix} \right\}.$$

Letting I_r denote the $r \times r$ identity matrix, the subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}$$

is the unipotent radical of \mathcal{P} . The subgroup

$$M^{\mathcal{P}} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}(n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

is the standard choice of Levi subgroup of \mathcal{P} .

Definition 4.4. (Maass form Φ associated to a parabolic \mathcal{P}) Let $n \geq 2$. Consider a partition $n = n_1 + \cdots + n_r$ with $1 < r < n$. Let $\mathcal{P} := \mathcal{P}_{n_1, n_2, \dots, n_r} \subset \mathrm{GL}(n, \mathbb{R})$. For $i = 1, 2, \dots, r$, let $\phi_i : \mathrm{GL}(n_i, \mathbb{R}) \rightarrow \mathbb{C}$ be either the constant function 1 (if $n_i = 1$) or a Maass cusp form for $\mathrm{SL}(n_i, \mathbb{Z})$ (if $n_i > 1$). The form $\Phi := \phi_1 \otimes \cdots \otimes \phi_r$ is defined on $\mathrm{GL}(n, \mathbb{R}) = \mathcal{P}(\mathbb{R})K$ (where $K = \mathrm{O}(n, \mathbb{R})$) by the formula

$$\Phi(nmk) := \prod_{i=1}^r \phi_i(m_i), \quad (n \in N^{\mathcal{P}}, m \in M^{\mathcal{P}}, k \in K)$$

where $m \in M^{\mathcal{P}}$ has the form $m = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_r \end{pmatrix}$, with $m_i \in \mathrm{GL}(n_i, \mathbb{R})$. In fact, this construction works equally well if some or all of the ϕ_i are Eisenstein series.

Definition 4.5. (Character of a parabolic subgroup) Let $n \geq 2$. Fix a partition $n = n_1 + n_2 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1, n_2, \dots, n_r}$. Define

$$(4.6) \quad \rho_{\mathcal{P}}(j) = \begin{cases} \frac{n-n_1}{2}, & j = 1 \\ \frac{n-n_j}{2} - n_1 - \cdots - n_{j-1}, & j \geq 2. \end{cases}$$

Let $s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$ satisfy $\sum_{i=1}^r n_i s_i = 0$. Consider the function

$$|\cdot|_{\mathcal{P}}^s := I(\cdot, \alpha)$$

on $\mathrm{GL}(n, \mathbb{R})$, where

$$\begin{aligned} \alpha = & \overbrace{\left(s_1 - \rho_{\mathcal{P}}(1) + \frac{1-n_1}{2}, s_1 - \rho_{\mathcal{P}}(1) + \frac{3-n_1}{2}, \dots, s_1 - \rho_{\mathcal{P}}(1) + \frac{n_1-1}{2} \right)}^{n_1 \text{ terms}} \\ & \overbrace{\left(s_2 - \rho_{\mathcal{P}}(2) + \frac{1-n_2}{2}, s_2 - \rho_{\mathcal{P}}(2) + \frac{3-n_2}{2}, \dots, s_2 - \rho_{\mathcal{P}}(2) + \frac{n_2-1}{2} \right)}^{n_2 \text{ terms}} \\ & \vdots \\ & \overbrace{\left(s_r - \rho_{\mathcal{P}}(r) + \frac{1-n_r}{2}, s_r - \rho_{\mathcal{P}}(r) + \frac{3-n_r}{2}, \dots, s_r - \rho_{\mathcal{P}}(r) + \frac{n_r-1}{2} \right)}^{n_r \text{ terms}}. \end{aligned}$$

The conditions $\sum_{i=1}^r n_i s_i = 0$ and $\sum_{i=1}^r n_i \rho_{\mathcal{P}}(i) = 0$ guarantee that α 's entries sum to zero. When $g \in \mathcal{P}$, with diagonal block entries $m_i \in \mathrm{GL}(n_i, \mathbb{R})$, one has

$$|g|_{\mathcal{P}}^s = \prod_{i=1}^r |\det(m_i)|^{s_i},$$

so that $|\cdot|_{\mathcal{P}}^s$ restricts to a character of \mathcal{P} which is trivial on $N^{\mathcal{P}}$.

Definition 4.7. (Langlands Eisenstein series twisted by Maass forms of lower rank) Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ with $n \geq 2$. Consider a parabolic subgroup $\mathcal{P} = \mathcal{P}_{n_1, \dots, n_r}$ of $\mathrm{GL}(n, \mathbb{R})$ and functions Φ and $|\cdot|_{\mathcal{P}}^s$ as given in Definitions 4.4 and 4.5, respectively. Let

$$s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r, \quad \text{where } \sum_{i=1}^r n_i s_i = 0.$$

The Langlands Eisenstein series determined by this data is defined by

$$(4.8) \quad E_{\mathcal{P}, \Phi}(g, s) := \sum_{\gamma \in (\mathcal{P} \cap \Gamma) \backslash \Gamma} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s + \rho_{\mathcal{P}}}$$

as an absolutely convergent sum for $\mathrm{Re}(s_i)$ sufficiently large, and extends to all $s \in \mathbb{C}^r$ by meromorphic continuation.

For $k = 1, 2, \dots, r$, let $\alpha^{(k)} := (\alpha_{k,1}, \dots, \alpha_{k,n_k})$ denote the Langlands parameters of ϕ_k . We adopt the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. Then the Langlands parameters of $E_{\mathcal{P}, \Phi}(g, s)$ (denoted $\alpha_{\mathcal{P}, \Phi}(s)$) are

$$(4.9) \quad \left(\overbrace{\alpha_{1,1} + s_1, \dots, \alpha_{1,n_1} + s_1}^{n_1 \text{ terms}}, \overbrace{\alpha_{2,1} + s_2, \dots, \alpha_{2,n_2} + s_2}^{n_2 \text{ terms}}, \dots, \overbrace{\alpha_{r,1} + s_r, \dots, \alpha_{r,n_r} + s_r}^{n_r \text{ terms}} \right).$$

Proposition 4.10. (The M^{th} Fourier coefficient of $E_{\mathcal{P}, \Phi}$) Let $s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$, where $\sum_{i=1}^r n_i s_i = 0$. Consider $E_{\mathcal{P}, \Phi}(*, s)$ with associated Langlands parameters $\alpha_{\mathcal{P}, \Phi}(s)$ as defined in (4.9). Let $M = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}_+^{n-1}$. Then the M^{th} term in the Fourier-Whittaker expansion of $E_{\mathcal{P}, \Phi}$ is

$$\int_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} E_{\mathcal{P}, \Phi}(ug, s) \overline{\psi_M(u)} du = \frac{A_{E_{\mathcal{P}, \Phi}}(M, s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}} W_{\alpha_{\mathcal{P}, \Phi}(s)}(Mg),$$

where $A_{E_{\mathcal{P}, \Phi}}(M, s) = A_{E_{\mathcal{P}, \Phi}}((1, \dots, 1), s) \cdot \lambda_{E_{\mathcal{P}, \Phi}}(M, s)$, and

$$(4.11) \quad \lambda_{E_{\mathcal{P}, \Phi}}((m, 1, \dots, 1), s) = \sum_{\substack{1 \leq c_1, c_2, \dots, c_r \in \mathbb{Z} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r}.$$

is the $(m, 1, \dots, 1)^{\mathrm{th}}$ (or more informally the m^{th}) Hecke eigenvalue of $E_{\mathcal{P}, \Phi}$.

Proof. The proof of (4.11) is given in [Gol06]. □

Theorem 4.12. (The first Fourier coefficient of $E_{\mathcal{P},\Phi}$) Assume that each Maass form ϕ_k (with $1 \leq k \leq r$) occurring in Φ has Langlands parameters $\alpha^{(k)} := (\alpha_{k,1}, \dots, \alpha_{k,n_k})$ with the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. We also assume that each ϕ_k is normalized to have Petersson norm $\langle \phi_k, \phi_k \rangle = 1$. Then the first coefficient of $E_{\mathcal{P},\Phi}$ is given by

$$A_{E_{\mathcal{P},\Phi}}((1, \dots, 1), s) = \prod_{\substack{k=1 \\ n_k \neq 1}}^r L^*(1, \text{Ad } \phi_k)^{-\frac{1}{2}} \prod_{1 \leq j < \ell \leq r} L^*(1 + s_j - s_\ell, \phi_j \times \phi_\ell)^{-1}$$

up to a non-zero constant factor with absolute value depending only on n . Here

$$L^*(1, \text{Ad } \phi_k) = L(1, \text{Ad } \phi_k) \prod_{1 \leq i \neq j \leq n_k} \Gamma\left(\frac{1 + \alpha_{k,i} - \alpha_{k,j}}{2}\right)$$

and

$$L^*(1 + s_j - s_\ell, \phi_j \times \phi_\ell) = \begin{cases} L^*(1 + s_j - s_\ell, \phi_j) & \text{if } n_\ell = 1 \text{ and } n_j \neq 1, \\ L^*(1 + s_j - s_\ell, \phi_\ell) & \text{if } n_j = 1 \text{ and } n_\ell \neq 1, \\ \zeta^*(1 + s_j - s_\ell) & \text{if } n_j = n_\ell = 1. \end{cases}$$

Otherwise, $L^*(1 + s_j - s_\ell, \phi_j \times \phi_\ell)$ is the completed Rankin-Selberg L -function.

Proof. To prove Theorem 4.12 we apply the template method introduced in [GMW21]. In the template protocol we replace each cusp form ϕ_k in Φ with a (smaller) Borel Eisenstein series

$$E_{\mathcal{B}}(*, \alpha^{(k)})$$

with the same Langlands parameters as ϕ_k . The only issue that remains is how to determine the correct normalization of $E_{\mathcal{B}}(*, \alpha^{(k)})$. Since ϕ_k has Petersson norm = 1, it follows from [GSW21] that the first Fourier coefficient of ϕ_k (denoted $A_{\phi_k}(1, \dots, 1)$) is given by

$$A_{\phi_k}(1, \dots, 1) = \begin{cases} L(1, \text{Ad } \phi)^{-\frac{1}{2}} \prod_{1 \leq j < k \leq n} \Gamma\left(\frac{1 + \alpha_j - \alpha_k}{2}\right)^{-1} & \text{if } n_k > 1, \\ 1 & \text{if } n_k = 1. \end{cases}$$

up to a non-zero constant factor with absolute value depending only on n . This together with (3.4) shows that

$$(4.13) \quad A_{\phi_k}(1, \dots, 1) \left(\prod_{1 \leq i < j \leq n_k} \zeta^*(1 + \alpha_{k,i} - \alpha_{k,j}) \right) \cdot E_{\mathcal{B}}(*, \alpha^{(k)})$$

has exactly the same first coefficient as ϕ_k up to a non-zero constant factor with absolute value depending only on n .

Recall the Langlands parameters of $E_{\mathcal{P},\Phi}(g, s)$ (denoted $\alpha_{\mathcal{P},\Phi}(s)$) given by

$$(4.14) \quad \left(\overbrace{\alpha_{1,1} + s_1, \dots, \alpha_{1,n_1} + s_1}^{n_1 \text{ terms}}, \overbrace{\alpha_{2,1} + s_2, \dots, \alpha_{2,n_2} + s_2}^{n_2 \text{ terms}}, \dots, \overbrace{\alpha_{r,1} + s_r, \dots, \alpha_{r,n_r} + s_r}^{n_r \text{ terms}} \right).$$

By replacing each ϕ_k with (4.13) we may form a new Borel Eisenstein series $E_{\mathcal{B},\text{new}}$ with Langlands parameters given by (4.14). We then apply (3.4) to obtain the first coefficient of $E_{\mathcal{B},\text{new}}$ which takes the form

$$\begin{aligned}
& \left[\prod_{\substack{1 \leq k \leq r \\ n_k \neq 1}} L(1, \text{Ad } \phi)^{-\frac{1}{2}} \prod_{1 \leq j < k \leq n} \Gamma\left(\frac{1 + \alpha_j - \alpha_k}{2}\right)^{-1} \left(\prod_{1 \leq i < j \leq k} \zeta^*(1 + \alpha_{k,i} - \alpha_{k,j}) \right) \right] \\
& \cdot \left(\prod_{1 \leq k \leq r} \prod_{1 \leq i < j \leq n_k} \zeta^*(1 + \alpha_{k,i} - \alpha_{k,j}) \right)^{-1} \left(\prod_{1 \leq k < \ell \leq r} \prod_{1 \leq i \leq n_k} \prod_{1 \leq j \leq n_\ell} \zeta^*(1 + s_k - s_\ell + \alpha_{k,i} - \alpha_{\ell,j}) \right)^{-1} \\
& = \left(\prod_{\substack{1 \leq k \leq r \\ n_k \neq 1}} L^*(1, \text{Ad } \phi_k)^{-\frac{1}{2}} \right) \cdot \left(\prod_{1 \leq k < \ell \leq r} \prod_{1 \leq i \leq n_k} \prod_{1 \leq j \leq n_\ell} \zeta^*(1 + s_k - s_\ell + \alpha_{k,i} - \alpha_{\ell,j}) \right)^{-1}
\end{aligned}$$

up to a non-zero constant factor with absolute value depending only on n .

By the template method, the occurrence of $\zeta^*(1 + s_k - s_\ell + \alpha_{k,i} - \alpha_{\ell,j})$ in the first coefficient of $E_{\mathcal{B}, \text{new}}$ tells us that $L^*(1 + s_k - s_\ell, \phi_k \times \phi_\ell)$ is the corresponding component of the first coefficient of $E_{\mathcal{P}, \Phi}(g, s)$ provided neither ϕ_k or ϕ_ℓ are the constant function one. The other cases (when one or both of ϕ_k, ϕ_ℓ equal 1) follow in a similar manner. \square

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027, USA

E-mail address: goldfeld@columbia.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO BOULDER, BOULDER, COLORADO 80309, USA

E-mail address: stade@colorado.edu

DEPARTMENT OF MATHEMATICS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN RD, PISCATAWAY, NJ 08854-8019, USA

E-mail address: michael.woodbury@rutgers.edu