NATURAL BOUNDARIES AND THE CORRECT NOTION OF INTEGRAL MOMENTS OF L–FUNCTIONS

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Abstract. It is shown that a large class of multiple Dirichlet series which arise naturally in the study of moments of L–functions have natural boundaries. As a remedy we consider a new class of multiple Dirichlet series whose elements have nice properties: a functional equation and meromorphic continuation. We believe this class reveals the correct notion of integral moments of L–functions.

§1. Introduction

The problem of obtaining asymptotic formulae (as $T \to \infty$) for the integral moments

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2r}\, dt \qquad (\text{for } r = 1, 2, 3, \dots) \tag{1.1}$$

is approximately 100 years old and very well known.. See [CFKRS] for a nice exposition of this problem and its history. Following [Be-Bu], we note that it was proved by Carlson that if $\sigma > 1 - \frac{1}{r}$ then

$$\int_0^T |\zeta(\sigma+it)|^{2r}\, dt \;\sim\; \left[\sum_{n=1}^{\infty} d_r(n)^2 n^{-2\sigma}\right]\cdot T, \qquad (T \to \infty).$$

Furthermore

$$\sum_{n=1}^{\infty} d_r(n)^2 n^{-s} = \zeta(s)^{r^2} \prod_p P_r\left(p^{-s}\right),$$

where

$$P_r(x) = (1-x)^{2r-1} \sum_{n=0}^{r-1} \binom{r-1}{n}^2 x^r.$$

Now Esterman [E] showed that the Euler product $\prod_p P_r(s)$ is absolutely convergent for $\Re(s) > \frac{1}{2}$, and that it has meromorphic continuation to $Re(s) > 0$. He also proved the disconcerting theorem that if $r \geq 3$ then the Euler product $\prod_p P_r(s)$ has a natural boundary on the line $\Re(s) = 0$. Estermann's result was later generalized by Kurokowa (see [K1, K2]) to a much larger class of Euler products. This situation, where an innocuous looking L–function has a natural boundary, is now

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called the Estermann phenomenon. A very interesting instance where the Estermann phenomenon occurs is for L–functions formed with the arithmetic Fourier coefficients $a(n), n = 1, 2, 3, \ldots$ of an automorphic form on $GL(2)$, say. The L–functions

$$\sum_{n=1}^{\infty} a(n)n^{-s}, \qquad \sum_{n=1}^{\infty} |a(n)|^2 n^{-s},$$

both have nice properties: meromorphic continuation and functional equation, but the L–function

$$\sum_{n=1}^{\infty} |a(n)|^r n^{-s} \tag{1.2}$$

will have a natural boundary if $r \geq 3$. Thus the L–function defined in (1.2) does not have the *correct* structure when $r \geq 3$. It is now generally believed that the "correct notion" of (1.2) is the r^{th} symmetric power L–function as in [S].

Another approach to obtain asymptotics for (1.1) is to study the meromorphic continuation (in the complex variable w) of the zeta integral

$$\mathcal{Z}_r(w) = \int_1^{\infty} |\zeta(\tfrac{1}{2} + it)|^{2r}\, t^{-w}\, dt, \tag{1.3}$$

for r a positive rational integer where this integral is easily shown to be absolutely convergent if $\Re(w)$ sufficiently large. Such an approach was pioneered by Ivić, Jutila and Motohashi [I, J, IJM, M3] and somewhat later in [DGH].

One of the aims of this paper is to give a rough sketch of a proof that the function $\mathcal{Z}_r(w)$, for $r \geq 3$, has a natural boundary at $\Re(w) = \frac{1}{2}$. For simplicity of exposition, we shall consider (1.3) only in the special case when $r = 3$. There is an infinite class of other examples of this phenomenon where our method of proof should generalize. For instance,

$$\int_1^{\infty} |\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2} + it)|^4\, t^{-w}\, dt \;=\; \int_1^{\infty} |\zeta(\tfrac{1}{2} + it)L(\tfrac{1}{2} + it, \chi_{-4})|^4\, t^{-w}\, dt,$$

which is compatible with $\mathcal{Z}_4(w)$, should also have a natural boundary.

In view of the fact that the Estermann phenomenon occurs for the integrals (1.1), (1.3) we believe that the *classical* $2r$-th integral moment of zeta

$$\int_0^{T} |\zeta(\tfrac{1}{2} + it)|^{2r}\, dt \tag{1.4}$$

does not have the *correct* structure when $r \geq 3$. It is therefore doubtful that *substantial* advances in the theory of the Riemann zeta-function will come from further investigations of (1.4).

The final goal of this paper is to provide an alternative to (1.4) in the same spirit that the symmetric power L–function is an alternative to (1.2). Accordingly, in §3, we introduce what we believe to be the "correct notion" of integral moment of L–functions.

§2. Multiple Dirichlet series with natural boundaries

For $s_1, \ldots, s_r$, and $w \in \mathbb{C}$ with sufficiently large real parts, let

$$Z(s_1, \ldots, s_r, w) = \int_1^\infty \zeta(s_1 + it)\zeta(s_1 - it) \cdots \zeta(s_r + it)\zeta(s_r - it)\, t^{-w}\, dt. \tag{2.1}$$

This multiple Dirichlet series was considered in [DGH], and it is more convenient to study this function rather than $\mathcal{Z}_r(w)$. Specializing $r = 3$, we can write

$$Z(s_1, s_2, s_3, w) \,=\, \sum_{m,n} \frac{1}{(mn)^{\Re(s_1)}} \int_1^\infty \left(\frac{m}{n}\right)^{it} \zeta(s_2 + it)\zeta(s_2 - it)\zeta(s_3 + it)\zeta(s_3 - it)\, t^{-w}\, dt.$$

The reason why $\mathcal{Z}_3(w)$ should have a natural boundary is based on a simple idea. The inner integral admits meromorphic continuation to $\mathbb{C}^3$. For $s_2 = s_3 = \frac{1}{2}$, this function should have infinitely many poles on the line $\Re(w) = \frac{1}{2}$, where the position of the poles depends on m, n. As $m, n \to \infty$ the number of poles in any fixed interval will tend to infinity. Summing over m, n "all these poles form" a natural boundary. Accordingly, the main difficulty is to meromorphically continue the integral

$$\int_1^\infty \left(\frac{m}{n}\right)^{it} \zeta(s_2 + it)\zeta(s_2 - it)\zeta(s_3 + it)\zeta(s_3 - it)\, t^{-w}\, dt, \tag{2.2}$$

as a function of s_2, s_3, w to $\mathbb{C}^3$ (see also Motohashi [M2] and [M3], where the integral (2.2) with t^{-w} replaced by a Gaussian weight is studied). When $m = n = 1$, the meromorphic continuation of (2.2) was already established by Motohashi in [M1]. Although this integral can certainly be studied by his method, the approach we follow is based on the more general ideas developed in [G], [Di-Go1], [Di-Go2], [Di-Ga1] and [Di-Ga-Go]. Using our techniques, it is possible to study in a *unified* way very general integrals attached to integral moments.

We remark that one can establish the meromorphic continuation of the slightly more general integral

$$\int_1^\infty \left(\frac{m}{n}\right)^{it} L(s_1 + it, f)\, L(s_2 - it, f)\, t^{-w}\, dt, \tag{2.3}$$

where f is an automorphic form on $GL_2(\mathbb{Q})$ and $L(s, f)$ is the L–function attached to f. Note that this implies the meromorphic continuation of an integral of type

$$\int_1^\infty L(s_1 + it, f)\, L(s_2 - it, f) \left| \sum_{n \le N} a_n n^{it} \right|^2 t^{-w}\, dt \qquad (\text{with } a_n \in \mathbb{C} \text{ for } 1 \le n \le N).$$

In fact, it is technically easier to study the integral (2.3) when f is a cuspform on $SL_2(\mathbb{Z})$ than the corresponding analysis of (2.2). Accordingly, to illustrate our point, we shall discuss, for simplicity, the case when f is a holomorphic cuspform of (even) weight κ for $SL_2(\mathbb{Z})$. Then f has a Fourier expansion

$$f(z) \,=\, \sum_{\ell=1}^\infty a_\ell e^{2\pi i \ell z}, \qquad (z = x + iy,\ y > 0).$$

For m, n two coprime positive integers, consider the congruence subgroup

$$\Gamma_{m,n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; b \equiv 0 \pmod{m},\ c \equiv 0 \pmod{n} \right\}.$$

Then, the function $F_{\frac{n}{m}}(z) := y^\kappa\, \overline{f\left(\frac{n}{m}z\right)} f(z)$ is $\Gamma_{m,n}$–invariant. For $v \in \mathbb{C}$, let $\varphi(z)$ be a function satisfying

$$\varphi(\rho z) = \rho^v \varphi(z) \qquad \text{(for } \rho > 0 \text{ and } z = x+iy,\ y>0\text{)},$$

and (formally) define the Poincaré series

$$P(z;\varphi) \;=\; \sum_{\gamma \in Z\backslash\Gamma_{m,n}} \varphi(\gamma z), \tag{2.4}$$

where Z is the center of $\Gamma_{m,n}$. To ensure convergence, one can choose for instance

$$\varphi(z) = y^v \left(\frac{y}{\sqrt{x^2+y^2}} \right)^w \tag{2.5}$$

where v, $w \in \mathbb{C}$ with sufficiently large real parts. This type of Poincaré series were introduced by Anton Good in [G].

Let $\langle\ ,\ \rangle$ denote the Petersson scalar product for automorphic forms for the group $\Gamma_{m,n}$. As in [Di-Go1], we have the following.

Proposition 2.6. *Let m and n be two coprime positive integers, and let $P(z;\varphi)$, $F_{\frac{n}{m}}$ and $\Gamma_{m,n}$ be as defined above. For $\sigma > 0$ sufficiently large and φ defined by (2.5), we have*

$$\Big\langle P(\cdot,\varphi),\ F_{\frac{n}{m}} \Big\rangle = \frac{\pi(2\pi)^{-(v+\kappa+1)}\Gamma(w+v+\kappa-1)}{2^{w+v+\kappa-2}} \cdot \left(\frac{m}{n}\right)^\sigma$$
$$\cdot \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{it} L(\sigma+it,f)L(v+\kappa-\sigma-it,f) \cdot \frac{\Gamma(\sigma+it)\Gamma(v+\kappa-\sigma-it)}{\Gamma\left(\frac{w}{2}+\sigma+it\right)\Gamma\left(\frac{w}{2}+v+\kappa-\sigma-it\right)}\,dt.$$

As we already pointed out, the above proposition (with appropriate modifications) remains valid if the cuspform f is replaced by a *truncation* of the usual Eisenstein series $E(z,s)$ (for instance, on the line $\Re(s) = \frac{1}{2}$), or a Maass form. On the other hand, using Stirling's formula, it can be shown that the kernel in the above integral is (essentially) asymptotic to t^{-w}, as $t \to \infty$. This fact holds whether f is holomorphic or not. It follows that the meromorphic continuation of (2.3) can be obtained from the meromorphic continuation (in $w \in \mathbb{C}$) of the Poincaré series (2.4).

The meromorphic continuation of the Poincaré series (2.4) can be obtained by spectral theory[1], as in [Di-Go1]. To describe the contribution from the discrete part of the spectrum, let

$$\eta(z) = y^{\frac{1}{2}} \sum_{\ell \neq 0} \rho(\ell)\, K_{i\mu}(2\pi|\ell|y)\, e^{2\pi i \ell x}$$

[1]The Poincaré series $P(z,\varphi)$ is *not* square-integrable. Just after an obvious Eisenstein series is subtracted, the remaining part is not only in L^2 but also has sufficient decay so that its integrals against Eisenstein series converge absolutely (see [Di-Go1], [Di-Go2] and [Di-Ga1]).

($K_\mu(y)$ is the K–Bessel function) be a Maass cuspform (for the group $\Gamma_{m,n}$) which is an eigenfunction of the Laplacian with eigenvalue $\frac{1}{4}+\mu^2$. We shall need the well known transforms

$$\int\limits_{-\infty}^{\infty} (x^2+1)^{-w} e^{-2\pi i\ell xy}\, dx \;=\; \frac{2\pi^w}{\Gamma(w)}\,(|\ell|y)^{w-\frac12}\, K_{\frac12-w}(2\pi|\ell|y), \qquad \Big(\Re(w)>\frac{1}{2}\Big),$$

and

$$\int\limits_{0}^{\infty} y^v K_{i\mu}(y)\, K_{\frac12-w}(y)\frac{dy}{y} \;=\; \frac{2^{v-3}\,\Gamma\Big(\frac{\frac12-i\mu+v-w}{2}\Big)\,\Gamma\Big(\frac{\frac12+i\mu+v-w}{2}\Big)\,\Gamma\Big(\frac{-\frac12-i\mu+v+w}{2}\Big)\,\Gamma\Big(\frac{-\frac12+i\mu+v+w}{2}\Big)}{\Gamma(v)},$$

which is valid provided $\Re(v+w)>\frac12$ and $\Re(w-v)<\frac12$. (**These conditions hold if μ is real, i.e., if we assume Selberg conjecture**). Unfolding the integral, and applying the above transforms, one obtains

(2.7)
$$\begin{aligned}
\frac{\langle P(\cdot,\varphi),\eta\rangle}{\langle\eta,\eta\rangle} &= \frac{1}{\langle\eta,\eta\rangle}\int\limits_0^\infty\int\limits_{-\infty}^{\infty} y^{v+\frac12}\left(\frac{y}{\sqrt{x^2+y^2}}\right)^w \sum_{\ell\neq0}\overline{\rho(\ell)}\, K_{i\mu}(2\pi|\ell|y)\, e^{-2\pi i\ell x}\,\frac{dxdy}{y^2}\\
&= \frac{1}{\langle\eta,\eta\rangle}\sum_{\ell\neq0}\overline{\rho(\ell)}\int\limits_0^\infty\int\limits_{-\infty}^{\infty} y^{v+\frac12}(1+x^2)^{-\frac{w}{2}} K_{i\mu}(2\pi|\ell|y)\, e^{-2\pi i\ell xy}\,\frac{dxdy}{y}\\
&= \frac{2\pi^{\frac{w}{2}}}{\langle\eta,\eta\rangle\cdot\Gamma(\frac{w}{2})}\sum_{\ell\neq0}\overline{\rho(\ell)}\,|\ell|^{\frac{w-1}{2}}\int\limits_0^\infty y^{v+\frac{w}{2}}\, K_{i\mu}(2\pi|\ell|y)\, K_{\frac{1-w}{2}}(2\pi|\ell|y)\,\frac{dy}{y}\\
&= \frac{\pi^{-v}}{2\langle\eta,\eta\rangle}L(v+\tfrac12,\,\overline{\eta})\cdot\frac{\Gamma\Big(\frac{\frac12-i\mu+v}{2}\Big)\,\Gamma\Big(\frac{\frac12+i\mu+v}{2}\Big)\,\Gamma\Big(\frac{-\frac12-i\mu+v+w}{2}\Big)\,\Gamma\Big(\frac{-\frac12+i\mu+v+w}{2}\Big)}{\Gamma(v+\frac{w}{2})\Gamma(\frac{w}{2})}.
\end{aligned}$$

Here $L(s,\eta)$ is the L–function associated to η. Note that the above computation is valid (all integrals and infinite sums converge absolutely) provided v,w have large real parts. The identity (2.7) then extends by analytic continuation. The ratio of products of gamma functions in the right hand side of (2.7) has simple poles at $v+w=\frac12\pm i\mu$ with corresponding residues

$$\frac{\pi^{-v}}{\langle\eta,\eta\rangle}\cdot\frac{\Gamma(\pm i\mu)\Gamma\Big(\frac{\frac12\mp i\mu+v}{2}\Big)}{\Gamma\Big(\frac{\frac12\pm i\mu-v}{2}\Big)}\cdot L(v+\tfrac12,\overline{\eta}).$$

For $\Re(w)\geq\frac12$, the above residues are almost always non-zero (**this should be true, but not easy to justify). There is also a factor $\langle\eta,F_{\frac{n}{m}}\rangle$ in the spectral decomposition, which should be non-zero for most η's. This can probably be verified when f is the Eisenstein series $E(z,s)$ on $SL_2(\mathbb{Z})$. Also, assume that the subspace with eigenvalue $\frac14+\mu^2$ is one dimensional. When $v=0$, we know that**

$$\frac{L(v+\frac12,\overline{\eta})}{\overline{\rho(1)}}\;\geq\;0$$

by Waldspurger. Furthermore, if the sign of the functional equation of $L(s,\bar{\eta})$ **is** $+1$, **we expect** $L(\frac{1}{2},\bar{\eta}) \neq 0$ **almost always.** It also follows from Weyl's law that the number of such poles with imaginary part in the interval $[-T,T]$ is $\approx T^2$ as $T \to \infty$. Summing over m, n, we see from the above argument that the function

$$\sum_{m,n} m^{-2\Re(s_1)} \Big\langle P(\cdot,\varphi),\ \ F_{\frac{n}{m}} \Big\rangle,$$

with the choices $\sigma = \kappa/2$ and $v = 0$ has a natural boundary at $\Re(w) = \frac{1}{2}$. In a similar manner one may show that the function $Z(s_1, 1/2, 1/2, w)$, in particular, has meromorphic continuation to *at most* $\Re(s_1) \geq \frac{1}{2}$ and $\Re(w) > \frac{1}{2}$.

§3. The correct notion of integral moment

In [Di-Ga-Go], we propose a mechanism to obtain asymptotics for integral moments of GL_r $(r \geq 2)$ automorphic L–functions over an arbitrary number field. In particular, it reveals what we believe should be the *correct* notion of integral moments. Our treatment follows the viewpoint of [Di-Ga1], where second integral moments for GL_2 are presented in a form enabling application of the structure of adele groups and their representation theory. We establish relations of the form

$$\text{moment expansion } = \int_{Z_{\mathbb{A}}GL_r(k)\backslash GL_r(\mathbb{A})} \text{Pé} \cdot |f|^2 \ = \text{ spectral expansion},$$

where Pé is a Poincaré series on GL_r over number field k, for cuspform f on $GL_r(\mathbb{A})$. Roughly, the *moment expansion* is a sum of weighted moments of convolution L–functions $L(s, f \otimes F)$, where F runs over a basis of cuspforms on GL_{r-1}, as well as further continuous-spectrum terms. Indeed, the moment-expansion side itself does involve a spectral decomposition on GL_{r-1}. The *spectral expansion* side follows immediately from the spectral decomposition of the Poincaré series, and (surprisingly) consists of only three parts: a leading term, a sum arising from cuspforms on GL_2, and a continuous part from GL_2. That is, no cuspforms on GL_ℓ with $2 < \ell \leq r$ contribute.

In the case of GL_2 over $\mathbb{Q}$, the above expression gives (for f spherical) the spectral decomposition of the classical integral moment

$$\int_{-\infty}^{\infty} |L(\tfrac{1}{2}+it, f)|^2 g(t)\, dt$$

for suitable smooth weights $g(t)$.

In the simplest case beyond GL_2, take f a spherical cuspform on GL_3 over $\mathbb{Q}$. We construct a weight function $\Gamma(s, v, w, f_\infty, F_\infty)$ depending upon complex parameters s, v, and w, and upon the *archimedean* data for both f and cuspforms F on GL_2, such that $\Gamma(s, v, w, f_\infty, F_\infty)$ has explicit asymptotic behavior, and such that the *moment expansion* arises as an integral

$$\int_{Z_{\mathbb{A}}GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \text{Pé}(g)\, |f(g)|^2\, dg \ \ = \sum_{F \text{ on } GL_2} \frac{1}{2\pi i} \int_{\Re(s)=\frac{1}{2}} |L(s, f\otimes F)|^2 \cdot \Gamma(s, 0, w, f_\infty, F_\infty)\, ds$$

$$+ \ \frac{1}{4\pi i}\frac{1}{2\pi i} \sum_{k\in\mathbb{Z}} \ \int_{\Re(s_1)=\frac{1}{2}} \ \int_{\Re(s_2)=\frac{1}{2}} |L(s_1, f\otimes E^{(k)}_{1-s_2})|^2 \cdot \Gamma(s_1, 0, w, f_\infty, E^{(k)}_{1-s_2,\infty})\, ds_2\, ds_1.$$

Here, for $\Re(s_2) = 1/2$, write $1 - s_2$ in place of $\bar{s}_2$, to maintain holomorphy in complex-conjugated parameters. In this vein, over $\mathbb{Q}$, it is reasonable to put

$$L(s_1, f \otimes \bar{E}^{(k)}_{s_2}) = L(s_1, f \otimes E^{(k)}_{1-s_2}) = \frac{L(s_1 - s_2 + \frac{1}{2}, f) \cdot L(s_1 + s_2 - \frac{1}{2}, f)}{\zeta(2 - 2s_2)} \qquad \text{(finite-prime part)}$$

since the natural normalization of the Eisenstein series $E^{(k)}_{s_2}$ on GL_2 contributes the denominator $\zeta(2s_2)$. In the above expression, F runs over an orthonormal basis for all level-one cuspforms on GL_2, with *no* restriction on the right K_∞–type. The Eisenstein series $E^{(k)}_s$ run over all level-one Eisenstein series for $GL_2(\mathbb{Q})$ with no restriction on K_∞–type denoted here by k. The weight function $\Gamma(s, v, w, f_\infty, F_\infty)$ can be described as follows. Let $U(\mathbb{R})$ denote the subgroup of $GL_3(\mathbb{R})$ of matrices of the form $\begin{pmatrix} I_2 & * \\ & 1 \end{pmatrix}$. For $w \in \mathbb{C}$, define φ on $U(\mathbb{R})$ by

$$\varphi\begin{pmatrix} I_2 & x \\ & 1 \end{pmatrix} \;=\; \left(1 + ||x||^2\right)^{-\frac{w}{2}}$$

and set

$$\psi\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \;=\; e^{2\pi i (x_1 + x_2)}$$

Then, the weight function is (essentially)

$$\begin{aligned}\Gamma(s, v, w, f_\infty, F_\infty) \;=\; |\rho_F(1)|^2 \cdot \int_0^\infty \int_0^\infty \int_{O_2(\mathbb{R})} \int_0^\infty \int_0^\infty \int_{O_2(\mathbb{R})} &(t^2 y)^{v-s+\frac{1}{2}} \cdot (t'^{\,2} y')^{s-\frac{1}{2}}\, \mathcal{K}(h,\, m) \\ &\cdot W_{f,\mathbb{R}}\begin{pmatrix} ty & & \\ & t & \\ & & 1 \end{pmatrix} W_{F,\mathbb{R}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot k\right) \\ &\cdot \overline{W}_{f,\mathbb{R}}\begin{pmatrix} t'y' & & \\ & t' & \\ & & 1 \end{pmatrix} \overline{W}_{F,\mathbb{R}}\left(\begin{pmatrix} y' & \\ & 1 \end{pmatrix} \cdot k'\right) \\ &\cdot dk\, \frac{dy}{y^2}\, \frac{dt}{t}\, dk'\, \frac{dy'}{y'^2}\, \frac{dt'}{t'},\end{aligned}$$

where: $\rho_F(1)$ is the first Fourier coefficient of F,

$$h \;=\; \begin{pmatrix} ty & & \\ & t & \\ & & 1 \end{pmatrix} \begin{pmatrix} k & \\ & 1 \end{pmatrix}, \qquad m \;=\; \begin{pmatrix} t'y' & & \\ & t' & \\ & & 1 \end{pmatrix} \begin{pmatrix} k' & \\ & 1 \end{pmatrix},$$

and

$$\mathcal{K}(h,\, m) \;=\; \int_{U(\mathbb{R})} \varphi(u)\, \psi\left(huh^{-1}\right) \overline{\psi}(mum^{-1})\, du.$$

Here $W_{f,\mathbb{R}}$ and $W_{F,\mathbb{R}}$ denote the Whittaker functions at ∞ attached to f and F, respectively.

To obtain higher moments of automorphic L–functions such as ζ, we replace the cuspform f by a truncated Eisenstein series or wavepacket of Eisenstein series. For example, for GL_3, the continuous part of the above moment expansion gives the following natural integral

$$\int\limits_{\Re(s)=\frac{1}{2}} \int_{-\infty}^{\infty} \left| \frac{\zeta(s+it)^3 \cdot \zeta(s-it)^3}{\zeta(1-2it)} \right|^2 M(s,t,w)\, dt\, ds$$

where M is the smooth weight obtained by summing over the K_∞–types k the function Γ above.

For applications to Analytic Number Theory, one finds it useful to present, in classical language, the derivation of the *explicit* moment identity, when $r = 3$ over $\mathbb{Q}$. To do so, let $G = GL_3(\mathbb{R})$, and define the standard subgroups:

$$P = \left\{ \begin{pmatrix} 2\times 2 & * \\ & 1\times 1 \end{pmatrix} \right\}, \quad U = \left\{ \begin{pmatrix} I_2 & * \\ & 1 \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} 2\times 2 & \\ & 1 \end{pmatrix} \right\}, \quad Z = \text{center of } G.$$

Let N be the unipotent radical of standard minimal parabolic in H, that is, the subgroup of upper-triangular unipotent elements in H, and set $K = O_3(\mathbb{R})$.

For $w \in \mathbb{C}$, define φ on U by

$$\varphi\begin{pmatrix} I_2 & x \\ & 1 \end{pmatrix} = \left(1 + ||x||^2\right)^{-\frac{w}{2}}.$$

We extend φ to G by requiring right K–invariance and left equivariance

$$\varphi(mg) = \left| \frac{\det A}{d^2} \right|^v \cdot \varphi(g) \qquad \left(v \in \mathbb{C},\, g \in G,\, m = \begin{pmatrix} A & \\ & d \end{pmatrix} \in ZH \right).$$

More generally, we can take *suitable* functions (see [Di-Ga1], [Di-Ga2]) φ on U, and extend them to G by right K–invariance and the same left equivariance.

For $\Re(v)$ and $\Re(w)$ sufficiently large, define the Poincaré series

$$\text{Pé}(g) = \text{Pé}(g; v, w) = \sum_{\gamma \in H(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \varphi(\gamma g) \qquad (g \in G) \tag{3.1}$$

where $H(\mathbb{Z})$ is the subgroup of $SL_3(\mathbb{Z})$ whose elements belong to H. Note that $H(\mathbb{Z}) \approx SL_2(\mathbb{Z})$. To see that the series defining Pé(g) converges absolutely and uniformly on compact subsets of G/ZK, one can use the Iwasawa decomposition to make a simple comparison with the maximal parabolic Eisenstein series.

For a cuspform f of type $\mu = (\mu_1, \mu_2)$ on $SL_3(\mathbb{Z})$ (right ZK–invariant), consider the integral

$$I = I(v,w) = \int\limits_{ZSL_3(\mathbb{Z})\backslash G} \text{Pé}(g)\, |f(g)|^2\, dg. \tag{3.2}$$

Unwinding the Poincaré series, we write

$$I = \int\limits_{ZH(\mathbb{Z})\backslash G} \varphi(g)\, |f(g)|^2\, dg.$$

Next, we will use the Fourier expansion (see [Go])

$$f(g) = \sum_{\gamma\in N(\mathbb{Z})\backslash H(\mathbb{Z})} \sum_{\ell_1=1}^{\infty} \sum_{\ell_2\neq 0} \frac{a(\ell_1,\,\ell_2)}{|\ell_1\ell_2|}\cdot W_\mu(L\gamma g) \qquad (\text{with } a(\ell_1,\,\ell_2)=a(\ell_1,-\ell_2)) \tag{3.3}$$

where $N(\mathbb{Z})$ is the subgroup of upper-triangular unipotent elements in $H(\mathbb{Z})$, $L=\mathrm{diag}(\ell_1\ell_2,\ell_1,1)$, and W_μ is the Whittaker function. Then the integral I further unwinds to

$$I = \sum_{\ell_1,\,\ell_2} \frac{a(\ell_1,\,\ell_2)}{|\ell_1\ell_2|} \int\limits_{ZN(\mathbb{Z})\backslash G} \varphi(g)\,W_\mu(Lg)\bar{f}(g)\,dg. \tag{3.4}$$

Now, let P_1 be the (minimal) parabolic subgroup of G of upper-triangular matrices, and let K_1 be the subgroup of K fixing the row vector $(0,0,1)$. Using the Iwasawa decomposition

$$G=P_1\cdot K, \qquad P=(HZ)\cdot U=P_1\cdot K_1,$$

we can write (up to a constant) the right hand side of (3.4) as

$$I = \sum_{\ell_1,\,\ell_2} \frac{a(\ell_1,\,\ell_2)}{|\ell_1\ell_2|} \int\limits_{(N(\mathbb{Z})\backslash H)\times U} \varphi(hu)\,W_\mu(Lhu)\bar{f}(hu)\,dh\,du. \tag{3.5}$$

The constant involved is $\left(\int_{K_1} 1\,dk\right)^{-1}$.

One of the **key** ideas is to decompose the left $H(\mathbb{Z})$–invariant function $\bar{f}(hu)$ along $H(\mathbb{Z})\backslash H$. Accordingly, we have the spectral decomposition

$$\begin{aligned}\bar{f}(hu) &= \int\limits_{(\eta)} \eta(h) \int\limits_{H(\mathbb{Z})\backslash H} \overline{\eta}(m)\bar{f}(mu)\,dm\,d\eta\\ &= \sum_{\ell_1',\,\ell_2'} \frac{\overline{a(\ell_1',\,\ell_2')}}{|\ell_1'\ell_2'|} \int\limits_{(\eta)} \eta(h) \int\limits_{N(\mathbb{Z})\backslash H} \overline{\eta}(m)\,\overline{W}_\mu(L'mu)\,dm\,d\eta.\end{aligned} \tag{3.6}$$

Plugging (3.6) into (3.5), we can decompose

$$I = \sum_{\ell_1,\,\ell_2} \sum_{\ell_1',\,\ell_2'} \frac{a(\ell_1,\,\ell_2)}{|\ell_1\ell_2|} \frac{\overline{a(\ell_1',\,\ell_2')}}{|\ell_1'\ell_2'|} I_{\ell_1,\,\ell_2,\,\ell_1',\,\ell_2'}, \tag{3.7}$$

where, for fixed $\ell_1,\,\ell_2,\,\ell_1',\,\ell_2'$,

$$I_{\ell_1,\,\ell_2,\,\ell_1',\,\ell_2'} = \int\limits_{(\eta)} \int\limits_{(N(\mathbb{Z})\backslash H)\times U} \int\limits_{N(\mathbb{Z})\backslash H} \varphi(hu)\,W_\mu(Lhu)\,\eta(h)\,\overline{W}_\mu(L'mu)\,\overline{\eta}(m)\,dh\,dm\,du\,d\eta. \tag{3.8}$$

The integral over U in (3.8) is

$$\begin{aligned}
&\int\limits_U \varphi(u)\, W_\mu(Lhu)\, \overline{W}_\mu(L'mu)\, du \\
&= W_\mu(Lh)\, \overline{W}_\mu(L'm) \int\limits_U \varphi(u)\, \psi\left(Lhuh^{-1}L^{-1}\right) \overline{\psi}(L'mum^{-1}L'^{\,-1})\, du \\
&= W_\mu(Lh)\, \overline{W}_\mu(L'm) \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \cdots \; dx_2\, dx_3 \\
&= W_\mu(Lh)\, \overline{W}_\mu(L'm)\, \mathcal{K}(Lh,\, L'm),
\end{aligned}$$

where

$$\psi\begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix} = e^{2\pi i(x_1+x_3)}.$$

Therefore,

$$I_{\ell_1,\,\ell_2,\,\ell_1',\,\ell_2'} = \int\limits_{(\eta)} \int\limits_{N(\mathbb{Z})\backslash H} \int\limits_{N(\mathbb{Z})\backslash H} \varphi(h)\, \mathcal{K}(Lh,\, L'm)\, W_\mu(Lh)\, \eta(h)\, \overline{W}_\mu(L'm)\, \overline{\eta}(m)\, dh\, dm\, d\eta. \tag{3.9}$$

For $n \in N$ and $h \in H$, we have:

$$\begin{aligned}
\varphi(nh) &= \varphi(h), \\
\mathcal{K}(Lnh,\, L'm) &= \mathcal{K}(Lh,\, L'm), \\
W_\mu(Lnh) &= \psi\left(LnL^{-1}\right) W_\mu(Lh).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int\limits_{N(\mathbb{Z})\backslash H} \int\limits_{N(\mathbb{Z})\backslash H} \varphi(h)\, \mathcal{K}(Lh,\, L'm)\, W_\mu(Lh)\, \eta(h)\, \overline{W}_\mu(L'm)\, \overline{\eta}(m)\, dh\, dm \\
&= \int\limits_{N\backslash H} \int\limits_{N\backslash H} \varphi(h)\, \mathcal{K}(Lh,\, L'm)\, W_\mu(Lh)\, \overline{W}_\mu(L'm) \\
&\qquad \cdot \int\limits_{N(\mathbb{Z})\backslash N} \psi\left(LnL^{-1}\right) \eta(nh)\, dn \;\; \cdot \int\limits_{N(\mathbb{Z})\backslash N} \overline{\psi}\left(L'n'L'^{\,-1}\right) \overline{\eta}(n'm)\, dn'\, dh\, dm.
\end{aligned} \tag{3.10}$$

To simplify (3.10), let

$$h = \begin{pmatrix} ty & & \\ & t & \\ & & 1 \end{pmatrix} \begin{pmatrix} k & \\ & 1 \end{pmatrix}, \qquad m = \begin{pmatrix} t'y' & & \\ & t' & \\ & & 1 \end{pmatrix} \begin{pmatrix} k' & \\ & 1 \end{pmatrix}, \qquad (k,\, k' \in O_2(\mathbb{R})).$$

The functions η above are of the form $|\det|^{-s} \otimes F$ with $s \in i\mathbb{R}$. In what follows, for convergence purposes, the real part of the parameter s will necessarily be shifted to a fixed (large) $\sigma = \Re(s)$. The shifting occurs in (3.6) (there is a hidden vertical integral in the integral over η).

Remark. For every K–type κ, we choose F in an orthonormal basis consisting of common eigenfunctions for all Hecke operators T_n. Furthermore, this basis is normalized as in Corollary 4.4 and (4.69) [DFI] with respect to Maass operators.

Note that

$$\int\limits_{N(\mathbb{Z})\backslash N} \psi\left(LnL^{-1}\right) F(nh)\, dn \;=\; \frac{\rho_{F}(-\ell_2)}{\sqrt{|\ell_2|}}\, W^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix} |\ell_2|\, y & \\ & 1\end{pmatrix}\cdot k\right), \tag{3.11}$$

$$\int\limits_{N(\mathbb{Z})\backslash N} \overline{\psi}\left(L'n'L'^{\,-1}\right) \bar{F}(n'm)\, dn' \;=\; \overline{\frac{\rho_{F}(-\ell_2')}{\sqrt{|\ell_2'|}}}\, \overline{W}^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix} |\ell_2'|\, y' & \\ & 1\end{pmatrix}\cdot k'\right), \tag{3.12}$$

where $W^{\pm}_{F,\,\mathbb{R}}$ are the GL_2 Whittaker functions attached to F. These functions can be expressed in terms of the *classical* Whittaker function

$$W_{\alpha,\,\beta}(y) \;=\; \frac{y^{\alpha}\, e^{-\frac{y}{2}}}{2\pi i} \int\limits_{-i\infty}^{i\infty} \frac{\Gamma(u)\,\Gamma(-u-\alpha-\beta+\frac12)\,\Gamma(-u-\alpha+\beta+\frac12)}{\Gamma(-\alpha-\beta+\frac12)\,\Gamma(-\alpha+\beta+\frac12)}\, y^{u}\, du,$$

where the contour has loops, if necessary, so that the poles of $\Gamma(u)$ and the poles of the function $\Gamma(-u-\alpha-\beta+\frac12)\Gamma(-u-\alpha+\beta+\frac12)$ are on opposite sides of it. For $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO_2(\mathbb{R})$, we have (see [DFI])

$$W^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\cdot k\right) \;=\; e^{i\kappa\theta}\, W^{\pm}_{F,\,\mathbb{R}}\begin{pmatrix} y & \\ & 1\end{pmatrix} \;=\; e^{i\kappa\theta}\, W_{\pm\frac{\kappa}{2},\, i\mu_F}(4\pi y) \qquad (y>0)$$

if F is an eigenfunction of

$$\Delta_\kappa \;=\; y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) \;-\; i\kappa y\frac{\partial}{\partial x}$$

with eigenvalue $\frac14 + \mu_F^2$. In (3.11) and (3.12), the Whittaker functions are determined by the signs of $-\ell_2$ and $-\ell_2'$, respectively. If F corresponds to a holomorphic, or anti-holomorphic, cuspform, there are no negative, or positive, respectively, terms in its Fourier expansion. We have

$$W^{+}_{F,\,\mathbb{R}}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\cdot k\right) \;=\; e^{i\kappa\theta}\, W^{+}_{F,\,\mathbb{R}}\begin{pmatrix} y & \\ & 1\end{pmatrix} \;=\; e^{i\kappa\theta}\, W_{\frac{\kappa}{2},\, \frac{\kappa_0-1}{2}}(4\pi y) \qquad (\text{for } \kappa \ge \kappa_0 \ge 12,\ y>0)$$

for F corresponding to a holomorphic cuspform of weight κ_0.

Then, making the substitutions

$$t \to \frac{t}{\ell_1}, \qquad y \to \frac{y}{|\ell_2|}, \qquad t' \to \frac{t'}{\ell_1'}, \qquad y' \to \frac{y'}{|\ell_2'|},$$

we can write (3.10) as

$$\begin{aligned}(3.13)\quad &\frac{\sqrt{|\ell_2|}\,\rho_F(-\ell_2)}{(\ell_1^2|\ell_2|)^{v-s}}\,\frac{\sqrt{|\ell_2'|}\,\overline{\rho_F(-\ell_2')}}{(\ell_1'^2|\ell_2'|)^{s}}\int_0^\infty\int_0^\infty\int_{H\cap K}\int_0^\infty\int_0^\infty\int_{H\cap K}(t^2y)^{v-s}\cdot(t'^{\,2}y')^s\,\mathcal{K}(h,\,m)\\ &\qquad\cdot W_\mu\begin{pmatrix}ty&&\\&t&\\&&1\end{pmatrix}W^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix}y&\\&1\end{pmatrix}\cdot k\right)\\ &\qquad\cdot \overline{W}_\mu\begin{pmatrix}t'y'&&\\&t'&\\&&1\end{pmatrix}\overline{W}^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix}y'&\\&1\end{pmatrix}\cdot k'\right)\\ &\qquad\cdot dk\,\frac{dy}{y^2}\,\frac{dt}{t}\,dk'\,\frac{dy'}{y'^2}\,\frac{dt'}{t'},\end{aligned}$$

where

$$\mathcal{K}(h,\,m)\;=\;\int_U\varphi(u)\,\psi\left(huh^{-1}\right)\overline{\psi}(mum^{-1})\,du.$$

Recall that the Rankin-Selberg convolution $L(s,f\otimes F)$ is given by

$$L(s,f\otimes F)\;=\;L(s,f\otimes F_0)\;\;=\sum_{\ell_1,\,\ell_2=1}^{\infty}\frac{a(\ell_1,\,\ell_2)\lambda_{F_0}(\ell_2)}{(\ell_1^2\,\ell_2)^s},$$

where F_0 is the basic ancestor of F, and $\lambda_{F_0}(\ell)$ is the corresponding eigenvalue of the Hecke operator T_ℓ. Since $a(\ell_1,\,\ell_2)=a(\ell_1,-\ell_2)$, it follows from (3.7), (3.9) and (3.13) that

$$\begin{aligned} I\;&=\int_{ZSL_3(\mathbb{Z})\backslash G}\text{Pé}(g)\,|f(g)|^2\,dg\\ &=\sum_{F\text{ in }GL_2}\frac{1}{2\pi i}\int_{\Re(s)=\sigma}L(v+1-s,\,f\otimes F)\,L(s,\,\bar f\otimes\bar F)\,\Gamma_\varphi(s)\,ds,\end{aligned}$$

where

(3.14)

$$\begin{aligned}\Gamma_\varphi(s)\;&=\;\Gamma_\varphi(s,\,v,\,w,\,f,\,F)\\ &=\;\sum_{\pm}\rho_F(\pm1)\overline{\rho_F(\pm1)}\cdot\int_0^\infty\int_0^\infty\int_{H\cap K}\int_0^\infty\int_0^\infty\int_{H\cap K}(t^2y)^{v-s+\frac12}\cdot(t'^{\,2}y')^{s-\frac12}\,\mathcal{K}(h,\,m)\\ &\qquad\cdot W_\mu\begin{pmatrix}ty&&\\&t&\\&&1\end{pmatrix}W^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix}y&\\&1\end{pmatrix}\cdot k\right)\\ &\qquad\cdot \overline{W}_\mu\begin{pmatrix}t'y'&&\\&t'&\\&&1\end{pmatrix}\overline{W}^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix}y'&\\&1\end{pmatrix}\cdot k'\right)\\ &\qquad\cdot dk\,\frac{dy}{y^2}\,\frac{dt}{t}\,dk'\,\frac{dy'}{y'^2}\,\frac{dt'}{t'},\end{aligned}$$

with all four possible sign choices in the sum. Note that we have also replaced s by $s-\frac{1}{2}$.

The kernel $\Gamma_\varphi(s)$ can be expressed as a Barnes type (multiple) integral. To see this, note that

$$\psi\big(huh^{-1}\big) \;=\; e^{2\pi i t(u_1\sin\theta+u_2\cos\theta)}, \qquad \overline{\psi}\big(mum^{-1}\big) \;=\; e^{-2\pi i t'(u_1\sin\theta'+u_2\cos\theta')},$$

with $0\le\theta,\,\theta'\le 2\pi$. Changing the variables $u_1=r\cos\phi,\ u_2=r\sin\phi$ ($r\ge 0$ and $0\le\phi\le 2\pi$), one can write

$$\mathcal{K}(h,\,m) \;=\; \int\limits_0^\infty\int\limits_0^{2\pi} r^2\varphi(r)\, e^{2\pi i r t\sin(\theta+\phi)}\, e^{-2\pi i r t'\sin(\theta'+\phi)}\, d\phi\,\frac{dr}{r}. \tag{3.15}$$

In (3.15), express the two exponentials using the Fourier expansion

$$e^{iu\sin\theta} \;=\; \sum_{\ell=-\infty}^{\infty} J_\ell(u)\, e^{i\ell\theta}.$$

Recalling that

$$W^{\pm}_{F,\,\mathbb{R}}\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\cdot k\right) \;=\; e^{i\kappa\theta}\, W^{\pm}_{F,\,\mathbb{R}}\begin{pmatrix} y & \\ & 1\end{pmatrix},$$

it follows that, up to a positive constant, $\Gamma_\varphi(s)$ is represented by

(3.16)

$$\sum_{\pm}\rho_F(\pm1)\overline{\rho_F(\pm1)}\cdot\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty (t^2y)^{v-s+\frac12}\ (t'^{\,2}y')^{s-\frac12}\cdot\int\limits_0^\infty r^2\varphi(r)\,J_\kappa(2\pi r t)\,J_\kappa(2\pi r t')\,\frac{dr}{r}$$
$$\cdot\, W_\mu\begin{pmatrix} ty & & \\ & t & \\ & & 1\end{pmatrix} W^{\pm}_{F,\,\mathbb{R}}\begin{pmatrix} y & \\ & 1\end{pmatrix}\overline{W}_\mu\begin{pmatrix} t'y' & & \\ & t' & \\ & & 1\end{pmatrix}\overline{W}^{\pm}_{F,\,\mathbb{R}}\begin{pmatrix} y' & \\ & 1\end{pmatrix}\frac{dy}{y^2}\,\frac{dt}{t}\,\frac{dy'}{y'^2}\,\frac{dt'}{t'}.$$

Here we have also used the well-known identity $J_{-\kappa}(z)=(-1)^\kappa J_\kappa(z)$.

To continue the computation, express both $GL_3(\mathbb{R})$ Whittaker functions in (3.16) as (see [Bu])

$$W_\mu\begin{pmatrix} ty & & \\ & t & \\ & & 1\end{pmatrix} = \frac{1}{(2\pi i)^2}\int\limits_{(\delta_1)}\int\limits_{(\delta_2)}\pi^{-\xi_1-\xi_2}\,V(\xi_1,\xi_2)\,t^{1-\xi_1}y^{1-\xi_2}\,d\xi_1\,d\xi_2,$$

where

$$V(\xi_1,\xi_2) \;=\; \frac14\,\frac{\Gamma\big(\frac{\xi_1+\alpha}{2}\big)\Gamma\big(\frac{\xi_1+\beta}{2}\big)\Gamma\big(\frac{\xi_1+\gamma}{2}\big)\Gamma\big(\frac{\xi_2-\alpha}{2}\big)\Gamma\big(\frac{\xi_2-\beta}{2}\big)\Gamma\big(\frac{\xi_2-\gamma}{2}\big)}{\Gamma\big(\frac{\xi_1+\xi_2}{2}\big)},$$

the vertical lines of integration being taken to the right of all poles of the integrand. We shall consider only the $(+,+)$ part of (3.16), assuming $\kappa\ge 0$ and

$$W^{+}_{F,\,\mathbb{R}}\begin{pmatrix} y & \\ & 1\end{pmatrix} \;=\; W_{\frac{\kappa}{2},\,i\mu_{F_0}}(4\pi y).$$

Interchanging the order of integration and applying standard integral formulas (see [GR]), we write the integrals of the $(+,+)$ part of (3.16) corresponding to the above choice of $W^{+}_{F,\mathbb{R}}$ as

$$\begin{aligned}\frac{\pi^{-3(1+v)}}{128}\frac{1}{(2\pi i)^4}\int\limits_{(\delta_1)}\int\limits_{(\delta_2)}\int\limits_{(\delta_1')}\int\limits_{(\delta_2')} V(\xi_1,\xi_2)\,\overline{V}(\xi_1',\xi_2')\,&\frac{\Gamma\big(1+\frac{\kappa}{2}-s-\frac{\xi_1}{2}+v\big)\,\Gamma\big(\frac{\kappa}{2}+s-\frac{\xi_1'}{2}\big)}{\Gamma\big(\frac{\kappa}{2}+s+\frac{\xi_1}{2}-v\big)\,\Gamma\big(\frac{\kappa}{2}+1-s+\frac{\xi_1'}{2}\big)}\\
&\cdot\Gamma\Big(\frac{1-s-\xi_2+v-i\mu_{F_0}}{2}\Big)\Gamma\Big(\frac{1-s-\xi_2+v+i\mu_{F_0}}{2}\Big)\\
&\cdot\Gamma\Big(\frac{s-\xi_2'-i\mu_{F_0}}{2}\Big)\Gamma\Big(\frac{s-\xi_2'+i\mu_{F_0}}{2}\Big)\\
&\cdot\frac{\Gamma\big(\frac{\xi_1+\xi_1'-2v}{2}\big)\Gamma\big(\frac{-\xi_1-\xi_1'+2v+w}{2}\big)}{\Gamma\big(\frac{w}{2}\big)}\,d\xi_2'\,d\xi_1'\,d\xi_2\,d\xi_1.\end{aligned}\tag{3.17}$$

This representation holds provided

$$\begin{aligned}&\delta_1,\ \delta_2,\ \delta_1',\ \delta_2'>0;\\
&\Re(v)-\Re(s)-\delta_2>-1;\quad \Re(s)-\delta_2'>0;\\
&\frac{3}{2}>2\Re(s)-\delta_1'>0;\quad -\frac{1}{2}>2\Re(v)-2\Re(s)-\delta_1>-2;\\
&\Re(w)>\delta_1+\delta_1'-2\Re(v)>0.\end{aligned}$$

We remark that for all the other choices of $W^{\pm}_{F,\mathbb{R}}$, one obtains similar expressions.

For fixed F_0 a Maass cuspform of weight zero, or a classical holomorphic (or anti-holomorphic) cuspform of weight κ_0, the corresponding *archimedean* sum over the K–types κ in the moment expansion can be evaluated using the effect of the Maass operators on F_0 given explicitly in [DFI] (see especially (4.70), (4.77), (4.78) and (4.83)).

We summarize the main result of this section in the following

Theorem 3.18. *Let $P\acute{e}(g)$ defined in (3.1) be the Poincaré series associated to φ. Then, for $s,\ v,\ w\in\mathbb{C}$ with sufficiently large real parts, and f a cuspform on $SL_3(\mathbb{Z})$, we have*

$$\int\limits_{ZSL_3(\mathbb{Z})\backslash G} P\acute{e}(g)\,|f(g)|^2\,dg \;=\; \sum_{F \text{ in } GL_2}\frac{1}{2\pi i}\int\limits_{\Re(s)=\sigma} L(v+1-s,\ f\otimes F)\,L(s,\ \bar f\otimes\bar F)\,\Gamma_\varphi(s)\,ds$$

where F runs over an orthonormal basis for all level-one cuspforms together with vertical integrals of all level-one Eisenstein series on $GL_2(\mathbb{Q})$, with no restriction on the right K–types. The weight function $\Gamma_\varphi(s)$ is given by

$$\begin{aligned}\Gamma_\varphi(s) \;=\; \sum_{\pm}\rho_F(\pm1)\overline{\rho_F(\pm1)}\cdot\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (t^2y)^{v-s+\frac12}\ (t'^{\,2}y')^{s-\frac12}\cdot\int_0^\infty r^2\varphi(r)\,J_\kappa(2\pi rt)\,J_\kappa(2\pi rt')\,\frac{dr}{r}\\
\cdot W_\mu\begin{pmatrix}ty&&\\&t&\\&&1\end{pmatrix}W^{\pm}_{F,\mathbb{R}}\begin{pmatrix}y&\\&1\end{pmatrix}\overline{W}_\mu\begin{pmatrix}t'y'&&\\&t'&\\&&1\end{pmatrix}\overline{W}^{\pm}_{F,\mathbb{R}}\begin{pmatrix}y'&\\&1\end{pmatrix}\frac{dy}{y^2}\,\frac{dt}{t}\,\frac{dy'}{y'^2}\,\frac{dt'}{t'},\end{aligned}$$

with all four possible sign choices in the sum.

§4. Spectral decomposition of Poincaré series

We begin by showing that our Poincaré series Pé(g) is a degenerate GL_3 object (i.e., the cuspforms on $SL_3(\mathbb{Z})$ **do not** contribute to its spectral decomposition). We have the following

Proposition 4.1. *The Poincaré series Pé(g) is orthogonal to the space of cuspforms on $SL_3(\mathbb{Z})$.*

Proof: Let f be a cuspform on $SL_3(\mathbb{Z})$ with Fourier expansion

$$f(g) \;=\; \sum_{\gamma\in N(\mathbb{Z})\backslash H(\mathbb{Z})} \sum_{\ell_1=1}^{\infty} \sum_{\ell_2\neq 0} \frac{a(\ell_1,\,\ell_2)}{|\ell_1\ell_2|}\cdot W(L\gamma g).$$

Unwinding twice, it follows, as before, that

$$\int\limits_{ZSL_3(\mathbb{Z})\backslash G} \text{Pé}(g)\bar{f}(g)\,dg \;=\; \sum_{\ell_1,\,\ell_2} \frac{\overline{a(\ell_1,\,\ell_2)}}{|\ell_1\ell_2|} \int\limits_{ZN(\mathbb{Z})\backslash G/K} \varphi(g)\,\overline{W}(Lg)\,dg. \tag{4.2}$$

Now, write $g\in G$ in Iwasawa form,

$$\begin{aligned} g &= \begin{pmatrix} 1 & x_1 & x_2\\ & 1 & x_3\\ & & 1\end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\begin{pmatrix} d & & \\ & d & \\ & & d\end{pmatrix} k \qquad (y_1,\,y_2>0,\;k\in K)\\ &= \begin{pmatrix} y_1y_2\,d & & \\ & y_1 d & \\ & & d\end{pmatrix}\begin{pmatrix} 1 & x_1/y_2 & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & (x_2-x_1x_3)/y_1y_2\\ 0 & 1 & x_3/y_1\\ 0 & 0 & 1\end{pmatrix} k. \end{aligned}$$

Then,

$$\varphi(g) \;=\; (y_1^2y_2)^v\,\varphi\begin{pmatrix} 1 & 0 & (x_2-x_1x_3)/y_1y_2\\ 0 & 1 & x_3/y_1\\ 0 & 0 & 1\end{pmatrix} \tag{4.3}$$

and

$$W(Lg) \;=\; e^{2\pi i(\ell_2x_1+\ell_1x_3)}\cdot W\begin{pmatrix} \ell_1y_1|\ell_2|y_2 & & \\ & \ell_1y_1 & \\ & & 1\end{pmatrix}. \tag{4.4}$$

Also, the integral in the right hand side of (4.2) can be written explicitly as

$$\int\limits_{ZN(\mathbb{Z})\backslash G/K} \cdots\; dg \;=\; \int\limits_{y_2=0}^{\infty}\int\limits_{y_1=0}^{\infty}\int\limits_{x_3=-\infty}^{\infty}\int\limits_{x_2=-\infty}^{\infty}\int\limits_{x_1=0}^{1} \cdots\; dx_1\,dx_2\,dx_3\,\frac{dy_1}{y_1^3}\,\frac{dy_2}{y_2^3}.$$

Letting

$$x_1 = t_1, \qquad x_2 = t_2 + t_1 t_3, \qquad x_3 = t_3,$$

the inner integral over t_1 is

$$\int_0^1 e^{-2\pi i \ell_2 t_1}\, dt_1 \;=\; 0$$

(since $\ell_2 \neq 0$). Thus,

$$\int\limits_{ZSL_3(\mathbb{Z})\backslash G} \mathrm{P\acute{e}}(g)\bar{f}(g)\,dg \;=\; 0. \qquad \square$$

Now write the Poincaré series as

$$\mathrm{P\acute{e}}(g) \quad = \sum_{\gamma\in H(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \varphi(\gamma g) \quad = \sum_{\gamma\in P(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \; \sum_{\beta\in U(\mathbb{Z})} \varphi(\beta\gamma g)$$

where $P(\mathbb{Z})$ denotes the subgroup of $SL_3(\mathbb{Z})$ with the bottom row $(0, 0, 1)$. By the Poisson summation formula, we have

$$\begin{aligned}
\sum_{\beta\in U(\mathbb{Z})} \varphi(\beta g) \quad &= \sum_{m_2,\, m_3=-\infty}^{\infty} \varphi\left(\begin{pmatrix} 1 & & m_2 \\ & 1 & m_3 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) \\
&= \sum_{m_2,\, m_3=-\infty}^{\infty} \varphi\left(\begin{pmatrix} 1 & x_1 & x_2+m_2 \\ & 1 & x_3+m_3 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) \\
&= \sum_{m_2,\, m_3=-\infty}^{\infty} C_\varphi^{(m_2,\, m_3)}(x_1,\, y_1,\, y_2)\, e^{2\pi i (m_2 x_2 + m_3 x_3)},
\end{aligned}$$

where $C_\varphi^{(m_2,\, m_3)}(x_1,\, y_1,\, y_2)$ is given by

$$\begin{aligned}
C_\varphi^{(m_2,\, m_3)}(x_1,\, y_1,\, y_2) \;&=\; (y_1^2 y_2)^v \int\limits_{\mathbb{R}^2} \varphi\begin{pmatrix} 1 & 0 & (u_2 - x_1 u_3)/y_1 y_2 \\ 0 & 1 & u_3/y_1 \\ 0 & 0 & 1 \end{pmatrix} e^{-2\pi i (m_2 u_2 + m_3 u_3)}\, du_2\, du_3 \\
&= (y_1^2 y_2)^{v+1} \int\limits_{\mathbb{R}^2} \varphi\begin{pmatrix} 1 & & t_2 \\ & 1 & t_3 \\ & & 1 \end{pmatrix} e^{-2\pi i [m_2 y_1 y_2 t_2 + (m_2 x_1 + m_3) y_1 t_3]}\, dt_2\, dt_3. \qquad (4.5)
\end{aligned}$$

Therefore, denoting $C_\varphi^{(m_2,\, m_3)}(x_1,\, y_1,\, y_2)\, e^{2\pi i (m_2 x_2 + m_3 x_3)}$ by $\widehat{\varphi}_g(m_2, m_3)$, we can write

$$\mathrm{P\acute{e}}(g) \quad = \sum_{\gamma\in P(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \; \sum_{m_2,\, m_3=-\infty}^{\infty} \widehat{\varphi}_{\gamma g}(m_2, m_3).$$

Thus, by (4.5) we can decompose the Poincaré series $\mathrm{P\acute{e}}(g)$ as

$$\mathrm{P\acute{e}}(g) \;=\; C(\varphi)\cdot E^{2,1}(g, v+1) \;+\; \mathrm{P\acute{e}}^*(g) \qquad (4.6)$$

where $E^{2,1}(g, v+1)$ is the maximal parabolic Eisenstein series on $SL_3(\mathbb{Z})$ and

$$C(\varphi) \;=\; \int\limits_{\mathbb{R}^2} \varphi\begin{pmatrix} 1 & & t_2 \\ & 1 & t_3 \\ & & 1 \end{pmatrix} dt_2\, dt_3. \tag{4.7}$$

To obtain a spectral decomposition, we need to present the Poincaré series Pé(g) with the maximal parabolic Eisenstein series on $SL_3(\mathbb{Z})$ removed in a more useful way. To do so, we first write

$$\begin{aligned} \text{Pé}^*(g) \;&=\; \sum_{\gamma\in P(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \;\sum_{\substack{m_2,\, m_3=-\infty \\ (m_2,m_3)\neq(0,0)}}^{\infty} \widehat{\varphi}_{\gamma g}(m_2, m_3) \\ &=\; \sum_{\gamma\in P(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \;\sum_{\substack{\psi\in(U(\mathbb{Z})\backslash U(\mathbb{R}))^{\wedge} \\ \psi\neq 1}} \widehat{\varphi}_{\gamma g}(\psi), \end{aligned}$$

where

$$\widehat{\varphi}_g(\psi) \;=\; \int_U \varphi(ug)\overline{\psi(u)}\, du.$$

For $\beta \in H(\mathbb{Z})$, we observe that

$$\begin{aligned} \widehat{\varphi}_{\beta g}(\psi) \;=\; \int_U \varphi(u\beta g)\overline{\psi(u)}\, du \;=\; \int_U \varphi(\beta\beta^{-1}u\beta g)\overline{\psi(u)}\, du \;&=\; \int_U \varphi(\beta^{-1}u\beta g)\overline{\psi(u)}\, du \\ &=\; \int_U \varphi(ug)\overline{\psi(\beta u\beta^{-1})}\, du, \end{aligned} \tag{4.8}$$

as $\varphi(\beta g) = \varphi(g)$ for $\beta \in H(\mathbb{Z})$ and $g \in G$. Setting $\psi^{\beta}(u) = \psi(\beta u\beta^{-1})$, the last integral in (4.8) is $\widehat{\varphi}_g(\psi^{\beta})$.

Consider the characters on $U(\mathbb{Z})\backslash U(\mathbb{R})$

$$\psi^m(u) = e^{2\pi i m u_3} \qquad \left(m \in \mathbb{Z}^{\times} \text{ and } u = \begin{pmatrix} 1 & & u_2 \\ & 1 & u_3 \\ & & 1 \end{pmatrix} \right).$$

Since every non-trivial character on $U(\mathbb{Z})\backslash U(\mathbb{R})$ is obtained as $(\psi^m)^{\beta}$, for unique $m \in \mathbb{Z}^{\times}$ and $\beta \in P^{1,1}(\mathbb{Z})\backslash H(\mathbb{Z})$, where $P^{1,1}(\mathbb{Z})$ is the parabolic subgroup of $H(\mathbb{Z})$, it follows from (4.8) that

$$\begin{aligned} \text{Pé}^*(g) \;&=\; \sum_{\gamma\in P(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \;\sum_{\beta\in P^{1,1}(\mathbb{Z})\backslash H(\mathbb{Z})} \;\sum_{m\in\mathbb{Z}^{\times}} \widehat{\varphi}_{\beta\gamma g}(\psi^m) \\ &=\; \sum_{\gamma\in P^{1,1,1}(\mathbb{Z})\backslash SL_3(\mathbb{Z})} \;\sum_{m\in\mathbb{Z}^{\times}} \widehat{\varphi}_{\gamma g}(\psi^m). \end{aligned}$$

Let

$$\Theta = \left\{ \begin{pmatrix} 1 & & \\ & * & * \\ & * & * \end{pmatrix} \right\}, \qquad U' = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}, \qquad U'' = \left\{ \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}.$$

Then

$$\text{Pé}^*(g) \;=\; \sum_{\gamma\in P^{1,2}(\mathbb{Z})\backslash SL_3(\mathbb{Z})}\;\sum_{\beta\in P^{1,1}(\mathbb{Z})\backslash\Theta(\mathbb{Z})}\;\sum_{m\in\mathbb{Z}^\times}\int_{U''}\overline{\psi}^m(u'')\cdot\left(\int_{U'}\varphi(u'u''\beta\gamma g)\,du'\right)du''.$$

Setting

$$\widetilde{\varphi}(g) \;=\; \int_{U'}\varphi(u'g)\,du',$$

the last expression of $\text{Pé}^*(g)$ becomes

$$\text{Pé}^*(g) \;=\; \sum_{\gamma\in P^{1,2}(\mathbb{Z})\backslash SL_3(\mathbb{Z})}\;\sum_{\beta\in P^{1,1}(\mathbb{Z})\backslash\Theta(\mathbb{Z})}\;\sum_{m\in\mathbb{Z}^\times}\int_{U''}\overline{\psi}^m(u'')\,\widetilde{\varphi}(u''\beta\gamma g)\,du''. \tag{4.9}$$

Let

$$\Phi(g) \;=\; \sum_{\beta\in P^{1,1}(\mathbb{Z})\backslash\Theta(\mathbb{Z})}\;\sum_{m\in\mathbb{Z}^\times}\int_{U''}\overline{\psi}^m(u'')\,\widetilde{\varphi}(u''\beta g)\,du''. \tag{4.10}$$

We shall need the following simple observation.

Lemma 4.11. *We have the equivariance*

$$\widetilde{\varphi}(pg) \;=\; |q|^{v+1}\cdot|a|^v\cdot|d|^{-2v-1}\cdot\widetilde{\varphi}(g), \qquad \left(\textit{for } p=\begin{pmatrix} q & b & c\\ & a & \\ & & d\end{pmatrix}\in GL_3(\mathbb{R})\right).$$

Proof: Indeed, since

$$\begin{pmatrix}1 & & t\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} q & b & c\\ & a & \\ & & d\end{pmatrix} = \begin{pmatrix} q & b & td+c\\ & a & \\ & & d\end{pmatrix} = \begin{pmatrix} q & b & \\ & a & \\ & & d\end{pmatrix}\begin{pmatrix}1 & & (td+c)/q\\ & 1 & \\ & & 1\end{pmatrix},$$

we have

$$\widetilde{\varphi}(pg) \;=\; \int_{U'}\varphi(u'pg)\,du' \;=\; \left|\frac{qa}{d^2}\right|^v\cdot\int_{\mathbb{R}}\varphi\left(\begin{pmatrix}1 & & (td+c)/q\\ & 1 & \\ & & 1\end{pmatrix}g\right)dt \;=\; |q|^{v+1}\cdot|a|^v\cdot|d|^{-2v-1}\widetilde{\varphi}(g). \qquad\square$$

Assuming g of the form

$$g \;=\; \begin{pmatrix} a & *\\ & g'\end{pmatrix} \qquad (a\in\mathbb{R}^\times \text{ and } g'\in GL_2(\mathbb{R})),$$

(we can always do using the Iwasawa decomposition), and decomposing it as

$$g \;=\; \begin{pmatrix} a & *\\ & I_2\end{pmatrix}\begin{pmatrix} 1 & \\ & g'\end{pmatrix},$$

we have

$$\widetilde{\varphi}(g) \;=\; |a|^{v+1}\cdot\widetilde{\varphi}\begin{pmatrix}1 & \\ & g'\end{pmatrix}.$$

Since

$$\begin{pmatrix}1 & \\ & D\end{pmatrix} g \;=\; \begin{pmatrix}a & * \\ & Dg'\end{pmatrix} \qquad (\text{for } D\in GL_2(\mathbb{R})),$$

it follows that $\Phi(g)$ defined in (4.10) descends to a GL_2 Poincaré series, with the corresponding Eisenstein series removed, of the type studied in [Di-Ga1], [Di-Go1], [Di-Go2]. Setting

$$\varphi^{(2)}\begin{pmatrix}1 & x\\ & 1\end{pmatrix} \;=\; \widetilde{\varphi}\begin{pmatrix}1 & & \\ & 1 & x\\ & & 1\end{pmatrix} \qquad (x\in\mathbb{R})$$

and extending it to $GL_2(\mathbb{R})$ by

$$\varphi^{(2)}\left(\begin{pmatrix}a & \\ & d\end{pmatrix} gk\right) \;=\; \left|\frac{a}{d}\right|^{\frac{3v+1}{2}}\cdot\;\varphi^{(2)}(g) \qquad (g\in GL_2(\mathbb{R}),\, k\in O_2(\mathbb{R})),$$

we can write

$$\Phi\begin{pmatrix}a & *\\ & g'\end{pmatrix} \;=\; |a|^{v+1}\cdot|\det g'|^{-\frac{v+1}{2}}\;\cdot \sum_{\beta\in P^{1,1}(\mathbb{Z})\backslash SL_2(\mathbb{Z})}\;\sum_{m\in\mathbb{Z}^\times}\int_N \overline{\psi}^{\,m}(n)\,\varphi^{(2)}(n\beta g')\,dn, \tag{4.12}$$

with N the subgroup of upper-triangular unipotent elements in $GL_2(\mathbb{R})$. Note that, for

$$\varphi\begin{pmatrix}I_2 & u\\ & 1\end{pmatrix} \;=\; \left(1+||u||^2\right)^{-\frac{w}{2}},$$

we have

$$\begin{aligned}\varphi^{(2)}\begin{pmatrix}1 & x\\ & 1\end{pmatrix} &= \widetilde{\varphi}\begin{pmatrix}1 & & \\ & 1 & x\\ & & 1\end{pmatrix} \;=\; \int_{U'}\varphi\left(u'\begin{pmatrix}1 & & \\ & 1 & x\\ & & 1\end{pmatrix}\right)du' \\ &= \int_{-\infty}^{\infty}\left(1+u^2+x^2\right)^{-\frac{w}{2}}du \;=\; \sqrt{\pi}\,\frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})}\cdot\left(1+x^2\right)^{\frac{1-w}{2}}.\end{aligned} \tag{4.13}$$

Then, by (2.2), (2.3) and (5.8) in [Di-Go1], it follows that, for an orthonormal basis of Maass cuspforms which are simultaneous eigenfunctions of all the Hecke operators, we have the spectral decomposition

$$\begin{aligned}\Phi\begin{pmatrix}a & *\\ & g'\end{pmatrix} \;=\; &\frac{1}{2}\sum_{F-\text{even}}\overline{\rho_F(1)}\,L\Big(\frac{3v}{2}+1,F\Big)\,\mathcal{G}\Big(\frac{1}{2}+i\mu_F;\frac{3v+1}{2},w-1\Big)\,|a|^{v+1}\,|\det g'|^{-\frac{v+1}{2}}\,F(g') \\ &+\;\frac{1}{4\pi i}\int_{\Re(s)=\frac{1}{2}}\frac{\zeta(\frac{3v}{2}+\frac12+s)\,\zeta(\frac{3v}{2}+\frac32-s)}{\pi^{-1+s}\,\Gamma(1-s)\,\zeta(2-2s)}\,\mathcal{G}\Big(1-s;\frac{3v+1}{2},w-1\Big)\,|a|^{v+1}\,|\det g'|^{-\frac{v+1}{2}}\,E(g',s)\,ds,\end{aligned}$$

where

$$\mathcal{G}(s;v,w) \;=\; \pi^{-v+\frac{1}{2}}\,\frac{\Gamma\big(\frac{-s+v+1}{2}\big)\Gamma\big(\frac{s+v}{2}\big)\Gamma\big(\frac{-s+v+w}{2}\big)\Gamma\big(\frac{s+v+w-1}{2}\big)}{\Gamma(\frac{w+1}{2})\Gamma\big(v+\frac{w}{2}\big)}.$$

This decomposition holds provided $\Re(v)$ and $\Re(w)$ are sufficiently large. Hence, by (4.9) and (4.10), $\text{Pé}^*(g)$ has the induced spectral decomposition from GL_2,

$$\text{Pé}^*(g) \;=\; \frac{1}{2}\sum_{F-\text{even}} \overline{\rho_F(1)}\, L\Big(\frac{3v}{2}+1,F\Big)\,\mathcal{G}\Big(\frac{1}{2}+i\mu_F;\frac{3v+1}{2},w-1\Big)\,E_F^{1,2}(g,v+1)$$
$$+\;\frac{1}{4\pi i}\int\limits_{\Re(s)=\frac{1}{2}} \frac{\zeta(\frac{3v}{2}+\frac{1}{2}+s)\,\zeta(\frac{3v}{2}+\frac{3}{2}-s)}{\pi^{-1+s}\,\Gamma(1-s)\,\zeta(2-2s)}\;\mathcal{G}\Big(1-s;\frac{3v+1}{2},w-1\Big)\,E^{1,1,1}\Big(g,\frac{v+1}{2}-\frac{s}{3},\frac{2s}{3}\Big)\,ds.$$

By Godement's criterion (see [Bo]), the minimal parabolic Eisenstein series $E^{1,1,1}$ inside the integral converges absolutely and uniformly on compact subsets of G/ZK for $\Re(v)$ sufficiently large. The meromorphic continuation of the Poincaré series $\text{Pé}(g)$ in $(v,w)\in\mathbb{C}^2$ follows by shifting the contour similarly to Section 5 of [Di-Go1], or Theorem 4.17 in [Di-Ga1].

We summarize the main result of this section in the following theorem.

Theorem 4.14. *For $\Re(v)$ and $\Re(w)$ sufficiently large, the Poincaré series Pé(g) associated to*

$$\varphi\begin{pmatrix} I_2 & u\\ & 1\end{pmatrix} \;=\; \big(1+||u||^2\big)^{-\frac{w}{2}}$$

has the spectral decomposition

$$\text{Pé}(g) \;=\; \frac{2\pi}{w-2}\cdot E^{2,1}(g,v+1)$$
$$+\;\frac{1}{2}\sum_{F-\text{even}} \overline{\rho_F(1)}\, L\Big(\frac{3v}{2}+1,F\Big)\,\mathcal{G}\Big(\frac{1}{2}+i\mu_F;\frac{3v+1}{2},w-1\Big)\,E_F^{1,2}(g,v+1)$$
$$+\;\frac{1}{4\pi i}\int\limits_{\Re(s)=\frac{1}{2}} \frac{\zeta(\frac{3v}{2}+\frac{1}{2}+s)\,\zeta(\frac{3v}{2}+\frac{3}{2}-s)}{\pi^{-1+s}\,\Gamma(1-s)\,\zeta(2-2s)}\;\mathcal{G}\Big(1-s;\frac{3v+1}{2},w-1\Big)\,E^{1,1,1}\Big(g,\frac{v+1}{2}-\frac{s}{3},\frac{2s}{3}\Big)\,ds.$$

Final Remark. Let φ on U be defined by

$$\varphi\begin{pmatrix} I_2 & u\\ & 1\end{pmatrix} \;=\; 2^{1-w}\,\sqrt{\pi}\,\frac{\Gamma(\frac{w}{2})\,\big(1+||u||^2\big)^{-\frac{w}{2}}\,F(\frac{w}{2},\frac{w}{2};w;\frac{1}{1+||u||^2})}{\Gamma(\frac{w-1}{2})},$$

and consider the Poincaré series $\text{Pé}(g)$ attached to this choice of φ. Representing the hypergeometric function by its power series,

$$F(\alpha,\beta;\gamma;z) \;=\; \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}\,\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)}\,z^m \qquad (|z|<1),$$

and using the last identity in (4.13), it follows, as in [Di-Ga2], Section 3, that the Poincaré series $\text{Pé}(g)$ with $v=0$ satisfies a shifted functional equation (involving an Eisenstein series) as $w\to 2-w$ (see also [G] and [Di-Go1]).

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